131 (2006) MATHEMATICA BOHEMICA No. 1, 39–48

MODAL OPERATORS ON MV-ALGEBRAS

Magdalena Harlenderová, Jiří Rachůnek, Olomouc

(Received April 26, 2005)

Abstract. Modal operators on Heyting algebras were introduced by Macnab. In this paper we introduce analogously modal operators on MV-algebras and study their properties. Moreover, modal operators on certain derived structures are investigated.

Keywords: MV-algebra, modal operator, closure operator, residuated ℓ-monoid, Heyting algebra

MSC 2000: 06D35, 06D25, 06F05

Modal operators on Heyting algebras, i.e. on lattices which are associated with intuitionistic logic, were introduced and studied by Macnab in [3]. In this paper, we deal with the possibility of introducing modal operators on MV-algebras which are an algebraic counterpart of the Łukasiewicz infinite valued logic, i.e. one of the most important logics behind fuzzy reasoning. (Boolean algebras which are an algebraic semantics of the classical two-valued logic are special cases both of Heyting algebras and MV-algebras.)

Let us recall the notion of an MV-algebra.

Definition. An algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ is called an MV-*algebra*, if it satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $\neg\neg x = x$,
(MV5) $x \oplus \neg 0 = \neg 0$,
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

The second author was supported by the Council of Czech Government, MSM 6198959214.

For arbitrary $x, y \in A$ we put:

$$x \odot y := \neg(\neg x \oplus \neg y),\ x \vee y := \neg(\neg x \oplus y) \oplus y,$$
$$x \wedge y := \neg(\neg x \vee \neg y),\ 1 := \neg 0.$$

Then $(A; \odot, 1)$ is an Abelian monoid, $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $(A; \oplus, 0, \wedge, \vee)$ and $(A; \odot, 1, \wedge, \vee)$ are lattice ordered monoids, and, moreover, $x \oplus y = \neg(\neg x \odot \neg y)$ and $x \vee y = \neg(\neg x \wedge \neg y)$. At the same time $x \odot y \leqslant x \wedge y$, $x \vee y \leqslant x \oplus y$. Hence the binary operations "$\oplus$" and "$\odot$" are mutually dual as well as the lattice operations "$\vee$" and "$\wedge$". Therefore we can take the binary operation "$\odot$" (together with the negation) as initial. Further, for possibility to compare with Heyting algebras, we will work with another binary operation "$\rightarrow$" on A, defined by $x \rightarrow y := \neg x \oplus y$ for each $x, y \in A$.

Let "$\leqslant$" be the ordering on A induced by the lattice $(A; \vee, \wedge)$. It is easy to prove that $x \rightarrow y$ is the greatest element $z \in A$ such that $z \odot x \leqslant y$. For other results from the theory of MV-algebras see [1] or [2].

The following lemma contains the necessary properties of the operation "$\rightarrow$".

Lemma 1. *Let $\mathscr{A}$ be an* MV-*algebra and $x, y \in A$. Then the following holds:*

(1) $x \odot z \leqslant y \Leftrightarrow z \leqslant x \rightarrow y$;

(2) $x \odot (x \rightarrow y) = x \wedge y$;

(3) $x \leqslant y \Leftrightarrow x \rightarrow y = 1$;

(4) $x \leqslant y \Rightarrow x \rightarrow z \geqslant y \rightarrow z$;

(5) $x \leqslant y \Rightarrow z \rightarrow x \leqslant z \rightarrow y$;

(6) $(x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z) = x \rightarrow (y \rightarrow z)$;

(7) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$;

(8) $\neg x = x \rightarrow 0$.

Definition. Let A be an MV-algebra and $f\colon A \rightarrow A$ be a mapping. Then f is called a *modal operator* on A, if for each $x, y \in A$:

1. $x \leqslant f(x)$;
2. $f(f(x)) = f(x)$;
3. $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called *strong*, if for each $x, y \in A$:

4. $f(x \oplus y) = f(x \oplus f(y))$.

