A NEW FORM OF FUZZY α-COMPACTNESS

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(Received February 9, 2005)

Abstract. A new form of α-compactness is introduced in L-topological spaces by α-open L-sets and their inequality where L is a complete de Morgan algebra. It doesn't rely on the structure of the basis lattice L. It can also be characterized by means of α-closed L-sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented and the relations between it and the other types of compactness are discussed. Countable α-compactness and the α-Lindelöf property are also researched.

Keywords: L-topology, compactness, α-compactness, countable α-compactness, α-Lindelöf property, α-irresolute map, α-continuous map

MSC 2000: 54A40, 54D35

1. Introduction

The notion of α-open sets was introduced in [13]. The concept of α-compactness for topological spaces was discussed in [12], and it was generalized to $[0,1]$-topological spaces by Thakur, Saraf and Jabalpur [18]. The definition in [18] is based on Chang's compactness which is not a good extension of ordinary compactness.

In [1], Aygün presented a new form of α-compactness which is based on Kudri's compactness [7] which is equivalent to strong compactness in [9], [19].

The concepts of SR-compactness and near SR-compactness were introduced by S. G. Li, S. Z. Bai and N. Li in terms of strongly semiopen L-sets [4], [8]. In fact, a strongly semiopen L-set is exactly an α-open set in the sense of [14]. Thus both SR-compactness and near SR-compactness are extensions of α-compactness. Moreover, the notion of SR-compactness was based on N-compactness and the notion of near

This work is supported by the National Natural Science Foundation of China (10371079) and the Basic Research Foundation of Beijing Institute of Technology.

SR-compactness was based on strong fuzzy compactness. This implies that near SR-compactness is equivalent to α-compactness in [1] when the basis lattice L is a complete distributive de Morgan algebra.

In [15], [16], a new definition of fuzzy compactness was presented in L-topological spaces by means of open L-sets and their inequality where L is a complete de Morgan algebra. This new definition doesn't depend on the structure of L. When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [9], [10], [19].

In this paper, following the lines of [15], [16], we will introduce a new form of α-compactness in L-topological spaces by means of α-open L-sets and their inequality. This new form of α-compactness has many characterizations if L is completely distributive. Moreover, we will compare our α-compactness with other types of α-compactness.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a complete de Morgan algebra, X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X. We often don't distinguish a crisp subset A of X and its character function χ_A.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. An element a in L is called co-prime if a' is prime [6]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [5]. In a completely distributive de Morgan algebra L, each element b is a sup of $\{a \in L;\ a \prec b\}$. A set $\{a \in L;\ a \prec b\}$ is called the greatest minimal family of b in the sense of [9], [19], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L;\ a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$ we use the following notations from [17].

$$
\begin{aligned}
&A_{[a]} = \{x \in X;\ A(x) \geqslant a\}, \quad A^{(a)} = \{x \in X;\ A(x) \not\leqslant a\},\\
&A_{(a)} = \{x \in X;\ a \in \beta(A(x))\}.
\end{aligned}
$$

An L-topological space (or L-space for short) is a pair $(X, \mathscr{T})$, where $\mathscr{T}$ is a subfamily of L^X which contains $\chi_\emptyset$, χ_X and is closed for any suprema and finite infima. $\mathscr{T}$ is called an L-topology on X. Members of $\mathscr{T}$ are called open L-sets and their complements are called closed L-sets.

Definition 2.1 ([9], [19]). An L-space $(X, \mathscr{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathscr{T}$, it follows that $A^{(a)} \in [\mathscr{T}]$, where $[\mathscr{T}]$ denotes the topology formed by all crisp sets in $\mathscr{T}$.

Definition 2.2 ([9], [19]). For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X;\ A^{(a)} \in \tau,\ a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X; in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ). A topologically generated L-space is also called an induced L-space.

