THE CONTINUOUS SOLUTIONS OF A GENERALIZED DHOMBRES FUNCTIONAL EQUATION

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Abstract. We consider the functional equation $f(xf(x)) = \varphi(f(x))$ where $\varphi\colon J \to J$ is a given increasing homeomorphism of an open interval $J \subset (0,\infty)$ and $f\colon (0,\infty) \to J$ is an unknown continuous function. In a series of papers by P. Kahlig and J. Smítal it was proved that the range of any non-constant solution is an interval whose end-points are fixed under φ and which contains in its interior no fixed point except for 1. They also provide a characterization of the class of monotone solutions and prove a necessary and sufficient condition for any solution to be monotone.

In the present paper we give a characterization of the class of continuous solutions of this equation: We describe a method of constructing solutions as pointwise limits of solutions which are piecewise monotone on every compact subinterval. And we show that any solution can be obtained in this way. In particular, we show that if there exists a solution which is not monotone then there is a continuous solution which is monotone on no subinterval of a compact interval $I \subset (0,\infty)$.

Keywords: iterative functional equation, equation of invariant curves, general continuous solution

MSC 2000: 39B12, 39B22, 26A18

1. Introduction

Throughout this paper by a function we always mean a c o n t i n u o u s function. We consider the functional equation

$$f(xf(x)) = \varphi(f(x)), \quad x \in (0,\infty). \tag{1.1}$$

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Throughout the paper we assume that $J = (p, q) \subset (0, \infty)$ is an open interval, $\varphi\colon J \to J$ is a given homeomorphism of J, i.e., an increasing surjective function, and f is an unknown function, defined for $x \in (0, \infty)$, with values in J. Thus, by a solution of (1.1) we always mean a *continuous* solution.

This equation is a special case of *equations of invariant curves*. A survey of some general results can be found in [6] (cf. also [7]). Concerning the equation (1.1), the case $\varphi(y) = y^2$ was considered by Dhombres in [1]. More general types of φ have been studied, e.g., in a series of papers [2]–[5], where other references can be found.

We recall the main results from [4]. The range R_f of any non-constant solution f of (1.1) must be a φ-invariant interval (i.e., $\varphi(R_f) = R_f$), and each of the sets $\{x; f(x) < 1\}$, $\{x; f(x) = 1\}$ and $\{x; f(x) > 1\}$ is an interval, possibly empty or degenerate. Moreover, R_f contains no fixed point of φ different from 1, and if $1 \in R_f$ then $\varphi(1) = 1$.

However, to describe the class of solutions of (1.1) it suffices to consider the case $R_f \subset (0, 1]$. This follows easily by the facts given above, since the transformation $f \mapsto \tilde{f}$, where $\widetilde{\varphi}$ is defined by $\widetilde{\varphi}(x) = 1/(\varphi(1/x))$, is a bijection between the solutions of (1.1) and of the conjugate equation $\tilde{f}(x\tilde{f}(x)) = \widetilde{\varphi}(\tilde{f}(x))$, $x \in (0, \infty)$. This transformation maps constant, increasing, decreasing or continuous functions to functions with the same respective properties (for details, see [4]).

Moreover, if $R_f = (0, 1]$, then $f(x) \in (0, 1)$ on an interval $(0, a)$ or (a, ∞) and $f(x) = 1$ on the complement, and vice versa: If a function g defined on an interval $I \subset (0, \infty)$ satisfies the equation and $R_g = (0, 1)$ then $I = (0, a)$ or (b, ∞), possibly with $a = \infty$, and g can be uniquely extended to a solution f of (1.1) by $f(x) = 1$ for $x \in (0, \infty) \setminus I$.

1.1 Theorem (Cf. [4].). *For $x > 0$ and $y \in J$ put $\Phi(x, y) = (xy, \varphi(y))$. Then Φ is a homeomorphism of $(0, \infty) \times J$ which maps the graph of any continuous solution f of (1.1) onto itself.*

By the above argument, the class $\mathcal{S}(J, \varphi)$ of solutions of (1.1) corresponding to an arbitrary open interval $J \subset (0, \infty)$ and a homeomorphism φ of J is determined by the classes $\mathcal{S}(J, \varphi)$ with J, φ satisfying the conditions

$$J = (p, q),\ 0 \leqslant p < q \leqslant 1, \quad \text{and} \quad \varphi(y) \neq y \text{ for } x \in J, \tag{1.2}$$

with the exception that, in the case $q = 1$, the domain of the solution may be a proper subinterval $(0, a)$ or (a, ∞) of $(0, \infty)$. We will assume (1.2) with this reservation throughout the remainder of the paper.