Lemma 2. *Let A be an* MV*-algebra. If f is a modal operator on A, then for each $x, y \in A$:*

(i) $x \leqslant y \Rightarrow f(x) \leqslant f(y)$;

(ii) $f(x \to y) \leqslant f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$;

(iii) $f(x) \leqslant (x \to f(0)) \to f(0)$;

(iv) $\neg x \odot f(x) \leqslant f(0)$;

(v) $x \oplus f(0) \geqslant f(x)$.

Proof. (i) Let $x \leqslant y$. Then $f(x \wedge y) = f(x)$, therefore by Lemma 1(2), $f(y \odot (y \to x)) = f(x)$. Hence by the condition 3 from the definition we obtain $f(y) \odot f(y \to x) = f(x)$ and this implies $f(x) \leqslant f(y)$.

(ii) Let $x, y \in A$. Then by the condition 3 from the definition, Lemma 1(2) and the property (i) it holds $f(x) \odot f(x \to y) = f(x \odot (x \to y)) = f(x \wedge y) \leqslant f(y)$, therefore by Lemma 1(1), $f(x \to y) \leqslant f(x) \to f(y)$. Further by the conditions 1 and 2 from the definition, Lemma 1(4) and the last proven inequality we have:

$$\begin{aligned} f(f(x) \to f(y)) &\leqslant f(f(x)) \to f(f(y)) = f(x) \to f(y) \\ &\leqslant x \to f(y) \leqslant f(x \to f(y)) \leqslant f(x) \to f(f(y)) = f(x) \to f(y) \\ &\leqslant f(f(x) \to f(y)), \end{aligned}$$

hence $f(x \to f(y)) = f(f(x) \to f(y)) = f(x) \to f(y) = x \to f(y)$.

(iii) By Lemma 1(2), $f(x) \odot (f(x) \to f(0)) = f(x) \wedge f(0) \leqslant f(0)$, hence by Lemma 1(1), $f(x) \leqslant (f(x) \to f(0)) \to f(0)$, and then by (ii) we obtain $f(x) \leqslant (x \to f(0)) \to f(0)$.

(iv) $0 \leqslant f(0)$, therefore by Lemma 1(5)(8) and the property (iii), $\neg x = x \to 0 \leqslant x \to f(0) = f(x) \to f(0)$, and according to Lemma 1(2) we obtain $\neg x \odot f(x) \leqslant f(x) \odot (f(x) \to f(0)) = f(x) \wedge f(0) \leqslant f(0)$.

(v) By the property (ii) and Lemma 1(5) it holds $x \oplus f(0) = \neg\neg x \oplus f(0) = \neg x \to f(0) = f(\neg x \to f(0)) \geqslant f(\neg x \to 0) = f(\neg\neg x) = f(x)$. □

Remark 3. By the conditions 1 and 2 from the definition of a modal operator and Lemma 2(i) it follows that f is a closure operator on the lattice $(A; \vee, \wedge)$.

Lemma 4. *If f is a strong modal operator on an* MV*-algebra $\mathscr{A}$, then for each $x, y \in A$ it holds:*

(vi) $f(x \oplus y) = f(f(x) \oplus f(y))$;

(vii) $x \oplus f(0) = f(x)$.

Proof. (vi) By the condition 4 from the definition we obtain $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus f(y))$.

(vii) According to the condition 4 it holds $f(x \oplus f(0)) = f(x \oplus 0) = f(x)$ and by the condition 1 and property (v) we have $f(x) = f(x \oplus f(0)) \geqslant x \oplus f(0) \geqslant f(x)$. □

Theorem 5. *Let $\mathscr{A}$ be an* MV*-algebra and $f\colon A \to A$ be a mapping. Then f is a modal operator on $\mathscr{A}$ if and only if for each $x, y \in A$ it holds:*
(1) $x \to f(y) = f(x) \to f(y)$;
(2) $f(x) \odot f(y) \geqslant f(x \odot y)$.