It is obvious that $(X, \omega_L(\tau))$ is weakly induced.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ and $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.3 ([15], [16]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is called (countably) compact if for every (countable) family $\mathscr{U} \subseteq \mathscr{T}$, it follows that

$$\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \Big) \leqslant \bigvee_{\mathscr{V} \in 2^{(\mathscr{U})}} \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{V}} A(x) \Big).$$

Definition 2.4 ([16]). Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to have the Lindelöf property if for every family $\mathscr{U} \subseteq \mathscr{T}$, it follows that

$$\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \Big) \leqslant \bigvee_{\mathscr{V} \in 2^{[\mathscr{U}]}} \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{V}} A(x) \Big).$$

Lemma 2.5 ([16]). *Let L be a complete Heyting algebra, let $f\colon X \to Y$ be a map and $f_L^{\rightarrow}\colon L^X \to L^Y$ the extension of f. Then for any family $\mathscr{P} \subseteq L^Y$, we have*

$$\bigvee_{y \in Y} \Big(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B \in \mathscr{P}} B(y) \Big) = \bigvee_{x \in X} \Big(G(x) \wedge \bigwedge_{B \in \mathscr{P}} f_L^{\leftarrow}(B)(x) \Big),$$

where $f_L^{\rightarrow}\colon L^X \to L^Y$ and $f_L^{\leftarrow}\colon L^Y \to L^X$ are defined as follows:

$$f_L^{\rightarrow}(G)(y) = \bigvee_{x \in f^{-1}(y)} G(x), \quad f_L^{\leftarrow}(B) = B \circ f.$$

The notion of an α-open set was introduced by Njåstad in [13] and generalized to $[0,1]$-topological spaces by Shahana in [14]. Analogously we can generalize it to L-fuzzy setting as follows:

Definition 2.6 ([14]). An L-set G in an L-space $(X, \mathscr{T})$ is called α-open if $G \leqslant \operatorname{int}(\operatorname{cl}(\operatorname{int}(G)))$. G is called α-closed if G' is α-open.

Definition 2.7 ([18])**.** Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ be two L-spaces. A map $f\colon (X, \mathscr{T}_1) \to (Y, \mathscr{T}_2)$ is called α-continuous if $f_L^{\leftarrow}(G)$ is α-open in $(X, \mathscr{T}_1)$ for every open L-set G in $(Y, \mathscr{T}_2)$.

It can be seen that an α-continuous map was also said to be strongly semi-continuous in [14].

Definition 2.8 ([18])**.** Let $(X, \mathscr{T}_1)$ and $(Y, \mathscr{T}_2)$ be two L-spaces. A map $f\colon (X, \mathscr{T}_1) \to (Y, \mathscr{T}_2)$ is called α-irresolute if $f_L^{\leftarrow}(G)$ is α-open in $(X, \mathscr{T}_1)$ for every α-open L-set G in $(Y, \mathscr{T}_2)$.

3. Definition and characterizations of α-compactness

Definition 3.1. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is called (countably) α-compact if for every (countable) family $\mathscr{U}$ of α-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{V}} A(x)\Big).$$

Definition 3.2. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to have the α-Lindelöf property (or be an α-Lindelöf L-set) if for every family $\mathscr{U}$ of α-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \leqslant \bigvee_{\mathscr{V}\in 2^{[\mathscr{U}]}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{V}} A(x)\Big).$$

Obviously we have the following theorem.

Theorem 3.3. *α-compactness implies countable α-compactness and the α-Lindelöf property. Moreover, an L-set having the α-Lindelöf property is α-compact if and only if it is countably α-compact.*

Since an open L-set is α-open, we have the following theorem.

Theorem 3.4. *α-compactness implies compactness, countable α-compactness implies countable compactness, and the α-Lindelöf property implies the Lindelöf property.*

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by simply using complements.