To simplify notation we will use in the sequel the following notation. For any $u, v \in J$, let

$$P(u,v) = \prod_{k=0}^{\infty} \frac{\varphi^k(u)}{\varphi^k(v)}, \quad \text{and} \quad Q(u,v) = \prod_{k=1}^{\infty} \frac{\varphi^{-k}(u)}{\varphi^{-k}(v)}. \tag{1.3}$$

The main results concerning monotone solutions are the following ones.

1.2 Theorem (Cf. [4].). *Assume* (1.2). *Then any monotone solution of* (1.1) *is non-decreasing if $\varphi(y) > y$ in J, and non-increasing otherwise (i.e., if $\varphi(y) < y$ in J).*

1.3 Theorem (Cf. [5].). *Assume* (1.2) *with $q = 1$. Then any continuous solution f of* (1.1) *is monotone.*

1.4 Theorem (Cf. [5].). *Assume* (1.2) *with $q < 1$, and $\varphi(y) < y$ in J. Then any continuous solution of* (1.1) *is monotone if and only if*

$$P(u,v) = \infty, \quad \text{for any } u > v \text{ in } J. \tag{1.3}$$

1.5 Theorem (Cf. [5].). *Assume* (1.2) *with $q < 1$, and $\varphi(y) > y$ in J. Then any continuous solution of* (1.1) *is monotone if and only if*

$$Q(u,v) = \infty, \quad \text{for any } u > v \text{ in } J. \tag{1.4}$$

1.6 Remark. Assume (1.2). Then neither (1.3) nor (1.4) can be satisfied if $p > 0$, cf. [5]. Thus, by the above theorems, non-monotone continuous solutions of (1.1) do exist if and only if one of the following three conditions is satisfied: (i) $0 < p < q < 1$; (ii) $0 = p < q < 1$, $\varphi(y) < y$ in J, and (1.3) is not true; (iii) $0 = p < q < 1$, $\varphi(y) > y$ in J, and (1.4) is not true.

Recall that by a piecewise monotone real-valued function defined on an interval we always mean a function with finite number of monotone pieces. Our main results read as follows.

1.7 Theorem. *Assume* (1.2) *with $q < 1$. Then for any solution f of* (1.1) *there is a sequence $\{f_n\}_{n=1}^{\infty}$ of solutions converging to f, uniformly on every compact subset of J, and such that any f_n is piecewise monotone on every compact interval.*

1.8 Theorem. *Assume* (1.2), *and let* (1.1) *have a solution which is not monotone. Then there is a solution f of* (1.1) *and a compact interval $I \subset (0,\infty)$ such that f is monotone on no subinterval of I.*

1.9 Remark. It is easy to see that the space $\mathcal{S}(J,\varphi)$ of solutions of (1.1) is closed with respect to the almost uniform convergence, i.e., convergence which is uniform on any compact subset of $(0,\infty)$. Consequently, by Theorem 1.7, the space $\mathcal{S}(J,\varphi)$ is the almost uniform closure of the set $\mathcal{M}(J,\varphi)$ of solutions piecewise monotone on every compact interval. Theorem 1.8, along with Theorems 1.4 and 1.5 imply that, for $q < 1$, this is a non-trivial statement.

The paper is organized as follows. In the next Section 2 we provide some technical results. The proofs of Theorems 1.7 and 1.8 are in Section 3.

2. Preliminaries

To simplify the notation, throughout the paper we identify any function with its graph. In particular, if $f\colon (0,\infty) \to (0,\infty)$ is a function then $\Phi(f)$ stands for $\{(x,f(x)); x \in (0,\infty)\}$. The following terminology is standard: a sequence $\{(x_n,y_n)\}_{n=-\infty}^{\infty}$ is the *orbit under* Φ of a point (x_0,y_0) in $(0,\infty)\times J$ provided $(x_{n+1},y_{n+1}) = \Phi(x_n,y_n)$ for any integer n. Induction yields the formulas

$$y_n = \varphi^n(y_0) \quad \text{for any } n, \tag{2.1}$$

$$x_n = x_0 y_0 \varphi(y_0)\ldots\varphi^{n-1}(y_0) \quad \text{for } n > 0, \tag{2.2}$$

$$x_{-n} = \frac{x_0}{\varphi^{-1}(y_0)\varphi^{-2}(y_0)\ldots\varphi^{-n}(y_0)} \quad \text{for } n > 0. \tag{2.3}$$

