MEASURES OF TRACEABILITY IN GRAPHS

Varaporn Saenpholphat, Bangkok, Futaba Okamoto,
Ping Zhang, Kalamazoo

(Received September 26, 2005)

Abstract. For a connected graph G of order $n \geqslant 3$ and an ordering $s\colon v_1, v_2, \dots, v_n$ of the vertices of G, $d(s) = \sum\limits_{i=1}^{n-1} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. The traceable number $t(G)$ of G is defined by $t(G) = \min\{d(s)\}$, where the minimum is taken over all sequences s of the elements of $V(G)$. It is shown that if G is a nontrivial connected graph of order n such that l is the length of a longest path in G and p is the maximum size of a spanning linear forest in G, then $2n-2-p \leqslant t(G) \leqslant 2n-2-l$ and both these bounds are sharp. We establish a formula for the traceable number of every tree in terms of its order and diameter. It is shown that if G is a connected graph of order $n \geqslant 3$, then $t(G) \leqslant 2n-4$. We present characterizations of connected graphs of order n having traceable number $2n-4$ or $2n-5$. The relationship between the traceable number and the Hamiltonian number (the minimum length of a closed spanning walk) of a connected graph is studied. The traceable number $t(v)$ of a vertex v in a connected graph G is defined by $t(v) = \min\{d(s)\}$, where the minimum is taken over all linear orderings s of the vertices of G whose first term is v. We establish a formula for the traceable number $t(v)$ of a vertex v in a tree. The Hamiltonian-connected number $\operatorname{hcon}(G)$ of a connected graph G is defined by $\operatorname{hcon}(G) = \sum\limits_{v \in V(G)} t(v)$. We establish sharp bounds for $\operatorname{hcon}(G)$ of a connected graph G in terms of its order.

Keywords: traceable graph, Hamiltonian graph, Hamiltonian-connected graph

MSC 2000: 05C12, 05C45

1. Introduction

We refer to the book [6] for graph-theoretical notation and terminology not described in this paper. Hamiltonian graphs can be defined as those graphs of order $n \geqslant 3$ for which there is a cyclic ordering $v_1, v_2, \dots, v_n$, $v_{n+1} = v_1$ of the vertices of G such that $\sum\limits_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. For a connected graph G of order $n \geqslant 3$ and a cyclic ordering $s\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the vertices of G, the number $d(s)$ is defined in [5] as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, $d(s) \geqslant n$ for each cyclic ordering s of $V(G)$. The *Hamiltonian number* $h(G)$ of G is defined in [5] by

$$h(G) = \min \{d(s)\},$$

where the minimum is taken over all cyclic orderings s of the vertices of G. Therefore, $h(G) = n$ if and only if G is Hamiltonian. To illustrate these concepts, consider the graph G of Figure 1. For the cyclic orderings $s_1\colon v_1, v_2, v_3, v_4, v_5, v_1$ and $s_2\colon v_1, v_3, v_2, v_4, v_5, v_1$ of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since G is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h(G) = 6$.

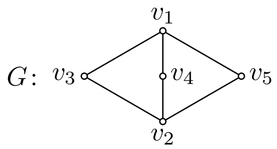

Figure 1. A graph G with $h(G) = 6$

In [8] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by $h(G)$. It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G. Consequently, this result justifies using the notation $h(G)$ for both the Hamiltonian number of a graph G and the length of a Hamiltonian walk in G. This concept was studied further in [4]. Hamiltonian walks were also studied by Asano, Nishizeki, and Watanabe [1], [2], [7], Bermond [3], Nebeský [9], and Vacek [11]. The following result appears in the papers [4], [5], [7], [8], [9].

Theorem A. *For every connected graph G of order $n \geqslant 2$,*

$$n \leqslant h(G) \leqslant 2n - 2.$$

Moreover, $h(G) = 2n - 2$ if and only if G is a tree.

In this paper, we study a natural related concept. A graph has been called *traceable* if it contains a Hamiltonian path. Therefore, every Hamiltonian graph is traceable.

The converse is not true of course. For a connected graph G of order $n \geqslant 3$ and an ordering (also called a *linear ordering*) $s\colon v_1, v_2, \ldots, v_n$ of the vertices of G, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}).$$

The *traceable number* $t(G)$ of G is defined by

$$t(G) = \min\{d(s)\},$$

where the minimum is taken over all sequences s of the elements of $V(G)$. Thus if G is a connected graph of order $n \geqslant 2$, then $t(G) \geqslant n-1$. Furthermore, $t(G) = n-1$ if and only if G is traceable. For example, since the graph G of Figure 1 is traceable and has order 5, it follows that $t(G) = 4$.

