ORTHOGONALLY ADDITIVE FUNCTIONALS ON BV

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Abstract. In this paper we give a representation theorem for the orthogonally additive functionals on the space BV in terms of a non-linear integral of the Henstock-Kurzweil-Stieltjes type.

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1. Introduction

Orthogonally additive functionals on function spaces have been studied by Orlicz and other authors; see references in [4]. In particular, Chew [4] proved a representation theorem for orthogonally additive functionals on the Denjoy space, the space of all Henstock-Kurzweil integrable functions on an interval $[a, b]$, in terms of a nonlinear Henstock-Kurzweil integral. Also, a representation theorem for boundedly continuous linear functionals defined on BV, the space of all functions of bounded variation, has been proved by Hildebrandt [3] using the left Cauchy integral. In this paper we prove a representation theorem for orthogonally additive functionals defined on BV, making use of the nonlinear integral and hence extending the result of Hildebrandt.

Let BV denote the space of functions of bounded variation on $[a, b]$, that is, $f \in BV$ if the total variation $V(f)$ of f on $[a, b]$ is finite. A functional F defined on BV is *orthogonally additive* if $F(f + g) = F(f) + F(g)$ for all f, $g \in BV$ such that $f(x)g(x) = 0$ except for finitely many x in $[a, b]$. A functional F is said to be *boundedly continuous* on BV if $F(f_n) \to F(f)$ as $n \to \infty$ whenever for every $x \in [a, b]$, $f_n(x) \to f(x)$ as $n \to \infty$ and there exists $M > 0$ such that $V(f_n) \leqslant M$ for every n. In this paper we shall prove that if F is an orthogonally additive and boundedly continuous functional on BV, then F can be represented by a non-linear integral of the Henstock-Kurzweil-Stieltjes type. The full detail is given in Theorem 3.

2. A non-linear integral

We introduce a non-linear integral of the Henstock-Kurzweil-Stieltjes type [4, p. 81]. Let $h = h(s, I)$ be a *point-interval function* defined for s being a real number and $I = [u, v] \subset [a, b]$. A real-valued function f is said to be *h-integrable* to A on a compact interval $[a, b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ for $\xi \in [a, b]$ such that for any division D of $[a, b]$ given by $a = x_0 < x_1 < \ldots < x_n = b$, with $\xi_1, \xi_2, \ldots, \xi_n$ satisfying $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i),\ \xi_i + \delta(\xi_i))$ for $i = 1, 2, \ldots, n$, we have

$$\left|\sum_{i=1}^{n} h(f(\xi_i), [x_{i-1}, x_i]) - A\right| < \varepsilon.$$

For brevity, we write $D = \{(\xi, [u, v])\}$ where $(\xi, [u, v])$ denotes a typical point-interval pair $(\xi_i, [x_{i-1}, x_i])$ in D, and also we write the Riemann sum in the form $(D) \sum h(f(\xi), [u, v])$. Here D is said to be *δ-fine* if the above condition holds. In short, f is *h-integrable* on $[a, b]$ if for every $\varepsilon > 0$, there is a positive function δ such that for any δ-fine division $D = \{(\xi, [u, v])\}$ of $[a, b]$ we have

$$\left|(D) \sum h(f(\xi), [u, v]) - A\right| < \varepsilon.$$

For simplicity, we write the h-integral $\int_a^b h(f(x), \mathrm{d}x) = A$. For example, when δ is a constant function and $h(f(x), [u, v]) = f(x)[g(v) - g(u)]$, the h-integral reduces to the well-known *Riemann-Stieltjes integral*.

We give a list of conditions on $h(s, I)$ which guarantee that the h-integral becomes meaningful.

(N1) $h(0, I) = 0$ for all intervals $I \subset [a, b]$.

(N2) $h(s, I)$, as a function of s, is continuous on the real line for all intervals $I \subset [a, b]$.