Theorem 3.5. *Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is (countably) α-compact if and only if for every (countable) family $\mathscr{B}$ of α-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x)\wedge \bigwedge_{B\in\mathscr{B}} B(x)\Big) \geqslant \bigwedge_{\mathscr{F}\in 2^{(\mathscr{B})}} \bigvee_{x\in X}\Big(G(x)\wedge \bigwedge_{B\in\mathscr{F}} B(x)\Big).$$

Theorem 3.6. *Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ has the α-Lindelöf property if and only if for every family $\mathscr{B}$ of α-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x)\wedge \bigwedge_{B\in\mathscr{B}} B(x)\Big) \geqslant \bigwedge_{\mathscr{F}\in 2^{[\mathscr{B}]}} \bigvee_{x\in X}\Big(G(x)\wedge \bigwedge_{B\in\mathscr{F}} B(x)\Big).$$

In order to present characterizations of α-compactness, countable α-compactness and the α-Lindelöf property, we generalize the notions of an a-shading and an a-R-neighborhood family in [15], [16] as follows:

Definition 3.7. Let $(X, \mathscr{T})$ be an L-space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathscr{A} \subseteq L^X$ is said to be

(1) an a-shading of G if for any $x \in X$, $\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \nleqslant a$.

(2) a strong a-shading of G if $\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathscr{U}} A(x)\Big) \nleqslant a$.

(3) an a-remote family of G if for any $x \in X$, $\Big(G(x) \wedge \bigwedge_{B\in\mathscr{P}} B(x)\Big) \ngeqslant a$.

(4) a strong a-remote family of G if $\bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\mathscr{P}} B(x)\Big) \ngeqslant a$.

It is obvious that a strong a-shading of G is an a-shading of G, a strong a-remote family of G is an a-remote family of G, and $\mathscr{P}$ is a strong a-remote family of G if and only if $\mathscr{P}'$ is a strong a'-shading of G. Moreover, a closed a-remote family is exactly an a-remote neighborhood family and a closed strong a-remote family is exactly an a^--remote neighborhood family in the sense of [19].

Definition 3.8. Let $a \in L \setminus \{0\}$ and $G \in L^X$. A subfamily $\mathscr{A}$ of L^X is said to have a weak a-nonempty intersection in G if $\bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{A\in\mathscr{A}} A(x)\Big) \geqslant a$. $\mathscr{A}$ is said to have the finite (countable) weak a-intersection property in G if every finite (countable) subfamily $\mathscr{F}$ of $\mathscr{A}$ has a weak a-nonempty intersection in G.

Definition 3.9. Let $a \in L \setminus \{0\}$ and $G \in L^X$. A subfamily $\mathscr{A}$ of L^X is said to be a weak a-filter relative to G if any finite intersection of members in $\mathscr{A}$ is weak a-nonempty in G. A subfamily $\mathscr{B}$ of L^X is said to be a weak a-filterbase relative to G if

$$\{A \in L^X;\ \text{ there exists } B \in \mathscr{B} \text{ such that } B \leqslant A\}$$

is a weak a-filter relative to G.

From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the following two results.

Theorem 3.10. *Let $(X, \mathscr{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is (countably) α-compact.*

(2) *For any $a \in L \setminus \{1\}$, each (countable) α-open strong a-shading $\mathscr{U}$ of G has a finite subfamily which is a strong a-shading of G.*

(3) *For any $a \in L \setminus \{0\}$, each (countable) α-closed strong a-remote family $\mathscr{P}$ of G has a finite subfamily which is a strong a-remote family of G.*

(4) *For any $a \in L \setminus \{0\}$, each (countable) family of α-closed L-sets which has the finite weak a-intersection property in G has a weak a-nonempty intersection in G.*

(5) *For each $a \in L \setminus \{0\}$, every α-closed (countable) weak a-filterbase relative to G has a weak a-nonempty intersection in G.*

Theorem 3.11. *Let $(X, \mathscr{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G has the α-Lindelöf property.*

(2) *For any $a \in L \setminus \{1\}$, each α-open strong a-shading $\mathscr{U}$ of G has a countable subfamily which is a strong a-shading of G.*

(3) *For any $a \in L \setminus \{0\}$, each α-closed strong a-remote family $\mathscr{P}$ of G has a countable subfamily which is a strong a-remote family of G.*