Proof. $\Rightarrow$: It follows from the definition of a modal operator and from Lemma 2(ii).

$\Leftarrow$: Let a mapping f satisfy the conditions (1) and (2).

1. If $x \in A$, then by (1) and Lemma 1(3), $x \to f(x) = f(x) \to f(x) = 1$, hence $x \leqslant f(x)$.

2. For each $x \in A$ by (1) it holds $1 = f(x) \to f(x) = f(f(x)) \to f(x)$, hence by Lemma 1(3), $f(f(x)) \leqslant f(x)$, therefore by 1 we obtain $f(f(x)) = f(x)$.

3. Let $x, y \in A$. Then by 1 it holds $x \odot y \leqslant f(x \odot y)$, hence by Lemma 1(1) and the condition (1), $y \leqslant x \to f(x \odot y) = f(x) \to f(x \odot y)$, which means $y \odot f(x) \leqslant f(x \odot y)$. From this, by Lemma 1(1) again, and by the condition (1), we get $f(x) \leqslant y \to f(x \odot y) = f(y) \to f(x \odot y)$ and thus $f(x) \odot f(y) \leqslant f(x \odot y)$.

Since f satisfies the condition (2), we obtain the equality $f(x) \odot f(y) = f(x \odot y)$. $\square$

Now, recall some relations between MV-algebras and Boolean algebras. On the one hand, we can view every Boolean algebra as an MV-algebra, in which the operation "$\oplus$" coincides with the operation "$\vee$" and the operation "$\odot$" coincides with the operation "$\wedge$". On the other hand, every MV-algebra $\mathscr{A}$ contains the subalgebra $B(\mathscr{A}) = \{x \in A\colon a \oplus a = a\} = \{x \in A\colon a \odot a = a\}$ which is a Boolean algebra and it is the greatest of all Boolean subalgebras of the MV-algebra $\mathscr{A}$. Moreover, elements from $B(\mathscr{A})$ are just all complemented elements of the lattice $(A; \vee, \wedge, 0, 1)$. Elements from $B(\mathscr{A})$ can be described also as follows: An element $a \in A$ belongs to $B(\mathscr{A})$ if and only if $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$ for each $x, y \in A$, or equivalently, if and only if $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$ for each $x, y \in A$.

For an arbitrary element $a \in A$ denote by $g_a\colon A \to A$ the mapping such that $g_a(x) = a \oplus x$ for each $x \in A$.

Theorem 6. *If $a \in A$, then g_a is a (strong) modal operator on $\mathscr{A}$ if and only if $a \in B(\mathscr{A})$.*

Proof. a) Let $a \in B(\mathscr{A})$. Then for arbitrary elements $x, y \in A$ it holds:

1. $x \leqslant a \oplus x = g_a(x)$.
2. $g_a(g_a(x)) = a \oplus (a \oplus x) = a \oplus x = g_a(x)$.
3. $g_a(x \odot y) = a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y) = g_a(x) \odot g_a(y)$.
4. $g_a(x \oplus y) = a \oplus (x \oplus y) = a \oplus (x \oplus (a \oplus y)) = g_a(x \oplus g_a(y))$.

So g_a is a strong modal operator on $\mathscr{A}$.

b) Let $a \in A$ and g_a be a modal operator on $\mathscr{A}$. Then for each $x, y \in A$ it holds $g_a(x \odot y) = g_a(x) \odot g_a(y)$, hence $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$, that is $a \in B(\mathscr{A})$. Moreover, by the part a) of the proof we get that g_a is strong. □

Remark 7. Therefore, by Lemma 4 and Theorem 6, a modal operator on $\mathscr{A}$ is strong if and only if $f = g_{f(0)}$.

Let $\mathscr{A}$ be an MV-algebra and $a \in A$. Put $h_a(x) := a \to x$, $k_a(x) := (x \to a) \to a$ for each element $x \in A$. The mappings of type h_a and k_a form important classes of modal operators in the theory of Heyting algebras (see [3]).