(N3) $h(s, I)$, as a function of I, is additive, i.e., $h(s, I_1 \cup I_2) = h(s, I_1) + h(s, I_2)$ whenever I_1, I_2 are nonoverlapping and adjacent, for all I_1, $I_2 \subset [a, b]$. That is, $I_1 \cup I_2$ is again an interval.

(N4) For every $M > 0$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left|\sum_{i=1}^{p} h(s_i, I_i) - \sum_{i=1}^{p} h(t_i, I_i)\right| < \varepsilon$$

whenever $|s_i - t_i| < \eta$, $|s_i| \leqslant M$, $|t_i| \leqslant M$ for every i and $I_1, I_2, \ldots, I_p$ are pairwise nonoverlapping.

Here we note that N2 follows from N4. For easy reference, we keep to the same labelling of conditions as in P. Y. Lee [4, p. 82] for N1 to N4. In our case, N5 in P. Y. Lee [4, p. 82] is not used. For reference, we state N5 here.

(N5) For every $M > 0$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|\sum_{i=1}^{p} h(s_i, I_i)| < \varepsilon$ whenever $\sum_{i=1}^{p} |I_i| < \eta$, $|s_i| \leqslant M$, for every i and $I_1, I_2, \ldots, I_p$ are nonoverlapping.

Furthermore, we state one more condition, namely N6, as required for our case.

(N6) For every $M > 0$ and $|s| \leqslant M$, the limit $\lim_{u \uparrow c} h(s, [u, c])$ exists for $c \in (a, b]$ and so does $\lim_{v \downarrow c} h(s, [c, v])$ for $c \in [a, b)$.

We remark that N5 is an essential condition in [4]. Here it is N6 that we need. Note that N5 implies N6 but not conversely. In what follows and throughout the paper, we assume that $h(s, I)$ is fixed and satisfies N1–N4 and N6. For other papers on the nonlinear integral, see references in [4], [5].

A function g^* defined on $[a, b]$ is said to be a *normalized function* of g if $g^*(x) = \frac{1}{2}[g(x+) + g(x-)]$ for every $x \in (a, b)$ and $g^*(a) = g(a+)$, $g^*(b) = g(b-)$.

Lemma 1. *If $h(s, I)$ satisfies* N1–N4 *and* N6, *then $\int_a^b h(\varphi(x), \mathrm{d}x)$ exists for any step function φ.*

P r o o f. It is sufficient to prove the lemma for $\varphi = \chi_{[c,d]}$, for some $[c, d] \subseteq [a, b]$. In view of N6, we can prove that

$$\int_a^b h(\varphi(x), \mathrm{d}x) = \lim_{u \to c-} h(s, [u, c]) + h(s, [c, d]) + \lim_{v \to d+} h(s, [d, v]).$$

$\square$

Theorem 1. *Let $\{f_n\}$ be a sequence of h-integrable functions uniformly bounded on $[a, b]$. If $\{f_n\}$ is uniformly convergent to f on $[a, b]$, then f is h-integrable on $[a, b]$ and $\lim_{n \to \infty} \int_a^b h(f_n(x), \mathrm{d}x) = \int_a^b h(f(x), \mathrm{d}x)$.*

The proof is standard and therefore omitted [5].

A function f defined on $[a, b]$ is said to be a *regulated function* [1] if f is the limit of a uniformly convergent sequence of step functions on $[a, b]$.

Corollary 1. *If f is a regulated function, then it is h-integrable on $[a, b]$.*

We introduce two more continuity concepts of functionals on BV. A sequence $\{f_n\}$ is *bounded* in BV if there is $M > 0$ such that $V(f_n) \leqslant M$ for all n. Also we write $\|f\| = \sup\{|f(x)|\colon\ a \leqslant x \leqslant b\}$. A functional F is said to be *uniformly continuous* on BV if $F(f_n) - F(g_n) \to 0$ as $n \to \infty$ whenever $\|f_n - g_n\| \to 0$ as $n \to \infty$ with $\{f_n\}$ and $\{g_n\}$ bounded in BV. Further, a functional F is said to be *two-norm continuous* on BV if it is uniformly continuous with g_n replaced by f. It is obvious that if F is uniformly continuous on BV, then F is two-norm continuous on BV. It is well-known that functions of bounded variation satisfy *Helly's choice property* [2], [8]. It states that if a sequence $\{g_n\}$ is bounded in BV, then there is a subsequence $\{f_k\}$ of $\{g_n\}$ and a function $f \in BV$ such that for every $x \in [a,b]$, $f_k(x) \to f(x)$ as $n \to \infty$.