(4) *For any $a \in L \setminus \{0\}$, each family of α-closed L-sets which has the countable weak a-intersection property in G has a weak a-nonempty intersection in G.*

4. Properties of (countable) α-compactness

Theorem 4.1. *Let L be a complete Heyting algebra. If both G and H are (countably) α-compact, then $G \vee H$ is (countably) α-compact.*

Proof. For any (countable) family $\mathscr{P}$ of α-closed L-sets, we have by Theorem 3.5 that

$$\begin{aligned}
&\bigvee_{x \in X}\Big((G \vee H)(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x)\Big) \\
&\quad = \Big\{\bigvee_{x \in X}\Big(G(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x)\Big)\Big\} \vee \Big\{\bigvee_{x \in X}\Big(H(x) \wedge \bigwedge_{B \in \mathscr{P}} B(x)\Big)\Big\}
\end{aligned}$$

$$
\geqslant \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right) \right\} \vee \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left(H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right) \right\}
$$
$$
= \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left((G \vee H)(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right).
$$

This shows that $G \vee H$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 4.2. *Let L be a complete Heyting algebra. If both G and H have the α-Lindelöf property, then so does $G \vee H$.*

Theorem 4.3. *If G is (countably) α-compact and H is α-closed, then $G \wedge H$ is (countably) α-compact.*

P r o o f. For any (countable) family $\mathscr{P}$ of α-closed L-sets, we have by Theorem 3.5 that

$$
\bigvee_{x\in X} \left((G \wedge H)(x) \wedge \bigwedge_{B\in\mathscr{P}} B(x) \right)
$$
$$
= \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathscr{P}\cup\{H\}} B(x) \right) \geqslant \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P}\cup\{H\})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right)
$$
$$
= \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right) \right\} \wedge \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right) \right\}
$$
$$
= \left\{ \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right) \right\}
$$
$$
= \bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}} \bigvee_{x\in X} \left((G \wedge H)(x) \wedge \bigwedge_{B\in\mathscr{F}} B(x) \right).
$$

This shows that $G \wedge H$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 4.4. *If G has the α-Lindelöf property and H is α-closed, then $G \wedge H$ has the α-Lindelöf property.*

Theorem 4.5. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be an α-irresolute map. If G is an α-compact (or a countably α-compact, an α-Lindelöf) L-set in $(X,\mathscr{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y,\mathscr{T}_2)$.*

P r o o f. We only prove that the theorem is true for α-compactness. Suppose that $\mathscr{P}$ is a family of α-closed L-sets in $(Y,\mathscr{T}_2)$. Then by Lemma 2.5 and α-compactness of G we have that

$$\begin{aligned}\bigvee_{y\in Y}\Big(f_L^{\rightarrow}(G)(y)\wedge\bigwedge_{B\in\mathscr{P}}B(y)\Big)&=\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{P}}f_L^{\leftarrow}(B)(x)\Big)\\ \geqslant\bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}}\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{B\in\mathscr{F}}f_L^{\leftarrow}(B)(x)\Big)&=\bigwedge_{\mathscr{F}\in 2^{(\mathscr{P})}}\bigvee_{y\in Y}\Big(f_L^{\rightarrow}(G)(y)\wedge\bigwedge_{B\in\mathscr{F}}B(y)\Big).\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is α-compact. $\square$

Analogously we have the following result.

Theorem 4.6. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be an α-continuous map. If G is an α-compact (a countably α-compact, an α-Lindelöf) L-set in $(X,\mathscr{T}_1)$, then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y,\mathscr{T}_2)$.*

Definition 4.7. Let $(X,\mathscr{T}_1)$ and $(Y,\mathscr{T}_2)$ be two L-spaces. A map $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ is called strongly α-irresolute if $f_L^{\leftarrow}(G)$ is open in $(X,\mathscr{T}_1)$ for every α-open L-set G in $(Y,\mathscr{T}_2)$.

It is obvious that a strongly α-irresolute map is α-irresolute and continuous.

Analogously we have the following result.