2.1 Lemma (Cf. [4].). *Let $J \subset (0,1)$ be an open interval and let ψ be an increasing function from J onto J such that $\psi(y) \neq y$ in J. For any $y,z \in J$ denote $R(y,z) := \prod\limits_{k=1}^{\infty} (\psi^k(y)/\psi^k(z))$. Then*

(i) *$R(y,z) \geqslant 1$ whenever $y > z$.*

(ii) *If $R(y,z)$ is finite for some $z < y$ then $t \mapsto P(y,z)$ is continuous and strictly increasing function in $[z,y]$.*

(iii) *If $\lim\limits_{n\to\infty} \psi^n(x) \neq 0$ for some $x \in J$ then, for any $y > z$ in J, $R(y,z) < \infty$.*

For any $(x,y) \in (0,\infty)\times J$ and any integer n, let $\pi_n(x,y)$ be the first coordinate of $\Phi^n(x,y)$.

2.2 Lemma. *Let f be a solution of* (1.1). *Then, for any n, $x \mapsto \pi_n(x, f(x))$ is a continuous strictly increasing function.*

P r o o f. For simplicity, put $x_n = \pi_n(x, f(x))$, $t_n = \pi_n(t, f(t))$ and assume, e.g., that $x < t$ but, for some n, $x_n = t_n$. Then, by (2.1)–(2.3), $f(x) \neq f(y)$ and hence, since φ is a homeomorphism, $f(x_n) \neq f(t_n)$. Thus, f would have two different values at x_n which is impossible. The case $x_n > t_n$ can be reduced to the previous one, by the continuity of π_n. □

2.3 Lemma. *Assume* (1.2) *with $q < 1$. Let $\{u_n, y_n\}_{n=-\infty}^{\infty}$ be the orbit of a point (u_0, y_0), and let $g\colon [u_1, u_0] \to J$ be a function such that $g(u_0) = y_0$ and $g(u_1) = y_1$. Assume that, for any n, $x \mapsto \pi_n(x, g(x))$ is strictly increasing on $[u_1, u_0]$. Then g can be extended to a solution $f\colon (0, \infty) \to J$ of* (1.1) *such that $f = \bigcup_{n=-\infty}^{\infty} \Phi^n(g)$.*

P r o o f. Since $\Phi^n(x, g(x)) = (\pi_n(x, g(x)), \varphi^n(g(x))$ and both $\pi_n(x, g(x))\colon [u_1, u_0] \to (0, \infty)$ and $\varphi^n(y)\colon J \to J$ are strictly increasing and continuous, f is (the graph of) a continuous function $I \to J$ where I is an interval. Since φ has no fixed points in $J = (0, q)$ and $q < 1$, (2.2) and (2.3) imply that $I = (0, \infty)$. □

2.4 Definition. Assume (1.2) with $q < 1$. We say that the points $T_0, T_1, \ldots, T_k$ in $(0, \infty) \times J$, $T_i = (x^i, y^i)$, are *consistent* if $x^0 > x^1 > \ldots > x^k$, $x^k = x^0 y^0$,

$$Q(y^i, y^{i-1}) \geqslant \frac{x^i}{x^{i-1}} \quad \text{if} \ \ y^{i-1} > y^i, \tag{2.4}$$

and

$$P(y^{i-1}, y^i) \geqslant \frac{x^i}{x^{i-1}} \quad \text{if} \ \ y^{i-1} < y^i, \tag{2.5}$$

whenever $0 < i \leqslant k$. We say that the points $T_0, T_1, \ldots T_k$ are *strictly consistent* if the inequalities $\geqslant$ in (2.4) and (2.5) are replaced by the strict inequalities $>$.

Clearly, the infinite products in (2.4) and (2.5) are finite and positive, and both conditions are satisfied also for $y^{i-1} = y^i$.