Proposition 8. *If $\mathscr{A}$ is an* MV-*algebra and $a \in B(\mathscr{A})$, then the mappings h_a and k_a are strong modal operators on $\mathscr{A}$.*

Proof. a) By the definition of the operation "$\to$" it holds $a \to x = \neg a \oplus x$ for each $x \in A$, thus $h_a = g_{\neg a}$.

b) If $x \in A$, then $(x \to a) \to a = (a \oplus \neg x) \to a = \neg(a \oplus \neg x) \oplus a = (\neg a \odot x) \oplus a = (\neg a \oplus a) \odot (x \oplus a) = a \oplus x$, hence $k_a = g_a$. □

Proposition 9. *If f is an arbitrary modal operator on an* MV-*algebra $\mathscr{A}$, then the restriction of f to the Boolean subalgebra $B(\mathscr{A})$ is a strong modal operator on $B(\mathscr{A})$.*

Proof. If $a \in B(\mathscr{A})$, then $f(a) \odot f(a) = f(a \odot a) = f(a)$, thus $f(a) \in B(\mathscr{A})$. Therefore the restriction $f|B(\mathscr{A})$ is a mapping of $B(\mathscr{A})$ into $B(\mathscr{A})$. (In particular, $f(0) \in B(\mathscr{A})$ for each modal operator f.) Conditions 1–3 from the definition of a modal operator for $f|B(\mathscr{A})$ are satisfied trivially.

Let now $a, b \in B(\mathscr{A})$. Then $f(a \oplus f(b)) = f(a \vee f(b)) \leqslant f(f(a \vee b)) = f(a \vee b) = f(a \oplus b)$, and since $f(a \oplus f(b)) \geqslant f(a \oplus b)$, we get the equality $f(a \oplus b) = f(a \oplus f(b))$. Consequently $f|B(\mathscr{A})$ is a strong modal operator on $B(\mathscr{A})$. □

For an arbitrary MV-algebra $\mathscr{A}$ denote by $\mathscr{M}(\mathscr{A})$ and $\mathscr{M}_s(\mathscr{A})$ the set of all modal and all strong modal operators on $\mathscr{A}$, respectively.

Theorem 10. *If $f_1, f_2 \in \mathscr{M}(\mathscr{A})$, or $f_1, f_2 \in \mathscr{M}_s(\mathscr{A})$, then $f_1 f_2 \in \mathscr{M}(\mathscr{A})$, or $f_1 f_2 \in \mathscr{M}_s(\mathscr{A})$, respectively, if and only if $f_1 f_2 = f_2 f_1$.*

Proof. By [4], Theorem 6, the composition of two closure operators on an arbitrary ordered set is a closure operator if and only if these operators commute.

Therefore it suffices to prove that if $\mathscr{A}$ is an MV-algebra, $f_1, f_2 \in \mathscr{M}(\mathscr{A})$ and $f_1 f_2 = f_2 f_1$, then $f_1 f_2$ satisfies the condition 3 from the definition of a modal operator, and if, moreover, $f_1, f_2 \in \mathscr{M}_s(\mathscr{A})$, then $f_1 f_2$ satisfies also the condition 4 from the same definition.

a) We will show that the composition of every couple of modal operators on $\mathscr{A}$ satisfies the condition 3. Let $x, y \in A$. Then $f_1 f_2(x \odot y) = f_1(f_2(x) \odot f_2(y)) = f_1 f_2(x) \odot f_1 f_2(y)$.

b) Let f_1 and f_2 be strong and commute. Then $f_1 f_2(x \oplus y) = f_1 f_2(x \oplus f_2(y)) = f_2 f_1(x \oplus f_2(y)) = f_2 f_1(x \oplus f_1 f_2(y)) = f_1 f_2(x \oplus f_1 f_2(y))$. $\square$

Suppose that the set $\mathscr{M}(\mathscr{A})$ is ordered pointwise, i.e. for arbitrary $f_1, f_2 \in \mathscr{M}(\mathscr{A})$ it holds $f_1 \leqslant f_2 \Leftrightarrow \forall x \in A; f_1(x) \leqslant f_2(x)$. Similarly for $\mathscr{M}_s(\mathscr{A})$.