As with Hamiltonian numbers of graphs, we now see that there is an alternative way to define the traceable number of a connected graph. Denote the length of a walk W in a graph by $L(W)$.

Proposition 1.1. *Let G be a nontrivial connected graph. Then $t(G)$ is the minimum length of a spanning walk in G.*

Proof. Suppose that the minimum length of a spanning walk in a graph G is l. Furthermore, let $s\colon v_1, v_2, \ldots, v_n$ be a sequence of the vertices of G such that $d(s) = t(G)$. For each integer i with $1 \leqslant i \leqslant n-1$, let P_i be a $v_i - v_{i+1}$ path of length $d(v_i, v_{i+1})$ in G. Let W' be the $v_1 - v_n$ spanning walk of G obtained by proceeding along the paths $P_1, P_2, \ldots, P_{n-1}$ in the given order. Thus the length of W' is $L(W') = d(s) = t(G)$. Since $l \leqslant L(W')$, it follows that $l \leqslant t(G)$.

Next, let W be a spanning walk of minimum length in G. Thus the length of W is l. Suppose that $W\colon x_1, x_2, \ldots, x_{l+1}$, where then $l+1 \geqslant n$. Define $u_1 = x_1$ and $u_2 = x_2$. For $3 \leqslant i \leqslant n$, define u_i to be x_{j_i}, where j_i is the smallest positive integer such that $x_{j_i} \notin \{u_1, u_2, \ldots, u_{i-1}\}$. Then $s\colon u_1, u_2, \ldots, u_n$ is an ordering of the vertices of G. For each integer i with $1 \leqslant i \leqslant n-1$, let W_i be the $u_i - u_{i+1}$ subwalk of W determined by the terms u_i and u_{i+1} in s. Thus $d(u_i, u_{i+1}) \leqslant L(W_i)$. Since

$$t(G) \leqslant d(s) = \sum_{i=1}^{n-1} d(u_i, u_{i+1}) \leqslant \sum_{i=1}^{n-1} L(W_i) = L(W) = l,$$

it follows that $t(G) \leqslant l$, giving the desired result. □

In Theorem A it is stated that for every connected graph G of order $n \geqslant 2$, the Hamiltonian number $h(G) \leqslant 2n-2$. As expected, there is a smaller upper bound for the traceable number of G.

Theorem 2.1. *If G is a nontrivial connected graph of order n the length of whose longest path is l, then*

$$t(G) \leqslant 2n-2-l.$$

Proof. To show that $t(G) \leqslant 2n-2-l$, we proceed by induction on n. Since it is straightforward to see that $t(G) = 2n-2-l$ for every connected graph G of order n with $2 \leqslant n \leqslant 4$, the inequality holds for every connected graph of order n with $2 \leqslant n \leqslant 4$. Assume, for every connected graph H of order $n-1 \geqslant 4$ the length of whose longest path is l', that $d(H) \leqslant 2n-4-l'$. Let G be a connected graph of order n, the length of whose longest path is l. We show that $t(G) \leqslant 2n-2-l$. If G contains a Hamiltonian path, then $l = n-1$ and $t(G) = n-1$; so $t(G) = 2n-2-l$. Hence we may assume that G does not contain a Hamiltonian path. Let P be a path of length $l < n-1$ in G. Among the vertices of G not on P, let w be a vertex of G such that the length of a path from w to a vertex on P is maximum. Thus $G-w$ has order $n-1$, is connected, and the length of a longest path in $G-w$ is l. By the induction hypothesis, $t(G-w) \leqslant 2n-4-l$. Let $s\colon v_1, v_2, \ldots, v_{n-1}$ be a sequence of the vertices of $G-w$ for which $d(s) = t(G-w)$. Suppose that w is adjacent to v_i $(1 \leqslant i \leqslant n-1)$. If $i = n-1$, then let $s'\colon v_1, v_2, \ldots, v_{n-1}, w$. Thus

$$\begin{aligned} t(G) \leqslant d(s') &= d(s) + d(v_{n-1}, w) = d(s) + 1 \\ &= t(G-w) + 1 \leqslant (2n-4-l) + 1 < 2n-2-l. \end{aligned}$$