Lemma 2. *If F is boundedly continuous on BV, then F is uniformly continuous on BV.*

Proof. Suppose F is boundedly continuous on BV. Take two sequences $\{f_n\}$ and $\{g_n\}$ in BV such that $\|f_n - g_n\| \to 0$ as $n \to \infty$ and $\{f_n\}$, $\{g_n\}$ are bounded in BV. Since BV satisfies Helly's choice property, we obtain a subsequence of $\{f_n\}$, denoted by $\{f_{n_i}\}$, such that f_{n_i} is boundedly convergent to f on $[a,b]$. Further, take a subsequence of $\{g_n\}$, denoted by $\{g_{n_i}\}$, such that for every $x \in [a,b]$, $g_{n_i}(x) \to g(x)$. In view of $\|f_n - g_n\| \to 0$, we have for every $x \in [a,b]$, $g_{n_i}(x) \to f(x)$ as $n_i \to \infty$.

Since F is boundedly continuous, $F(f_{n_i}) - F(f) \to 0$ and $F(g_{n_i}) - F(f) \to 0$ as $n_i \to \infty$. Therefore, $F(f_{n_i}) - F(g_{n_i}) \to 0$ as $n_i \to \infty$. Consequently, F is uniformly continuous on BV. □

Theorem 2. *Suppose $h(s,I)$ satisfies* N1–N4 *and* N6. *If a functional F defined on BV is given by $F(f) = \int_a^b h(f(x), \mathrm{d}x)$ for every $f \in BV$, then F is orthogonally additive and uniformly continuous on BV. Furthermore, F is two-norm continuous on BV.*

Proof. The orthogonal additivity follows from N1. Next, take $f_n, g_n \in BV$ such that $\|f_n - g_n\| \to 0$ as $n \to \infty$ and $\{f_n\}$, $\{g_n\}$ are bounded in BV. Since f_n and g_n are h-integrable on $[a,b]$, for every $\varepsilon > 0$ and every n there exists a function $\delta_n(\xi) > 0$ such that for every δ_n-fine division $D = \{(\xi, [u,v])\}$ of $[a,b]$ we have

$$\Bigl|F(f_n) - (D)\sum h(f_n(\xi), [u,v])\Bigr| < \frac{\varepsilon}{3} \text{ and } \Bigl|F(g_n) - (D)\sum h(g_n(\xi), [u,v])\Bigr| < \frac{\varepsilon}{3}.$$

Since $\|f_n - g_n\| \to 0$ as $n \to \infty$ and N4 holds with $s_i = f_n(\xi_i)$ and $t_i = g_n(\xi_i)$, we can show that $|(D)\sum h(f_n(\xi), [u,v]) - (D)\sum h(g_n(\xi), [u,v])| < \frac{\varepsilon}{3}$ for large n.

Consequently, $F(f_n) - F(g_n) \to 0$ as $n \to \infty$. Therefore F is a uniformly continuous functional on BV.

Finally, the two-norm continuity follows from the definition. $\square$

We remark that, as shown in [4], if $h(s, I)$ satisfies N1–N5, then the functional F defined by the h-integral is boundedly continuous. Since we have not had N5 in Theorem 2, bounded continuity does not follow.

4. Representation theorem on BV

In this section we give a series of lemmas leading to the main theorem which is the representation theorem for boundedly continuous orthogonally additive functionals on BV.

Lemma 3. *If $f \in BV$, then there exists a sequence of step functions f_n such that $\|f_n - f\| \to 0$ as $n \to \infty$ and $\{f_n\}$ is bounded in BV.*

The proof is standard and therefore omitted.