Theorem 11. *If $\mathscr{A}$ is an* MV-*algebra, $a \in B(\mathscr{A})$ and $f \in \mathscr{M}(\mathscr{A})$, then $f \leqslant g_a$ if and only if $f(a) = a$.*

P r o o f. Let $f \in \mathscr{M}(\mathscr{A})$ and $f \leqslant g_a$. Then for each $x \in A$, $f(x) \leqslant a \oplus x$, thus $f(a) \leqslant a \oplus a = a$, and hence $f(a) = a$.

Conversely, let $f(a) = a$. Then by Lemma 1(1) and Lemma 2(ii) we get, for each $x \in A$, $f(x) \leqslant (f(x) \to f(a)) \to f(a) = (x \to f(a)) \to f(a) = (x \to a) \to a = k_a(x) = g_a(x)$, that means $f \leqslant g_a$. $\square$

If id_A denotes the identity on A, then $\mathrm{id}_A = g_0$, hence $\mathrm{id}_A \in \mathscr{M}_s(\mathscr{A})$. Further, for the modal operator g_1 it holds that $g_1(x) = 1$ for each $x \in A$. Therefore we get as a consequence of the previous theorem:

Corollary 12. *The ordered sets $\mathscr{M}(\mathscr{A})$ and $\mathscr{M}_s(\mathscr{A})$ have the least element g_0 and the greatest element g_1.*

Now we will deal with the sets of fixed elements of modal operators. First recall the notion of a bounded commutative $R\ell$-monoid.

Definition. *A bounded commutative residuated ℓ-monoid ($R\ell$-monoid)* is an algebra $\mathscr{M} = (M; \odot, \vee, \wedge, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ satisfying the following conditions.

(i) $(M; \odot, 1)$ is a commutative monoid.
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.
(iii) $x \odot y \leqslant z$ iff $x \leqslant y \to z$, for each $x, y, z \in M$.
(iv) $x \odot (x \to y) = x \wedge y$, for each $x, y \in M$.

On the basis of properties of MV-algebras we can see that every MV-algebra with the mentioned signature is a special case of a bounded commutative $R\ell$-monoid. If $\mathscr{M}$ is a bounded commutative $R\ell$-monoid, define the unary operation $\neg\colon M \to M$ on M such that $\neg x := x \to 0$ for each $x \in M$. Then by [5], [6] it holds that $\mathscr{M}$ is an MV-algebra if and only if it satisfies the identity $\neg\neg x = x$.

Let now $\mathscr{A}$ be an MV-algebra and f be a modal operator on $\mathscr{A}$. Put $\mathrm{Fix}(f) = \{x \in A\colon f(x) = x\}$. From the condition 2 of the definition of a modal operator it follows that $\mathrm{Fix}(f) = \mathrm{Im}(f) = \{f(x) \colon x \in M\}$. Set $C = \mathrm{Fix}(f)$. Then C is the set of all closed elements of the closure operator f on the lattice $(A; \vee, \wedge)$, therefore $(C; \vee_C, \wedge)$, where $y \vee_C z = f(y \vee z)$ for each $y, z \in C$, is a lattice.

Theorem 13. *If f is a modal operator on an* MV*-algebra $\mathscr{A}$ and $C = \mathrm{Fix}(f)$, then $C = (C; \odot, \vee_C, \wedge, \to, f(0), 1)$, where "$\odot$", "$\wedge$" and "$\to$" are the induced operations, is a bounded commutative $R\ell$-monoid.*

Proof. (i) If $x, y \in C$, then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, therefore $x \odot y \in C$. Hence $(C; \odot, 1)$ is a commutative monoid.

(ii) Since f is a closure operator on the lattice $(A; \vee, \wedge, 0, 1)$, it holds that $(C; \vee_C, \wedge, f(0), 1)$ is a bounded lattice.