Theorem 4.8. *Let L be a complete Heyting algebra and let $f\colon (X,\mathscr{T}_1)\to(Y,\mathscr{T}_2)$ be a strongly α-irresolute map. If G is a compact (countably compact, Lindelöf) L-set in $(X,\mathscr{T}_1)$, then $f_L^{\rightarrow}(G)$ is an α-compact (a countably α-compact, an α-Lindelöf) L-set in $(Y,\mathscr{T}_2)$.*

5. Further characterizations of α-compactness and goodness

In this section we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of a β_a-open cover and a Q_a-open cover [16] as follows:

Definition 5.1. Let $(X, \mathscr{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathscr{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x))$. $\mathscr{U}$ is called a strong β_a-cover of G if $a \in \beta\Big(\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x)\Big)\Big)$.

Definition 5.2. Let $(X, \mathscr{T})$ be an L-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathscr{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathscr{U}} A(x) \geqslant a$.

It is obvious that a strong β_a-cover of G is a β_a-cover of G, and a β_a-cover of G is a Q_a-cover of G.

Analogously to the proof of Theorem 2.9 in [16] we can obtain the following theorem.

Theorem 5.3. *Let $(X, \mathscr{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G is α-compact.*

(2) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each α-closed strong a-remote family of G has a finite subfamily which is an a-remote (a strong a-remote) family of G.*

(3) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any α-closed strong a-remote family $\mathscr{P}$ of G, there exist a finite subfamily $\mathscr{F}$ of $\mathscr{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathscr{F}$ is a (strong) b-remote family of G.*

(4) *For any $a \in L \setminus \{1\}$ (or $a \in P(L)$), each α-open strong a-shading of G has a finite subfamily which is an a-shading (a strong a-shading) of G.*

(5) *For any $a \in L \setminus \{1\}$ (or $a \in P(L)$) and any α-open strong a-shading $\mathscr{U}$ of G, there exist a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathscr{V}$ is a (strong) b-shading of G.*

(6) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$), each α-open strong β_a-cover of G has a finite subfamily which is a (strong) β_a-cover of G.*

(7) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any α-open strong β_a-cover $\mathscr{U}$ of G, there exist a finite subfamily $\mathscr{V}$ of $\mathscr{U}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathscr{V}$ is a (strong) β_b-cover of G.*

(8) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$, each α-open Q_a-cover of G has a finite subfamily which is a Q_b-cover of G.*

(9) *For any $a \in L \setminus \{0\}$ (or $a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$ (or $b \in \beta^*(a)$), each α-open Q_a-cover of G has a finite subfamily which is a (strong) β_b-cover of G.*

Analogously we also can present characterizations of countable α-compactness and the α-Lindelöf property.

Now we consider the goodness of α-compactness.

For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \qquad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

Theorem 5.4 ([17]). *For an L-set $A \in L^X$, the following facts are true.*
(1) $A = \bigvee_{a \in L} (a \wedge A_{(a)}) = \bigvee_{a \in L} (a \wedge A_{[a]})$.
(2) $A = \bigwedge_{a \in L} (a \vee A^{(a)}) = \bigwedge_{a \in L} (a \vee A^{[a]})$.

Theorem 5.5 [17]. *Let $(X, \omega_L(\tau))$ be the L-space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold.*
(1) $\mathrm{cl}(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^-) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-)$;
(2) $\mathrm{cl}(A)_{(a)} \subset (A_{(a)})^- \subset (A_{[a]})^- \subset \mathrm{cl}(A)_{[a]}$;
(3) $\mathrm{cl}(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^-) = \bigwedge_{a \in L} (a \vee (A^{[a]})^-)$;
(4) $\mathrm{cl}(A)^{(a)} \subset (A^{(a)})^- \subset (A^{[a]})^- \subset \mathrm{cl}(A)^{[a]}$;
(5) $\mathrm{int}(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^\circ) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ)$;
(6) $\mathrm{int}(A)_{(a)} \subset (A_{(a)})^\circ \subset (A_{[a]})^\circ \subset \mathrm{int}(A)_{[a]}$;
(7) $\mathrm{int}(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^\circ) = \bigwedge_{a \in L} (a \vee (A^{[a]})^\circ)$;
(8) $\mathrm{int}(A)^{(a)} \subset (A^{(a)})^\circ \subset (A^{[a]})^\circ \subset \mathrm{int}(A)^{[a]}$, *where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ) and so on,* $\mathrm{cl}(A)$ *and* $\mathrm{int}(A)$ *denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.*