If $1 \leqslant i \leqslant n-2$, then insert w immediately after v_i in s, producing the sequence

$$s^*\colon v_1, v_2, \ldots, v_i, w, v_{i+1}, \ldots, v_{n-1}.$$

Thus

$$\begin{aligned} d(s^*) &= d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_{i+1}) \\ &\leqslant d(s) - d(v_i, v_{i+1}) + d(v_i, w) + d(w, v_i) + d(v_i, v_{i+1}) \\ &= t(G-w) + 2 \leqslant (2n-4-l) + 2 = 2n-2-l. \end{aligned}$$

Since $t(G) \leqslant d(s^*)$, it follows that $t(G) \leqslant 2n-2-l$. □

A graph is a *linear forest* if each of its components is a path. The following result gives a lower bound for the traceable number of a connected graph in terms of its order and the maximum size of a spanning linear forest.

Proposition 2.2. *If G is a nontrivial connected graph of order n such that the maximum size of a spanning linear forest in G is p, then*

$$t(G) \geqslant 2n - 2 - p.$$

Proof. Let $s\colon v_1, v_2, \ldots, v_n$ be an arbitrary sequence of the vertices of G. Since the maximum size of a spanning linear forest in G is p, at most p of the $n-1$ numbers $d(v_i, v_{i+1})$ $(1 \leqslant i \leqslant n-1)$ are 1 and the remaining $n-1-p$ numbers are at least 2. Thus

$$d(s) \geqslant p \cdot 1 + (n-1-p) \cdot 2 = p + 2n - 2 - 2p = 2n - 2 - p.$$

Therefore, $t(G) \geqslant 2n - 2 - p$. $\square$

The following corollary is an immediate consequence of Theorem 2.1 and Proposition 2.2.

Corollary 2.3. *Let G be a nontrivial connected graph of order n such that l is the length of a longest path in G and p is the maximum size of a spanning linear forest in G. Then*

$$2n - 2 - p \leqslant t(G) \leqslant 2n - 2 - l.$$

The graph G of Figure 2 has order $n = 11$. The length of a longest path in G is $l = 6$ and the maximum size of a spanning linear forest in G is $p = 8$. By Corollary 2.3, $12 \leqslant t(G) \leqslant 14$. Actually, $t(G) = 13$ and $s\colon v_1, v_2, \ldots, v_{11}$ is a linear ordering of the vertices of G such that $d(s) = 13$.

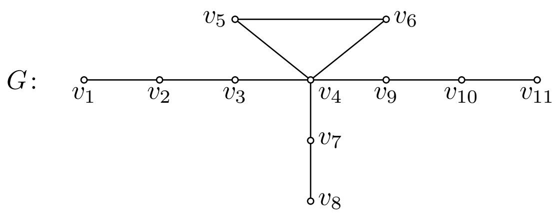

Figure 2. A graph G with $2n - 2 - p < t(G) < 2n - 2 - l$

Proposition 2.4. *If G is a nontrivial connected graph of order n and diameter 2 such that the maximum size of a spanning linear forest in G is p, then*

$$t(G) = 2n - 2 - p.$$

P r o o f. Since the maximum size of a spanning linear forest in G is p, there exists a sequence $s\colon v_1, v_2, \ldots, v_n$ of the vertices of G such that p of the $n-1$ distances $d(v_i, v_{i+1})$ $(1 \leqslant i \leqslant n-1)$ are 1 and the remaining $n-1-p$ numbers are 2. Thus $d(s) = p \cdot 1 + (n-1-p) \cdot 2 = p + 2n - 2 - 2p = 2n - 2 - p$. Hence $t(G) \leqslant 2n-2-p$. Since $t(G) \geqslant 2n-2-p$ by Proposition 2.2, it follows that $t(G) = 2n-2-p$. $\square$

Each of the graphs G_1 and G_2 of Figure 3 has order $n = 10$ and the maximum size of a spanning linear forest of each graph is $p = 7$. Such a spanning linear forest F_i of G_i $(i = 1, 2)$ is also shown in Figure 3.