2.5 Lemma. *Assume* (1.2) *with $q < 1$. Let f be a solution of* (1.1) *such that $f(v_0) = z_0$. Let $x^0 = v_0 > x^1 > \ldots > x^k = v_1 = v_0 z_0$. Then the points $T_i = (x^i, f(x^i))$, $0 \leqslant i \leqslant k$, are consistent.*

P r o o f. Put $f(x^i) = y_i$ and $x_n^i = \pi_n(x^i, y^i)$, and assume that $y^{i-1} > y^i$, for some i. By Lemma 2.2, $x_n^i < x_n^{i-1}$ for any n. Thus, by (2.3),

$$1 < \frac{x_{-n}^{i-1}}{x_{-n}^i} = \frac{x^{i-1}}{x^i} \prod_{k=1}^{n} \frac{\varphi^{-k}(y^i)}{\varphi^{-k}(y^{i-1})} \quad \text{for any} \ \ n > 0,$$

whence

$$1 \leqslant \frac{x^{i-1}}{x^i} Q(y^i, y^{i-1})$$

and (2.4) follows.

On the other hand, if $y^{i-1} < y^i$ then similarly, by (2.2),

$$1 < \frac{x_n^{i-1}}{x_n^i} = \frac{x^{i-1}}{x^i} \prod_{k=0}^{n} \frac{\varphi^k(y^{i-1})}{\varphi^k(y^i)} \qquad \text{for any} \quad n > 0,$$

and letting $n \to \infty$ we get (2.5). $\square$

2.6 Lemma. *Assume* (1.2) *with* $q < 1$. *Let* $[u_0, v_0] \subset (0, \infty)$, $[z_0, y_0] \subset J$, *and let* $P(z_0, y_0) > u_0/v_0$. *Then there is a decreasing function* τ *from* $[u_0, v_0]$ *onto* $[z_0, y_0]$ *such that*

$$P(\tau(v), \tau(u)) > \frac{u}{v}, \qquad \textit{whenever} \quad u_0 \leqslant u < v \leqslant v_0. \tag{2.6}$$

Proof. Denote $P(z_0, y_0) = \alpha$. Then $0 < \alpha < 1$. Therefore there is a function ϱ from $[\alpha, 1]$ onto $[u_0, v_0]$ such that $\varrho(x)/x$ is increasing. E.g., let ϱ be the affine function connecting the points (α, u_0) and $(1, v_0)$. Put $\lambda(x) = \varrho(P(z_0, x))$. Since $P(z_0, x)$ is decreasing (cf. Lemma 2.1) and ϱ is increasing, λ is a decreasing function and by Lemma 2.1(i), it is continuous. It maps $[z_0, y_0]$ onto $[u_0, v_0]$. Put $\tau = \lambda^{-1}$.

Since $\varrho(x)/x$ is increasing and $P(z_0, y) < P(z_0, z)$, for $z < y$ in $[z_0, y_0]$,

$$\frac{\varrho(P(z_0, y))}{P(z_0, y)} < \frac{\varrho(P(z_0, z))}{P(z_0, z)},$$

hence

$$\frac{\lambda(y)}{\lambda(z)} = \frac{\varrho(P(z_0, y))}{\varrho(P(z_0, z)} < \frac{P(z_0, y)}{P(z_0, z)} < 1.$$

This implies (2.6). $\square$

2.7 Lemma. *Assume* (1.2) *with* $q < 1$. *Let* $[u_0, v_0] \subset (0, \infty)$, $[y_0, z_0] \subset J$, *and let* $Q(y_0, z_0) > u_0/v_0$. *Then there is an increasing function* τ *from* $[u_0, v_0]$ *onto* $[y_0, z_0]$ *such that*

$$Q(\tau(u), \tau(v)) > \frac{u}{v}, \qquad \textit{whenever } u_0 \leqslant u < v \leqslant v_0. \tag{2.7}$$

Proof. It is similar to the previous one. Denote $Q(y_0, z_0) = \beta$. Then $0 < \beta < 1$, and there is a function ϱ from $[\beta, 1]$ onto $[u_0, v_0]$ such that $\varrho(x)/x$ is increasing. Put

$\lambda(x) = \varrho(Q(x, z_0))$. Then λ is increasing and, by Lemma 2.1(i), it is continuous, mapping $[y_0, z_0]$ onto $[u_0, v_0]$. Put $\tau = \lambda^{-1}$. Since $\varrho(x)/x$ is increasing, for $y < z$ in $[y_0, z_0]$ we have

$$\frac{\lambda(y)}{\lambda(z)} = \frac{\varrho(Q(y, z_0))}{\varrho(Q(z, z_0))} < \frac{Q(y, z_0)}{Q(z, z_0)} = Q(y, z) < 1.$$

This implies (2.7). □

2.8 Lemma. *Assume* (1.2) *with* $q < 1$. *Let* $T_0, T_1, \ldots, T_k$ *be strictly consistent points, cf. Definition* 2.4. *Then there is a solution of* (1.1) *passing through all these points such that* f *is monotone on any interval* $[x^i, x^{i-1}]$.