Lemma 5.6. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). If A is an α-open set in (X, τ), then χ_A is an α-open L-set in $(X, \omega_L(\tau))$. If B is an α-open L-set in $(X, \omega_L(\tau))$, then $B_{(a)}$ is an α-open set in (X, τ) for every $a \in L$.*

Proof. If A is an α-open set in (X, τ), then $A \subseteq ((A^\circ)^-)^\circ$. Thus we have

$$\chi_A \leqslant \chi_{((A^\circ)^-)^\circ} = \mathrm{int}(\chi_{(A^\circ)^-}) = \mathrm{int}(\mathrm{cl}(\chi_{A^\circ})) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\chi_A))).$$

This shows that χ_A is α-open in $(X, \omega_L(\tau))$.

If B is an α-open L-set in $(X, \omega_L(\tau))$, then $B \leqslant \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$. From Theorem 5.5 we have

$$B_{(a)} \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))_{(a)} \subseteq (\mathrm{cl}(\mathrm{int}(B))_{(a)})^\circ \subseteq ((\mathrm{int}(B)_{(a)})^-)^\circ \subseteq (((B_{(a)})^\circ)^-)^\circ.$$

This shows that $B_{(a)}$ is an α-open set in (X, τ). □

The next two theorems show that α-compactness, countable α-compactness and the α-Lindelöf property are good extensions.

Theorem 5.7. *Let $(X,\omega_L(\tau))$ be generated topologically by (X,τ). Then $(X,\omega_L(\tau))$ is (countably) α-compact if and only if (X,τ) is (countably) α-compact.*

Proof. (Necessity) Let $\mathscr{A}$ be an α-open cover (a countable α-open cover) of (X,τ). Then $\{\chi_A;\ A\in\mathscr{A}\}$ is a family of α-open L-sets in $(X,\omega_L(\tau))$ with $\bigwedge\limits_{x\in X}\Big(\bigvee\limits_{A\in\mathscr{U}}\chi_A(x)\Big)=1$. From (countable) α-compactness of $(X,\omega_L(\tau))$ we know that

$$1\geqslant\bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}}\bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{V}}\chi_A(x)\Big)\geqslant\bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{U}}\chi_A(x)\Big)=1.$$

This implies that there exists $\mathscr{V}\in 2^{(\mathscr{U})}$ such that $\bigwedge\limits_{x\in X}(\bigvee\limits_{A\in\mathscr{V}}\chi_A(x))=1$. Hence $\mathscr{V}$ is a cover of (X,τ). Therefore (X,τ) is (countably) α-compact.

(Sufficiency) Let $\mathscr{U}$ be a (countable) family of α-open L-sets in $(X,\omega_L(\tau))$ and let $\bigwedge\limits_{x\in X}(\bigvee\limits_{B\in\mathscr{U}}B(x))=a$. If $a=0$, then we obviously have

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}}B(x)\Big)\leqslant\bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}}\bigwedge_{x\in X}\Big(\bigvee_{A\in\mathscr{V}}B(x)\Big).$$

Now we suppose that $a\neq 0$. In this case, for any $b\in\beta(a)\setminus\{0\}$ we have

$$b\in\beta\Big(\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}}B(x)\Big)\Big)\subseteq\bigcap_{x\in X}\beta\Big(\bigvee_{B\in\mathscr{U}}B(x)\Big)=\bigcap_{x\in X}\bigcup_{B\in\mathscr{U}}\beta\left(B(x)\right).$$