Lemma 4. *If F is an orthogonally additive and boundedly continuous functional on BV, then $h(s, I)$ satisfies* N1–N4 *and* N6, *where $h(s, I) = F(s\chi_I^*)$ and χ_I^* is the normalized function of the characteristic function χ_I of $I = [u, v]$.*

Proof. The proofs of N1 and N3 follow from the orthogonal additivity and the proof of N2 from the fact that F is a boundedly continuous functional on BV.

We now prove N4. Suppose it is false. Then we shall deduce that F is not uniformly continuous on BV. In view of Lemma 2, this contradicts the fact that F is boundedly continuous on BV.

If N4 does not hold, then there exist $M > 0$ and $\varepsilon > 0$ for every $\eta > 0$ such that there exist x_i, y_i and I_i, $1 \leqslant i \leqslant k$, pairwise nonoverlapping, satisfying $|x_i| \leqslant M$, $|y_i| \leqslant M$, $|x_i - y_i| < \eta$ for every i and $|\sum\limits_{i=1}^{k} h(x_i, I_i) - \sum\limits_{i=1}^{k} h(y_i, I_i)| \geqslant \varepsilon$. Take $\eta = \frac{1}{n}$. Then there exist $x_{n,i}$, $y_{n,i}$ and $I_{n,i}$, $1 \leqslant i \leqslant k_n$, such that $|x_{n,i} - y_{n,i}| < \frac{1}{n}$ for every i, $|x_{n,i}| \leqslant M$, $|y_{n,i}| \leqslant M$ and

$$\left| \sum_{i=1}^{k_n} h(x_{n,i}, I_{n,i}) - \sum_{i=1}^{k_n} h(y_{n,i}, I_{n,i}) \right| \geqslant \varepsilon.$$

Put $f_n = \sum\limits_{i=1}^{k_n} x_{n,i}\chi_{I_{n,i}}^*$ and $g_n = \sum\limits_{i=1}^{k_n} y_{n,i}\chi_{I_{n,i}}^*$. Then we have $\|f_n\| \leqslant M$, $\|g_n\| \leqslant M$ for every n and $\|f_n - g_n\| \leqslant \frac{1}{n} \to 0$ as $n \to \infty$, but $|F(f_n) - F(g_n)| \geqslant \varepsilon$. That is, F is not uniformly continuous on BV.

We now prove N6. Take $u_n \uparrow c$ as $n \to \infty$. Then $s\chi^*_{[u_n,c]}$ is pointwise convergent to $s\chi^*_{[c,c]}$ and $\|s\chi^*_{[u_n,c]}\| \leqslant |s|$ for every n. Put $f_n = s\chi^*_{[u_n,c]}$ and $f = s\chi^*_{[c,c]}$. Obviously, f_n is boundedly convergent to f in BV. Since F is a boundedly continuous functional on BV, we have $|F(f_n) - F(f)| \to 0$ as $n \to \infty$. That is, $\lim\limits_{u_n \uparrow c} h(s,[u_n,c])$ exists and similarly, we can prove that $\lim\limits_{v_n \downarrow c} h(s,[c,v_n])$ exists. $\square$

Note that we require χ^*_I in the definition of $h(s,I)$ in order to prove N3. Replacing χ^*_I by χ_I would not be sufficient.

Lemma 5. *Suppose F is an orthogonally additive and boundedly continuous functional on BV and $h(s,I)$ satisfies* N1–N4 *and* N6 *where $h(s,I) = F(s\chi^*_I)$. Then the h-integral exists and $F(f) = \int_a^b h(f(x),\mathrm{d}x)$ for every normalized $f \in BV$.*