Proof. Denote $I_i = [x^i, x^{i-1}]$, and $I = [x^k, x^0]$. Apply Lemmas 2.6 and 2.7 to get a function $\tau \colon I \to J$ such that, for any i, $\tau(x^i) = y^i$ and, for $u_0 = x^i$ and $v_0 = x^{i-1}$, (2.6) is satisfied if $y^i > y^{i-1}$, and (2.7) is satisfied otherwise. (If $y^{i-1} = y^i$ we let τ be constant on I_i.) Thus, for any i,

$$Q(\tau(u), \tau(v)) > \frac{u}{v} \quad \text{if } u < v \text{ in } I_i \text{ and } y^{i-1} > y^i, \tag{2.8}$$

and

$$P(\tau(v), \tau(u)) > \frac{u}{v} \quad \text{if } u < v \text{ in } I_i \text{ and } y^{i-1} < y^i. \tag{2.9}$$

Then $x \mapsto \pi_n(x, \tau(x))$ is strictly increasing on I. Indeed, if τ is nondecreasing on I_i then $x \mapsto \pi_n(x, \tau(x))$ is strictly increasing obviously for $n > 0$, and by (2.8) for $n < 0$. Similarly, if τ is nonincreasing on I_i then $\pi_n(x, \tau(x))$ is strictly increasing obviously for $n < 0$, and by (2.9) for $n > 0$. The result now follows by Lemma 2.3. □

The following lemma is technical. To make it simpler we use the following terminology. Let f be a real-valued function on a compact interval $I = [a, b]$. We say that $x \in I$ is an *M-point* if f has a local maximum at x, and x is an *m-point* if f has a local minimum at x. The M-points and m-points are *extremal points*.

Let $\mathcal{P}_k$ denote the proposition: There are points $x^0 = b > x^1 > \ldots > x^k = a$ such that

(i) If $x^{i-1} - x^i > \frac{2}{k}(b - a)$ then there is an interval $U \subset [x^i, x^{i-1}]$ such that f is monotone on U and the measure of $[x^i, x^{i-1}] \setminus U$ is less than $\frac{2}{k}(b - a)$.

(ii) Either $f(x^i) > \max\{f(x^{i-1}), f(x^{i+1})\}$ or $f(x^i) < \min\{f(x^{i-1}), f(x^{i+1})\}$, for any i, $0 < i < k$.

(iii) Any x^i, $0 < i < k$ is an extremal point.

(iv) For no i the points x^i, x^{i+1} are both M-points, or both m-points.

2.9 Lemma. *Let f be a real-valued function on a compact interval $[a,b]$ such that $f(a) \neq f(b)$. If the points a and b are both M-points or both m-points, then $\mathcal{P}_k$ is true for any even $k > 0$. If one of the points a, b is an M-point and the other one an m-point then $\mathcal{P}_k$ is true for any odd $k > 0$. Finally, if one of the points a, b is not extremal then $\mathcal{P}_k$ is true for any $k > 0$.*

Proof. Clearly $\mathcal{P}_1$ is true. Assume by induction that $\mathcal{P}_k$ is true whenever $k < n$, with the specification for even and odd k. Moreover, assume for simplicity that a is an m-point; otherwise the argument is similar. Consider three cases.

Case 1. The point b is an M-point and n is odd. If f is monotone and, hence, nondecreasing on an interval $[x,b]$ we may assume that $[x,b]$ is a maximal such interval. Then $x > a$ since f is not piecewise monotone. If f is monotone on an interval $[y,x]$, $y < x$, then f is nonincreasing on $[y,x]$, and strictly decreasing at x from the left, by the maximality of $[x,b]$; put $x^1 = x$. Then x^1 is an m-point and by the hypothesis, applied to the interval $[a,x^1]$ there are points $x^n = a < x^{n-1} < \ldots < x^1$ satisfying along with x^0 the conditions (i)–(iv).