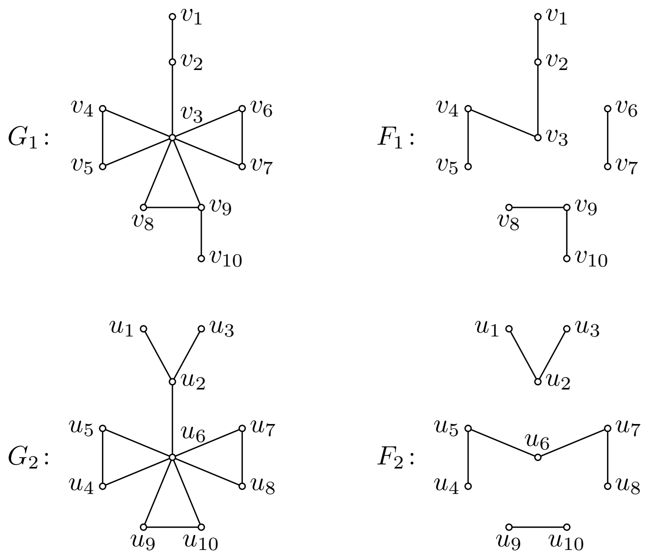

Figure 3. The graphs G_1 and G_2 and a spanning linear forest in each

By Proposition 2.2, $t(G_i) \geqslant 2n-2-p = 11$ for $i = 1, 2$. While $t(G_1) = 11$, it turns out that $t(G_2) = 12$. In the sequence $s_1\colon v_1, v_2, \ldots, v_{10}$ of the vertices of G_1, exactly $p = 7$ of the 9 distances $d(v_i, v_{i+1})$ $(1 \leqslant i \leqslant 9)$ are 1 and the other distances are 2. On the other hand, there is no sequence of the vertices of G_1 with this property and so $t(G_2) \geqslant 12$. Because $d(s_2) = 12$ for the sequence $s_2\colon u_1, u_2, \ldots, u_{10}$, it follows that $t(G_2) = 12$.

The following lemma establishes expected upper and lower bounds for $h(G)-t(G)$ for a nontrivial connected graph G. The *diameter* $\operatorname{diam}(G)$ of a connected graph G is the largest distance between two vertices in G.

Lemma 2.5. *For every nontrivial connected graph G,*

$$1 \leqslant h(G) - t(G) \leqslant \operatorname{diam}(G).$$

Proof. The lower bound is immediate. To verify the upper bound, let $s\colon v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that $d(s) = t(G)$ and let $s_c\colon v_1, v_2, \ldots, v_n, v_1$ be the cyclic ordering of the vertices of G obtained from s. Then

$$h(G) \leqslant d(s_c) = d(s) + d(v_n, v_1) \leqslant t(G) + \operatorname{diam}(G).$$

Therefore, $h(G) - t(G) \leqslant \operatorname{diam}(G)$. □

We now determine all connected graphs G for which $h(G) - t(G) = 1$.

Proposition 2.6. *For a nontrivial connected graph G,*

$$h(G) - t(G) = 1 \text{ if and only if } G \text{ is Hamiltonian.}$$

Proof. Observe first that if G is a Hamiltonian graph of order n, then $h(G) = n$ and $t(G) = n-1$; so $h(G)-t(G) = 1$. For the converse, assume that G is a connected graph such that $h(G)-t(G) = 1$. Let $s_c\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ be a cyclic ordering of the vertices of G with $d(s_c) = h(G)$. We show that $d_G(v_i, v_{i+1}) = 1$ for $1 \leqslant i \leqslant n$, which implies that $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle of G. Consider the linear ordering $s_l\colon v_1, v_2, \ldots, v_n$ of the vertices of G obtained from s_c. Since

$$d(s_l) = d(s_c) - d(v_1, v_n) = h(G) - d(v_1, v_n),$$

it follows that $t(G) \leqslant d(s_l) = h(G)-d(v_1, v_n)$ and so $1 \leqslant d(v_1, v_n) \leqslant h(G)-t(G) = 1$. Thus $d(v_1, v_n) = 1$. Consequently, $d(v_{i-1}, v_i) = 1$ for $2 \leqslant i \leqslant n$ as well. Therefore, $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle of G and so G is Hamiltonian. □

If G is a connected graph and H is a connected spanning subgraph of G, then $d_G(u,v) \leqslant d_H(u,v)$ for all $u,v \in V(G) = V(H)$. Thus for every linear ordering $s\colon v_1, v_2, \ldots, v_n$ of the vertices of G (or H),

$$d_G(s) = \sum_{i=1}^{n-1} d_G(v_i, v_{i+1}) \leqslant \sum_{i=1}^{n-1} d_H(v_i, v_{i+1}) = d_H(s)$$

and so $t(G) \leqslant t(H)$. We state this useful observation below.