P r o o f. In view of Lemma 3, it is sufficient to prove the assertion for every normalized step function φ. Let φ be the step function which we have defined in Lemma 1. Then by Lemma 1, $\int_a^b h(\varphi(x),\mathrm{d}x)$ exists and $\int_a^b h(\varphi(x),\mathrm{d}x) = A$. Then there exist $\delta_n(\xi) > 0$ for $\xi \in [a,b]$ and a δ_n-fine division $D_n = \{(\xi,[u_n,v_n])\}$ of $[a,b]$ such that

$$\Big|(D_n)\sum h(\varphi(\xi),[u_n,v_n]) - A\Big| < \frac{1}{n}.$$

Denote $\varphi_n = (D_n)\sum \varphi(\xi)\chi^*_{[u_n,v_n]}$. We may choose D_n so that for every $x \in [a,b]$, $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$. Then φ_n is boundedly convergent to φ in BV. Since F is a boundedly continuous functional on BV, we have $F(\varphi_n) \to F(\varphi)$ as $n \to \infty$. Since $h(s,I) = F(s\chi^*_I)$, we have $(D_n)\sum h(\varphi(\xi),[u_n,v_n]) = F((D_n)\sum \varphi(\xi)\chi^*_{[u_n,v_n]})$ and therefore $F(\varphi) = \lim\limits_{n\to\infty} F(\varphi_n) = A = \int_a^b h(\varphi(x),\mathrm{d}x)$. $\square$

We state the main theorem of this paper as follows:

Theorem 3. *If F is an orthogonally additive and boundedly continuous functional on BV, then there exists $h(s,I)$ satisfying* N1–N4 *and* N6 *such that*

$$F(f) = \int_a^b h(f^*(x),\mathrm{d}x) + \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$$

for every $f \in BV$, where $\delta_t(x) = 1$ when $x = t$ and 0 otherwise, t_i, $i = 1,2,\ldots$, are the discontinuity points of f, and f^* is the normalized function of f.

P r o o f. Let f^* be the normalized function of f. Then $F(f) = F(f^*) + F(f - f^*)$. It follows from Lemma 5 that $F(f^*) = \int_a^b h(f^*(x),\mathrm{d}x)$ and it remains to prove $F(f - f^*) = \sum\limits_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$.

Let t_i, $i = 1, 2, \ldots$, be the discontinuity points of f. Then we have $(f - f^*)(x) = \sum\limits_{i=1}^{\infty}[f(t_i) - f^*(t_i)]\delta_{t_i}(x)$. Since $\sum\limits_{i=1}^{\infty}[f(t_i) - f^*(t_i)]\delta_{t_i}(x)$ converges for every x, we have $\sum\limits_{i=n}^{\infty}[f(t_i) - f^*(t_i)]\delta_{t_i}(x) \to 0$ as $n \to \infty$. Also, $\{\sum\limits_{i=n}^{\infty}[f(t_i) - f^*(t_i)]\delta_{t_i}\}$ is bounded in BV. By Lemma 2 and Helly's choice property, F is two-norm continuous on BV and therefore $\lim\limits_{n\to\infty} F(\sum\limits_{i=n}^{\infty}[f(t_i) - f^*(t_i)]\delta_{t_i}) = 0$. That is, $\sum\limits_{i=n}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$ converges and

$$F(f - f^*) = \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i}).$$

The fact that $h(s, I)$ satisfies N1–N4 and N6 follows from Lemma 4. □

When F is boundedly continuous and linear in Theorem 3, we have $h(s, I) = sg_1(I)$ where $g_1(I) = F(s\chi_I)$. Here $F(s\chi_I^*) = sF(\chi_I)$. In view of N6 and the fact that f is a regulated function if and only if it has one-sided limits, we obtain that g_1 is a regulated function. Further, write $g_2(t) = F(\delta_t)$ for $t \in [a, b]$. We can prove by contradiction to the bounded continuity of F that g_2 is bounded on $[a, b]$. Hence we obtain a corollary of Theorem 3 as follows:

Corollary 2. *If F is a linear and boundedly continuous functional on BV, then there exist a regulated function g_1 and a bounded function g_2 such that*

$$F(f) = \int_a^b f^* \,\mathrm{d}g_1 + \sum_{i=1}^{\infty}[f(t_i) - f^*(t_i)]g_2(t_i)$$

for every $f \in BV$, where t_i, $i = 1, 2, \ldots$, are the discontinuity points of f and f^* is the normalized function of f.