On the other hand, if f is monotone on no interval $[y,x]$ where $y < x$, there is an m-point $y < x$ such that $x - y < \frac{2}{n}(b-a)$ and $f(y) < f(b)$. Then put $x^1 = y$ and apply the hypothesis to the interval $[a,x^1]$ as before.

Case 2. The point b is an m-point and n is even. Then the argument is similar as in the previous case.

Case 3. The point b is not extremal and n is arbitrary. Then there are points u,v in $(b - \frac{2}{n}(b-a), b)$ such that u is an M-point with $f(u) > f(b)$, and v an m-point with $f(v) < f(b)$. If n is even let $x^1 = u$, otherwise let $x^1 = v$, and apply the hypothesis to the interval $[a,x^1]$. □

2.10 Lemma. *Assume (1.2) with $q < 1$. Let $T_i = (x^i, y^i)$, $0 \leqslant i \leqslant k$ be strictly consistent points. Let $i > 0$ with $y^i < y^{i-1}$ and $x' \in (x^i, x^{i-1})$. Assume that $P(y,z) < \infty$ for any $y,z \in (y^i, y^{i-1})$. Then there are points $T' = (x', y')$, $T'' = (x'', y'')$ which form with the points T_i a strictly consistent system such that $x' < x'' < x^{i-1}$, and*

$$y^i < y'' < y' < y^{i-1}. \tag{2.10}$$

Proof. By (2.4) and Lemma 2.7 there is a point $y' \in (y^i, y^{i-1})$ such that

$$Q(y^i, y') > \frac{x^i}{x'} \quad \text{and} \quad Q(y', y^{i-1}) > \frac{x'}{x^{i-1}}. \tag{2.11}$$

It remains to find T'' such that

$$P(y'', y') > \frac{x'}{x''} \quad \text{and} \quad Q(y'', y^{i-1}) > \frac{x''}{x^{i-1}}. \tag{2.12}$$

By the second inequality in (2.11), $Q(y', y^{i-1}) > x''/x^{i-1}$, whenever x'' is sufficiently close to x'. By Lemma 2.2(i), $P(y', y)$ is continuous for $y \in (y^i, y^{i-1})$ hence, having fixed x'', (2.12) is satisfied for any y'' sufficiently close to y'. □

2.11 Lemma. *Assume* (1.2) *with* $q < 1$. *Let* $T_i = (x^i, y^i)$, $0 \leqslant i \leqslant k$ *be strictly consistent points. Let* $i > 0$ *with* $y^i > y^{i-1}$ *and* $x' \in (x^i, x^{i-1})$. *Assume that* $Q(y,z) < \infty$ *for any* $y, z \in (y^{i-1}, y^i)$. *Then there are points* $T' = (x', y')$, $T'' = (x'', y'')$ *which form with the points* T_i *a strictly consistent system such that* $x' < x'' < x^{i-1}$, *and*

$$y^i > y'' > y' > y^{i-1}. \tag{2.13}$$

P r o o f. By (2.5) and Lemma 2.6 there is a point $y' \in (y^{i-1}, y^i)$ such that

$$P(y', y^i) > \frac{x^i}{x'} \quad \text{and} \quad P(y^{i-1}, y') > \frac{x'}{x^{i-1}}. \tag{2.14}$$

It remains to find T'' such that

$$Q(y', y'') > \frac{x'}{x''}, \quad \text{and} \quad P(y^{i-1}, y'') > \frac{x''}{x^{i-1}}.$$

But this is possible since, by the second inequality in (2.14), $P(y^{i-1}, y') > x''/x^{i-1}$, whenever x'' is sufficiently close to x'. □

2.12 R e m a r k. If $\varphi(y) < y$ in J then $Q(u,v) < \infty$ for any $u, v \in J$. This follows by Lemma 2.2(ii). Hence in this case the condition "$Q(y,z) < \infty$ for any $y, z \in (y^{i-1}, y^i)$" in Lemma 2.11 need not be assumed. Similarly with Lemma 2.11 if $\varphi(y) < y$ in J.