(iii) Let $y, z \in C$. Then by Lemma 2(2) we obtain $y \to z = f(y) \to f(z) = f(f(y) \to f(z)) = f(y \to z)$, therefore $y \to z \in C$.

Hence, if $x, y, z \in C$ then $x \odot y \in C$ and $y \to z \in C$ and it holds in C that $x \odot y \leqslant z$ if and only if $x \leqslant y \to z$.

(iv) Now it is obvious that the identity $x \odot (x \to y) = x \wedge y$ also holds in C. □

Proposition 14. *An $R\ell$-monoid C is an* MV*-algebra if and only if $(x \to f(0)) \to f(0) = x$ for each $x \in C$.*

Proof. Put $\neg_C x := x \to f(0)$ for each $x \in C$. By [5], [6], an arbitrary commutative bounded $R\ell$-monoid is an MV-algebra if and only if it satisfies the identity $\neg\neg x = x$. Now, in our case, C is an MV-algebra if and only if it satisfies the identity $\neg_C \neg_C x = (x \to f(0)) \to f(0) = x$. □

Introduce now on the set $C = \mathrm{Fix}(f)$ the binary operation "$\oplus_C$" for $x, y \in C$ as follows: $x \oplus_C y := \neg_C(\neg_C x \odot \neg_C y)$.

Proposition 15. *An $R\ell$-monoid C is an* MV*-algebra if and only if $x \oplus_C f(0) = x$ for each $x \in C$.*

Proof. If $\mathscr{M} = (M; \odot, \vee, \wedge, \to, 0, 1)$ is an arbitrary bounded commutative $R\ell$-monoid, put $x \oplus y := \neg(\neg x \odot \neg y)$ for arbitrary $x, y \in M$. By [7], Remark 2.12, $\mathscr{M}$ is an MV-algebra if and only if it satisfies the identity $x \oplus 0 = x$. From this we obtain our proposition for the $R\ell$-monoid C. □

Recall the notion of a filter of an MV-algebra.

Definition. Let $\mathscr{A}$ be an MV-algebra and $\emptyset \neq F \subseteq A$. Then F is called a *filter* in $\mathscr{A}$, if it holds:

1. $\forall x, y \in F; x \odot y \in F$;
2. $\forall x \in F, z \in A; x \leqslant z \Longrightarrow z \in F$.

Lemma 16. *If f is a modal operator on an* MV*-algebra* $\mathscr{A}$ *and* $K_f = \mathrm{Ker}(f) = \{x \in A\colon\ f(x) = 1\}$, *then* K_f *is a filter in* $\mathscr{A}$.

P r o o f. $1 \in K_f$, thus $K_f \neq \emptyset$.
1. Let $x, y \in K_f$. Then $f(x \odot y) = f(x) \odot f(y) = 1$, therefore $x \odot y \in K_f$.
2. If $x \in K_f$, $z \in A$ and $x \leqslant z$, then $1 = f(x) \leqslant f(z)$, therefore $f(z) = 1$, that means $z \in \mathrm{Ker}(f)$. □

Definition. A filter F of an MV-algebra $\mathscr{A}$ is called *modal* if there exists a modal operator f on $\mathscr{A}$ such that $F = \mathrm{Ker}(f)$.

Proposition 17. *If* $a \in B(\mathscr{A})$ *then the interval* $[a, 1] = \{x \in A\colon\ a \leqslant x \leqslant 1\}$ *is a modal filter in* $\mathscr{A}$.

