KURZWEIL'S PU INTEGRAL AS THE LEBESGUE INTEGRAL

Rudolf Výborný, University of Queensland

(Received August 26, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. For a merely continuous partition of unity the PU integral is the Lebesgue integral.

Keywords: Kurzweil's PU integral, Lebesgue integral, McShane integral

MSC 2000: 26A39, 26A42, 28A99

1. Introduction

J. Kurzweil and J. Jarník [2] introduced the PU integral based on partition of unity. Kurzweil and others studied this integral in an effort to establish strong theorems of Gauss-Green-Stokes type [3], [4]. The PU integral lies somewhere between the Lebesgue integral and Kurzweil-Henstock integral, it is more general then the Lebesgue integral and the class of PU integrable functions is a proper subset of KH integrable functions. In Kurzweil's treatment the functions forming partitions of unity are subject to some differentiability conditions in order to achieve the desired goal. In this note we show that if the functions are merely continuous the resulting integral is the Lebesgue integral.

2. Basic definitions

Let us recall some definitions. A set of couples

$$D \equiv \{(x_i, [t_{i-1}, t_i]);\ i = 1, 2, \ldots, n\}$$

constitutes a tagged division of a compact interval $[a,b]$ if $x_i \in [a,b]$,

$$\bigcup_{i=1}^{n} [t_{i-1}, t_i] = [a,b]$$

and the intervals $[t_{i-1}, t_i]$ do not overlap. We shall abbreviate the Riemann sums as

$$\sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) = \sum_{D} f.$$

If $\delta\colon\ [a,b] \mapsto (0,\infty)$ then a tagged division D is, by definition, δ-fine if

$$[t_{i-1}, t_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$$

for $i = 1, 2, \dots, n$.

A function $f\colon\ [a,b] \mapsto \mathbb{R}$ is said to be McShane integrable and I is the McShane integral of f, if for every positive ε there exists a positive function δ such that for every δ-fine tagged division D of $[a,b]$

$$\left| \sum_{D} f - I \right| < \varepsilon.$$

We denote the McShane integral by the symbol $\mathrm{Mc}\!\int_a^b f$. It is well known [1], [5] that f is McShane integrable if and only if it is Lebesgue integrable.

A set of couples

$$\Phi \equiv \{(\xi_i, \varphi_i);\ i = 1, 2, \dots, n\}$$

constitutes a tagged partition of unity of a compact interval $[a,b]$ if $\xi_i \in [a,b]$ and $\{\varphi_i;\ i = 1, 2, \dots, n\}$ form a partition of unity on $[a,b]$. That is, the functions φ_i have compact support and $\sum\limits_{1}^{n} \varphi_i = 1$ on $[a,b]$. We assume throughout the paper that the functions φ_i are continuous in $\mathbb{R}$. Similarly as with the tagged division we shall abbreviate the following sums as

$$\sum_{i=1}^{n} f(\xi_i) \int_{-\infty}^{\infty} \varphi_i = \sum_{\Phi} f.$$

If $\delta\colon\ [a,b] \mapsto (0,\infty)$ then a tagged partition of unity is, by definition, δ-fine if

$$\operatorname{supp} \varphi_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$$

for $i = 1, 2, \dots, n$.

A function $f\colon [a,b] \mapsto \mathbb{R}$ is said to be PU integrable and J is the PU integral of f, if for every postive ε there exists a positive function δ such that for every δ-fine tagged partition of unity Φ

$$\Bigl|\sum_{\Phi} f - J\Bigr| < \varepsilon.$$

We denote the PU integral as $\mathrm{PU}\!\int_a^b f$.

3. The theorem

Theorem 1. *A function f is PU integrable if and only if it is McShane integrable.*

P r o o f. Let f be PU integrable and let δ_{pu} correspond to δ from the definition of the PU integral. Let $D \equiv \{(x_i, [t_{i-1}, t_i]),\ i = 1, 2, \ldots, n\}$ be a δ_{pu}-fine tagged division of $[a,b]$. There is a positive h such that for all $[t_{i-1}, t_i]$ of D the interval $[t_{i-1} - h, t_i + h]$ is δ_{pu}-fine. Let

$$\varphi_i(x) = \begin{cases} 0 & \text{for } x < t_{i-1} - h,\\ \dfrac{x - t_{i-1} + h}{2h} & \text{for } t_{i-1} - h \leqslant x \leqslant t_{i-1} + h,\\ 1 & \text{for } t_{i-1} + h < x < t_i - h,\\ \dfrac{t_i + h - x}{2h} & \text{for } t_i - h \leqslant x \leqslant t_i + h,\\ 0 & \text{for } t_i + h < x. \end{cases}$$

The function φ_i has been defined in such a way that

$$\int_{-\infty}^{\infty} \varphi_i = t_i - t_{i-1}$$

and consequently

$$\sum_D f(x_i)(t_i - t_{i-1}) = \sum_{\Phi} f.$$

This proves that a PU integrable f is McShane integrable, and the integrals agree.

Let now f be McShane integrable and let δ_{M} correspond to δ from the definition of the McShane integral. Let $\Phi \equiv \{(\xi_i, \varphi_i);\ i = 1, 2, \ldots, n\}$ be a δ_{M}-fine tagged partition of unity. The support of the function φ_i is contained in a smallest compact interval, say $[u_{\varphi_i}, v_{\varphi_i}]$. These intervals define a division

$$t_0 < t_1 < \ldots < t_n$$

of $[a,b]$ in the sense that between two consecutive points of this division there is no u_{φ_i} or v_{φ_i}. Let $i_1, i_2, \ldots, i_r$ be the indices for which

$$(\operatorname{supp}\varphi_{i_j}) \cap [t_i, t_{i+1}] \neq \emptyset.$$

For some ξ_{i_j} the value of f is maximal, denote this ξ_{i_j} by x_i. The tagged division $(\{(x_i, [t_{i-1}, t_i]);\ i = 1, 2 \ldots, n\}$ is δ_{M}-fine. It follows that

$$\sum_{\Phi} f \leqslant \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}) \leqslant \mathrm{Mc}\int_a^b f + \varepsilon.$$

The inequality

$$\sum_{\Phi} f(x) \int \varphi_i \geqslant \mathrm{Mc}\int_a^b f - \varepsilon,$$

can be proved similarly. □

References

[1] *Russel A. Gordon*: The Integrals of Lebesgue, Denjoy, Perron, and Henstock. American Mathematical Society, 1994. Zbl 0807.26004

[2] *J. Kurzweil, J. Jarník*: A non absolutely convergent integral which admits transformation and can be used on manifolds. Czechoslovak Math. J. *35* (1985), 116–139. Zbl 0614.26007

[3] *J. Kurzweil, J. Jarník*: A new and more powerful concept of the PU integral. Czechoslovak Math. J. *38* (1988), 8–48. Zbl 0669.26006

[4] *J. Kurzweil, J. Mawhin, W. Pfeffer*: An integral defined by approximating BV partitions of unity. Czechoslovak Math. J. *41* (1991), 695–712. Zbl 0763.26007

[5] *Lee, Peng Yee, Rudolf Výborný*: The Integral: An Easy Approach after Kurzweil and Henstock. Cambridge University Press, Cambridge, UK, 2000. Zbl 0941.26003

Author's address: *Rudolf Výborný*, Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia, e-mail: `rv@maths.uq.edu.au`.