This is equivalent to a result by Hildebrandt [3]. In his version, he expressed it in terms of the left or right Cauchy integral.

5. The space of regulated functions

A corresponding result of Theorem 3 holds true for the space RF of all regulated functions. We shall sketch a proof in this section. We shall define the boundedness of a sequence in RF and the bounded continuity of a functional on RF.

It is known [2, p. 48] that $f \in RF$ if and only if for every $\varepsilon > 0$ *the bounded ε-variation* $V_\varepsilon(f)$ of f on $[a, b]$ is finite, where

$$V_\varepsilon(f) = \inf\{V(g)\colon\ g \in BV \text{ and } |f(x) - g(x)| \leqslant \varepsilon \text{ for every } x \in [a, b]\},$$

where ε is given and fixed. A sequence $\{f_n\}$ is said to be *bounded in RF* if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\|f_n\| \leqslant M_\varepsilon$ and $V_\varepsilon(f_n) \leqslant M_\varepsilon$ for all n. Then a functional F on RF is said to be *boundedly continuous* if $F(f_n) \to F(f)$ as $n \to \infty$ whenever for every $x \in [a,b]$, $f_n(x) \to f(x)$ as $n \to \infty$ and $\{f_n\}$ is bounded in RF. Furthermore, *Helly's choice theorem* for RF was proved by Dana Franková in [2, Theorem 3.8, p. 51].

Theorem 4. *Suppose $\{g_n\}$ is bounded in RF. Then there is a subsequence $\{f_k\}$ of $\{g_n\}$ and a function $f \in RF$ such that for every $x \in [a,b]$, $f_k(x) \to f(x)$ as $n \to \infty$.*

Then following the same argument as above, we obtain results analogous to Theorem 3 and Corollary 2 with BV replaced by RF.

Theorem 5. *If F is an orthogonally additive and boundedly continuous functional on RF, then there exists $h(s,I)$ satisfying* N1–N4 *and* N6 *such that*

$$F(f) = \int_a^b h(f^*(x), \mathrm{d}x) + \sum_{i=1}^{\infty} F([f(t_i) - f^*(t_i)]\delta_{t_i})$$

for every $f \in RF$, where $\delta_t(x) = 1$ when $x = t$ and 0 otherwise, t_i, $i = 1, 2, \ldots$, are the discontinuity points of f, and f^* is the normalized function of f.

Corollary 3. *If F is a linear and boundedly continuous functional on RF, then there exist a function $g_1 \in BV$ and a function g_2 such that the infinite series below converges and*

$$F(f) = \int_a^b f^* \,\mathrm{d}g_1 + \sum_{i=1}^{\infty} [f(t_i) - f^*(t_i)] g_2(t_i)$$

for every $f \in RF$, where t_i, $i = 1, 2, \ldots$, are the discontinuity points of f, and f^* is the normalized function of f.

A special case of Corollary 3 has been proved by Tvrdý [7], where every function in RF is assumed to be normalized.

References

[1] *J. Dieudonné*: Foundations of Modern Analysis. Academic Press, N. Y., 1960.
[2] *D. Franková*: Regulated functions. Math. Bohem. *116* (1991), 20–59.
[3] *T. H. Hildebrandt*: Linear continuous functionals on the space (BV) with weak topologies. Proc. Amer. Math. Soc. *17* (1966), 658–664.
[4] *P. Y. Lee*: Lanzhou Lectures on Henstock Integration. World Scientific, 1989.
[5] *P. Y. Lee, R. Výborný*: The Integral: An Easy Approach after Kurzweil and Henstock. Cambridge University Press, 2000.

[6] *W. Orlicz*: Linear Functional Analysis. World Scientific, 1992.
[7] *M. Tvrdý*: Linear bounded functionals on the space of regular regulated functions. Tatra Mt. Math. Publ. *8* (1996), 203–210.
[8] *A. Zygmund*: Trigonometric Series I and II. Cambridge University Press, 1977.

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