P r o o f. Since $a \in B(\mathscr{A})$, it holds that $[a, 1]$ is the principal filter in $\mathscr{A}$ generated by the element a. We will show that $[a, 1] = \mathrm{Ker}(h_a)$. It is known that if $a \in B(\mathscr{A})$, then a is complemented in the lattice $(A; \vee, \wedge, 0, 1)$ and its negation $\neg a$ is its complement. Hence if $x \in A$, then $x \in \mathrm{Ker}(h_a)$ if and only if $1 = h_a(x) = \neg a \oplus x = \neg a \vee x$, which occurs if and only if $a \leqslant x$, i.e. $x \in [a, 1]$. □

If $\mathscr{A}$ is an MV-algebra and $a \in B(\mathscr{A})$, denote $I(a) := [0, a] = \{x \in A\colon\ 0 \leqslant x \leqslant a\}$. For arbitrary elements $x, y \in I(a)$ define $x \oplus_a y := x \oplus y$ and $\neg_a x := \neg x \odot a$. Then it holds (see e.g. [8]) that $I(a) = ([0, a]; \oplus_a, \neg_a, 0)$ is an MV-algebra and that for the operation of multiplication "$\odot_a$" in $I(a)$ it holds $x \odot_a y = x \odot y$ for each $x, y \in I(a)$. For arbitrary $a \in B(\mathscr{A})$ define the mapping $f^a\colon\ I(a) \to I(a)$ such that $f^a(x) = f(x) \odot a = f(x) \wedge a$ for each $x \in I(a)$.

Theorem 18. *If* $\mathscr{A}$ *is an* MV*-algebra,* $a \in B(\mathscr{A})$ *and* f *is a modal or a strong modal operator on* $\mathscr{A}$, *then* f^a *is a modal or a strong modal operator on the* MV*-algebra* $I(a)$, *respectively.*

P r o o f. Let $x, y \in I(a)$.
1. It holds $x \leqslant f(x)$ and $x \leqslant a$, hence $x \leqslant f(x) \wedge a = f^a(x)$.
2. $f^a(f^a(x)) = f^a(f(x) \wedge a) = f(f(x) \wedge a) \wedge a = f(f(x)) \wedge f(a) = f(x) \wedge a = f^a(x)$.
3. $f^a(x \odot y) = f(x \odot y) \wedge a = (f(x) \odot f(y)) \wedge a = f(x) \odot f(y) \odot a \odot a = (f(x) \odot a) \odot (f(y) \odot a) = f^a(x) \odot f^a(y)$.
4. Let f be strong. Then $f^a(x \oplus f^a(y)) = f(x \oplus (f(y) \wedge a)) \wedge a = f(x \oplus f(f(y) \wedge a)) \wedge a = f(x \oplus (f(f(y)) \wedge f(a))) \wedge a = f(x \oplus (f(y) \wedge f(a))) \wedge a = f(x \oplus f(y \wedge a)) \wedge a = f(x \oplus f(y)) \wedge a = f(x \oplus y) \wedge a = f^a(x \oplus y)$. □

Definition. a) An ordered pair $(\mathscr{A}, f)$ is called a *modal* MV*-algebra*, if $\mathscr{A}$ is an MV-algebra and f is a modal operator on $\mathscr{A}$.

b) If $(\mathscr{A}_1, f_1)$ and $(\mathscr{A}_2, f_2)$ are modal MV-algebras and $\varphi\colon A_1 \to A_2$ is a mapping, then φ is called a *modal* MV-*homomorphism*, if it is an MV-homomorphism and if $\varphi(f_1(x)) = f_2(\varphi(x))$ for each $x \in A_1$.

Theorem 19. *Let $(\mathscr{A}, f)$ be a modal* MV-*algebra and let $a \in B(\mathscr{A}) \cap \mathrm{Fix}(f)$. Put $\varphi_a(x) := a \odot x = a \wedge x$ for each $x \in A$. Then φ_a is a surjective modal* MV-*homomorphism of $(\mathscr{A}, f)$ onto $(I(a), f^a)$.*

Proof. By [8], Theorem 9, it holds that φ_a is a surjective homomorphism of the MV-algebra $\mathscr{A}$ onto the MV-algebra $I(a)$. Further, for each $x \in A$, $\varphi_a(f(x)) = a \odot f(x) = f(a) \odot f(x) = f(a \odot x \odot a) = f(\varphi_a(x) \odot a) = f(\varphi_a(x)) \odot f(a) = f(\varphi_a(x)) \odot a = f^a(\varphi_a(x))$, therefore the MV-homomorphism φ_a is modal. □

Proposition 20. *If $\mathscr{A}$ is an* MV-*algebra and f is a modal operator on $\mathscr{A}$, then the congruences on the modal* MV-*algebra $(\mathscr{A}, f)$ coincide with the congruences on the* MV-*algebra $\mathscr{A}$.*

Proof. It is obvious that every congruence on $(\mathscr{A}, f)$ is also a congruence on $\mathscr{A}$.