3. Proof of the main results

3.1 P r o o f of Theorem 1.7. Assume (1.2) with $q < 1$ and let $\varphi(y) < y$ in J. Let $f(v_0) = y_0$ and let $v_1 = v_0 y_0$. Denote by I the interval $[v_1, v_0]$. If $f|_I$ is piecewise monotone (i.e., with finite number of maximal monotone pieces) then it is piecewise monotone on every compact subinterval of $(0, \infty)$. This follows by Theorem 1.1 since φ and $x \mapsto \pi_k(x, f(x))$, for any k, are strictly increasing. Put $f_n = f$ for every $n \geqslant 1$.

So assume that $f|_I$ is not piecewise monotone. Put $\delta = v_0 - v_1$. By Lemma 2.9 applied to the interval I there are points $v_0 = x^0 > x^1 > x^2 > \ldots > x^k = v_1$, where k is arbitrarily large, with the following properties:

(i) If $x^{i-1} - x^i > 2\delta/k$ then there is an interval $U \subset [x^i, x^{i-1}]$ such that f is monotone on U and the measure of $[x^i, x^{i-1}] \setminus U$ is less than $2\delta/k$.

(ii) For every i, $f|\{x^{i-1}, x^i, x^{i+1}\}$ is not monotone. Define points y^i as follows. Put $y^0 = f(x^0)(= y_0)$ and $y^k = f(x^k)$. For $i \notin \{0, k\}$ let $y^i < f(x^i)$ if $f(x^{i-1}) < f(x^i)$, and $y^i > f(x^i)$ if $f(x^{i-1}) > f(x^i)$. It is easy to see that the points y^i can be chosen in such a way that the points $T_i = (x^i, y^i)$ be strictly consistent and $|f(x^i) - y^i| < 1/k$. This follows by Lemma 2.5. Now apply Lemma 2.8 to obtain a continuous solution f_k of (1.1) passing through the points T_i. This would imply $\lim\limits_{n\to\infty} f_n = f$ uniformly on I when f is monotone on no subinterval of I.

However, if $U = [u, v]$ is a maximal interval such that $f|_U$ is monotone, say, nondecreasing, then we have to ensure the uniform convergence of the sequence $\{f_n\}$ on U. Toward this we take points u^n, v^n such that $u < u^n < v^n < v$, $u^n \to u$ and $v^n \to v$. Let $I_i = [x^i, x^{i-1}]$ be the interval occurring in the construction of f_n, and containing U (cf. (i)). Then we modify f_n on I_i as follows: We let be $f_n(x) = f(x)$ for $x \in [u_n, v_n]$ and extend this map to a nondecreasing map on I_i (applying Lemma 2.8, or rather its particular case, Lemma 2.7) such that $f_n(x^i) = y^i$ and $f(x^{i-1}) = y^{i-1}$.

Such a modified sequence $\{f_n\}$ converges to f, uniformly on I and hence, by Theorem 1.1, on any compact subinterval of $(0, \infty)$. Similarly, any f_k is piecewise monotone on every compact interval. This completes the proof of the theorem in the case $\varphi(y) < y$ for $y \in J$. $\square$

3.2 Lemma. *Assume* (1.2) *with* $q < 1$, *and* $\varphi(y) < y$ *in* J. *Let* $y^0 < t$ *be points in* J *such that* $P(t, y^0) < \infty$, *and let* $x^0 > 0$ *be arbitrary. Put* $x^2 = x^0 y^0$. *Then there are* $x^1 \in (x^2, x^0)$, $y^1 \in (y^0, t)$ *such that the points* $T_i = (x^i, y^i)$, $0 \leqslant i \leqslant 2$, *are strictly consistent.*

Proof. Denote $y^2 = \varphi(y^0)$. Since

$$\prod_{k=1}^{n} \frac{\varphi^{-k}(y^2)}{\varphi^{-k}(y^0)} = \frac{y^0}{\varphi^{-k}(y^0)} \to \frac{y^0}{q} \quad \text{for } k \to \infty$$

we have $y^0 = x^2/x^0 < Q(y^2, y^0)$. Hence there are points $x^1 \in (x^2, x^0)$, $r \in (y^0, t)$ such that $x^2/x^1 < Q(y^2, r)$. On the other hand, by Lemma 2.2(i), $P(y^0, x)$ is continuous for $x \in [y^0, t]$, and $x^1/x^0 < 1 = P(y^0, y^0)$, hence, if $s > y^0$ is sufficiently close to y^0 then $x^1/x^0 < P(y^0, s)$. To finish the proof put $y^1 = \min\{r, s\}$. $\square$