By Lemma 5.6 this implies that $\{B_{(b)};\ B\in\mathscr{U}\}$ is an α-open cover of (X,τ). From (countable) α-compactness of (X,τ) we know that there exists $\mathscr{V}\in 2^{(\mathscr{U})}$ such that $\{B_{(b)};\ B\in\mathscr{V}\}$ is a cover of (X,τ). Hence $b\leqslant\bigwedge\limits_{x\in X}\Big(\bigvee\limits_{B\in\mathscr{V}}B(x)\Big)$. Further we have

$$b\leqslant\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}}B(x)\Big)\leqslant\bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}}\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}}B(x)\Big).$$

This implies that

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{U}}B(x)\Big)=a=\bigvee\{b;\ b\in\beta(a)\}\leqslant\bigvee_{\mathscr{V}\in 2^{(\mathscr{U})}}\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathscr{V}}B(x)\Big).$$

Therefore $(X,\omega_L(\tau))$ is (countably) α-compact. □

Analogously we have the following result.

Theorem 5.8. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ has the α-Lindelöf property if and only if (X, τ) has the α-Lindelöf property.*

6. The relations of α-compactness and other types of compactness

In this section we assume that L is again completely distributive.

Based on Kudri's compactness in [7], Aygün presented a definition of α-compactness in [1]. Since Kudri's compactness is equivalent to strong compactness in [9], [19], we shall also refer to Aygün's α-compactness as α-strong compactness. The following is its equivalent form.

Definition 6.1. Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be α-strongly compact if for any $a \in P(L)$, each α-open a-shading of G has a finite subfamily which is an a-shading of G.

In [4], [8], Bai and Li et al. introduced the notions of SR-compactness and near SR-compactness by means of strongly semiopen L-sets. In fact, a strongly semiopen L-set is equivalent to an α-open L-set. This implies that both SR-compactness and near SR-compactness are extensions of α-compactness in general topology. Their equivalent forms can be stated as follows:

Definition 6.2 ([4])**.** Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be SR-compact (we shall call it α-N-compact) if for each $a \in M(L)$, every α-closed a-remote family of G has a finite subfamily which is a strong a-remote family of G.

Definition 6.3 ([8])**.** Let $(X, \mathscr{T})$ be an L-space. $G \in L^X$ is said to be near SR-compact if for each $a \in M(L)$, every α-closed a-remote family of G has a finite subfamily which is an a-remote family of G.

It is obvious that Definition 6.3 is equivalent to Definition 6.1.

From Theorem 5.3 we easily obtain the following result.

Theorem 6.4. *For an L-set G in an L-space, the following implications hold.*

$$\begin{array}{ccccc}
\alpha\text{-N-compactness} & \Rightarrow & \alpha\text{-strong compactness} & \Rightarrow & \alpha\text{-compactness} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{N-compactness} & \Rightarrow & \text{strong compactness} & \Rightarrow & \text{compactness}
\end{array}$$

Notice that none of the above implications is invertible. We only present a counterexample which is α-compact but not α-strongly compact. The other examples can be found in [8], [9], [19] and in general topology.

E x a m p l e 6.5. Take $Y=\mathbb{N}$. For all $n\in Y$, define $B_n\in[0,1]^Y$ as follows:

$$B_n(y)=\begin{cases}(n+1)^{-1}, & y=n;\\ 0, & y\neq n.\end{cases}$$

Let $\mathscr{T}$ be the $[0,1]$-topology generated by the subbase $\mathscr{B}=\{B_n;\ n\in Y\}$. Obviously $\{B_n;\ n\in Y\}$ is an open 0-shading of χ_Y, but $\{B_n;\ n\in Y\}$ has no finite subfamily which is an open 0-shading of χ_Y. Therefore $(Y,\mathscr{T})$ is not strongly compact, of course it is not α-strongly compact either.

Now we prove that $(Y,\mathscr{T})$ is α-compact. It is easy to check that if A is an α-open L-set in $(Y,\mathscr{T})$ and $A\neq\chi_Y$, then $A\leqslant\bigvee\limits_{n\in Y}B_n$.