Let now θ be a congruence on $\mathscr{A}$. It holds that there exists a one to one correspondence between congruences and filters of MV-algebras. If F_θ is the filter corresponding to θ, then $\langle x, y\rangle \in \theta$ if and only if $(x \oplus \neg y) \odot (\neg x \oplus y) \in F_\theta$, i.e. if and only if $(y \to x) \odot (x \to y) \in F_\theta$, for each $x, y \in \mathscr{A}$.

Let $\langle x, y\rangle \in \theta$. Then $(x \to y) \odot (y \to x) \in F_\theta$, hence also $f(x \to y) \odot f(y \to x) = f((x \to y) \odot (y \to x)) \in F_\theta$. By Lemma 2(ii) it holds $f(x \to y) \leqslant f(x) \to f(y)$ and $f(y \to x) \leqslant f(y) \to f(x)$, so we get $(f(x) \to f(y)) \odot (f(y) \to f(x)) \in F_\theta$ and that means $\langle f(x), f(y)\rangle \in \theta$. Therefore θ is a congruence on $(\mathscr{A}, f)$. □

As an immediate consequence we get the following theorem.

Theorem 21. *Let $(\mathscr{A}, f)$ be a modal* MV-*algebra and θ be a congruence on $\mathscr{A}$. Let $\overline{f}(x/\theta) = f(x)/\theta$ for each $x \in A$. Then the mapping $\overline{f}$ is a modal operator on the factor* MV-*algebra $\mathscr{A}/\theta$. If f is a strong modal operator then also the modal operator $\overline{f}$ is strong.*

References

[1] *Cignoli, R. L. O., D'Ottaviano, I. M. L., Mundici, D.*: Algebraic Foundations of Many-valued Reasoning. Kluwer, Dordrecht, 2000. Zbl 0937.06009

[2] *Dvurečenskij, A., Pulmannová, S.*: New Trends in Quantum Structures. Kluwer Acad. Publ., Dordrecht, Ister Science, Bratislava, 2000. Zbl 0987.81005

[3] *Macnab, D. S.*: Modal operators on Heyting algebras. Algebra Univers. *12* (1981), 5–29. Zbl 0459.06005

[4] *Rachůnek, J.*: Modal operators on ordered sets. Acta Univ. Palacki. Olomuc., Fac. Rer. Nat., Math. *24* (1985), 9–14. Zbl 0621.06003

[5] *Rachůnek, J.*: $DR\ell$-semigroups and MV-algebras. Czechoslovak Math. J. *48* (1998), 365–372. Zbl 0952.06014

[6] *Rachůnek, J.*: MV-algebras are categorically equivalent to a class of $DR\ell_{1(i)}$-semigroups. Math. Bohem. *123* (1998), 437–441. Zbl 0934.06014

[7] *Rachůnek, J., Šalounová, D.*: Local bounded commutative residuated ℓ-monoids (submitted).

[8] *Rachůnek, J., Švrček, F.*: MV-algebras with additive closure operators. Acta Univ. Palacki. Olomuc., Fac. Rer. Mat., Math. *39* (2000), 183–189. Zbl 1039.06005

Authors' addresses: *M. Harlenderová*, AŽD Praha s.r.o., VaV Olomouc, Smetanova 29, 779 00 Olomouc, Czech Republic, e-mail: harlenderova.magdalena@azd.cz; *J. Rachůnek*, Department of Algebra and Geometry, Faculty of Sciences, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: rachunek@inf.upol.cz.