3.3 Proof of Theorem 1.8; the case $\varphi(y) < y$. Assume (1.2) with $q < 1$, $\varphi(y) < y$ in J, and let (1.1) have a solution which is not monotone. By Theorem 1.3, $q < 1$, and by Theorems 1.2 and 1.4, there are $u > v$ in J such that $P(u, v) < \infty$. Let $I = [a, b]$ be an arbitrary interval such that $bv = a$. Take $x^0 = b$, $y^0 = v$ and apply

Lemma 3.2 to get three strictly consistent points T_i. Keeping the notation from Lemma 3.2, let $c = x^1$ and $w = y^1$. Thus, $P(w, v) < \infty$ and hence, by Remark 2.12, Lemmas 2.10 and 2.11 are applicable to the interval $[c, b]$. Applying them we obtain a sequence $\{\mathcal{T}_k\}_{k=1}^{\infty}$ of sets of strictly consistent points in $[c, b] \times [v, w] \cup \{(a, \varphi(v)\}$, such that $\mathcal{T}_k$ has $2k+1$ points and $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, for any k. Applying Lemma 2.8 we obtain a sequence $\{f_k\}_{k=1}^{\infty}$ of piecewise monotone functions. The condition $\mathcal{T}_k \subset \mathcal{T}_{k+1}$ implies uniform convergence of f_n on I. Let f be the limit. Then f is a solution of (1.1), cf. Remark 1.9. Finally, to get f monotone on no subinterval of $[c, b]$ it suffices, at every step when applying Lemma 2.10 or 2.11, to choose x' in the middle of the corresponding interval. □

To prove Theorem 1.8 in the case $\varphi(y) > y$ we need a "dual" version of Lemma 3.2. The reason is that, by Lemma 2.2(ii), for any $u, v \in J$, $0 < Q(u, v) < \infty$ if $\varphi(y) < y$ in J, but $0 < P(u, v) < \infty$ if $\varphi(y) > y$ in J. See also Remark 2.12.

3.4 Lemma. *Assume* (1.2) *with $q < 1$ and $\varphi(y) > y$ in J. Let $y^0 > t$ be points in J such that $Q(y^0, t) < \infty$, and let $x^0 > 0$ be arbitrary. Put $x^2 = x^0 y^0$. Then there are $x^1 \in (x^2, x^0)$, $y^1 \in (t, y^0)$ such that the points $T_i = (x^i, y^i)$, $0 \leqslant i \leqslant 2$, are strictly consistent.*

Proof. Denote $y^2 = \varphi(y^0)$. Since

$$\prod_{k=0}^{n} \frac{\varphi^k(y^0)}{\varphi^k(y^2)} = \frac{y^0}{\varphi^{k+1}(y^0)} \longrightarrow \frac{y^0}{q} \quad \text{for} \quad k \to \infty$$

we have $y^0 = x^2/x^0 < P(y^0, y^2)$. Hence there are points $x^1 \in (x^2, x^0)$, $r \in (t, y^0)$ such that $x^2/x^1 < P(r, y^2)$. On the other hand, by Lemma 2.2(i), $Q(x, y^0)$ is continuous for $x \in [t, y^0]$, and $x^1/x^0 < 1 = Q(y^0, y^0)$, hence, if $s < y^0$ is sufficiently close to y^0 then $x^1/x^0 < Q(s, y^0)$. To finish the proof put $y^1 = \max\{r, s\}$. □

3.5 Proof of Theorem 1.8; the case $\varphi(y) > y$. Assume (1.2) with $q < 1$, $\varphi(y) > y$ in J, and let (1.1) have a solution which is not monotone. By Theorem 1.3, $q < 1$, and by Theorems 1.2 and 1.5, there are $u > v$ in J such that $Q(u, v) < \infty$. Let $I = [a, b]$ be an arbitrary interval such that $bu = a$. Take $x^0 = b$, $y^0 = u$ and apply Lemma 3.4 to get three strictly consistent points T_i. Keeping notation from Lemma 3.4, let $c = x^1$ and $w = y^1 < y^0$. Thus, $Q(u, w) < \infty$ and hence, by Remark 2.12, Lemmas 2.10 and 2.11 are applicable to the interval $[c, b]$.

The remainder of the proof is similar as in the case $\varphi(y) < y$ and we omit it. □

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