For each $a\in[0,1)$, suppose that $\mathscr{U}$ is an α-open strong a-shading of χ_Y. If $\chi_Y\in\mathscr{U}$, then $\{\chi_Y\}$ is a strong a-shading of χ_Y. Now we suppose that $\chi_Y\notin\mathscr{U}$. Then $\mathscr{U}$ is not a strong a-shading of χ_Y since $\bigwedge\limits_{y\in Y}\Big(\bigvee\limits_{A\in\mathscr{U}}A(y)\Big)\leqslant\bigwedge\limits_{y\in Y}\Big(\bigvee\limits_{n\in Y}B_n(y)\Big)=0$.

This shows that $(Y,\mathscr{T})$ is α-compact.

References

[1] *H. Aygün*: α-compactness in L-fuzzy topological spaces. Fuzzy Sets Syst. *116* (2000), 317–324. Zbl 0990.54009

[2] *K. K. Azad*: On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weak continuity. J. Math. Anal. Appl. *82* (1981), 14–32. Zbl 0511.54006

[3] *S. Z. Bai*: Fuzzy strongly semiopen sets and fuzzy strong semicontinuity. Fuzzy Sets Syst. *52* (1992), 345–351. Zbl 0795.54009

[4] *S. Z. Bai*: The SR-compactness in L-fuzzy topological spaces. Fuzzy Sets Syst. *87* (1997), 219–225. Zbl 0912.54009

[5] *P. Dwinger*: Characterization of the complete homomorphic images of a completely distributive complete lattice. I. Indagationes Mathematicae (Proceedings) *85* (1982), 403–414. Zbl 0503.06012

[6] *G. Gierz, et al.*: A Compendium of Continuous Lattices. Springer, Berlin, 1980. Zbl 0452.06001

[7] *S. R. T. Kudri*: Compactness in L-fuzzy topological spaces. Fuzzy Sets Syst. *67* (1994), 329–336. Zbl 0842.54011

[8] *S. G. Li, S. Z. Bai, N. Li*: The near SR-compactness axiom in L-topological spaces. Fuzzy Sets Syst. *147* (2004), 307–316. Zbl pre02118382

[9] *Y. M. Liu, M. K. Luo*: Fuzzy Topology. World Scientific, Singapore, 1997. Zbl 0906.54006

[10] *R. Lowen*: A comparison of different compactness notions in fuzzy topological spaces. J. Math. Anal. Appl. *64* (1978), 446–454. Zbl 0381.54004

[11] *S. N. Maheshwari, S. S. Thakur*: On α-irresolute mappings. Tamkang J. Math. *11* (1981), 209–214. Zbl 0485.54009

[12] *S. N. Maheshwari, S. S. Thakur*: On α-compact spaces. Bull. Inst. Math. Acad. Sinica *13* (1985), 341–347. Zbl 0591.54015

[13] *O. Njåstad*: On some classes of nearly open sets. Pacific J. Math. *15* (1965), 961–970. Zbl 0137.41903

[14] *A. S. B. Shahana*: On fuzzy strong semicontinuity and fuzzy precontinuity. Fuzzy Sets Syst. *44* (1991), 303–308. Zbl 0753.54001
[15] *F.-G. Shi*: Fuzzy compactness in L-topological spaces. Fuzzy Sets Syst., submitted.
[16] *F.-G. Shi*: Countable compactness and the Lindelöf property of L-fuzzy sets. Iran. J. Fuzzy Syst. *1* (2004), 79–88.
[17] *F.-G. Shi*: Theory of L_β-nested sets and L_α-nested and their applications. Fuzzy Systems and Mathematics *4* (1995), 65–72. (In Chinese.)
[18] *S. S. Thakur, R. K. Saraf*: α-compact fuzzy topological spaces. Math. Bohem. *120* (1995), 299–303. Zbl 0841.54005
[19] *G. J. Wang*: Theory of L-fuzzy Topological Spaces. Shanxi Normal University Press, Xian, 1988. (In Chinese.)

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