Observation 3.1. *If G is a connected graph and H is a connected spanning subgraph of G, then $t(G) \leqslant t(H)$. In particular, if G is a connected graph and T is a spanning tree of G, then $t(G) \leqslant t(T)$*

Observation 3.1 suggests the usefulness of knowing the traceable numbers of trees. Since a tree T is traceable if and only if T is a path, it follows for a tree T of order n that $t(T) = n-1$ if and only if $T = P_n$ and so $t(T) \geqslant n$ if $T \neq P_n$.

Since the length of a longest path in T is the diameter of T, we have the following consequence of Corollary 2.3.

Corollary 3.2. *If T is a nontrivial tree of order n such that the maximum size of a spanning linear forest in T is p, then*

$$2n - 2 - p \leqslant t(T) \leqslant 2n - 2 - \operatorname{diam}(T).$$

A *caterpillar* is a tree T the removal of whose end-vertices is a path. The trees T_1 and T_2 of Figure 4 are caterpillars of the same order $n = 10$. While the maximum size of a spanning linear forest of T_1 is $\operatorname{diam}(T_1)$, the maximum size of a spanning linear forest of T_2 is $\operatorname{diam}(T_2) + 1$. In Figure 4, F_i is a spanning linear forest of maximum size in T_i for $i = 1, 2$.

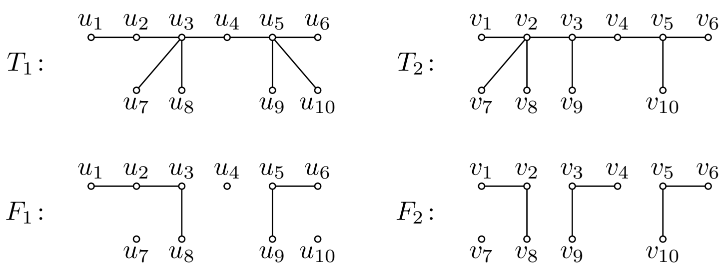

Figure 4. Spanning linear forests of maximum size in caterpillars

Since the maximum size of a spanning linear forest of T_1 is $\operatorname{diam}(T_1)$, it follows by Corollary 3.2 that $t(T_1) = 2n - 2 - \operatorname{diam}(T_1)$. In fact, $s_1\colon\ u_1, u_2, u_3, u_8, u_7, u_4, u_9, u_5, u_6, u_{10}$ is a linear ordering of the vertices of T_1 for which $d(s_1) = t(T_1)$. For the caterpillar T_2, however, the maximum size p of a spanning linear forest is $\operatorname{diam}(T_2)+1$. Consequently, by Corollary 3.2 either $t(T_2) = 2n-2-\operatorname{diam}(T_2)$ or $t(T_2) = 2n-3-\operatorname{diam}(T_2)$. The linear ordering $s_2\colon\ v_7, v_1, v_2, v_8, v_9, v_3, v_4, v_{10}, v_5, v_6$ of the vertices of T_2 has the property that $d(s_2) = 2n-2-\operatorname{diam}(T_2)$. A total of p of the $n-1$ terms in the sum $d(s_2)$ are 1. All of the remaining terms in $d(s_2)$ are 2, except for one which is 3. If fewer than p terms in the sum $d(s')$ for a linear ordering s' of the vertices of T_2 are 1, then $d(s') \geqslant 2n-2-\operatorname{diam}(T_2)$. Hence if there is a linear ordering s of the vertices of T_2 for which $d(s) = 2n-3-\operatorname{diam}(T_2)$, then there must be p terms in $d(s)$ equal to 1. We may assume that both v_1, v_2, v_8 (or v_8, v_2, v_1) and v_9, v_3, v_4 (or v_4, v_3, v_9) are subsequences of s. Assume, without loss of generality, that the vertices v_1, v_2, v_8 occur before v_9, v_3, v_4. Then the first vertex in s that follows the last vertex of v_1, v_2, v_8 or the last vertex of v_1, v_2, v_8, v_7 is a vertex whose distance is at least 3 from that vertex. Hence $d(s) \geqslant 2n-2-\operatorname{diam}(T_2)$ and so $t(T_2) = 2n-2-\operatorname{diam}(T_2)$. Proceeding in a similar manner for every caterpillar gives us the following result.

Corollary 3.3. *If T is a caterpillar of order n, then*

$$t(T) = 2n - 2 - \operatorname{diam}(T).$$

We now show that the formula presented in Corollary 3.3 for the traceable number of a caterpillar holds in fact for all trees.

Theorem 3.4. *If T is a nontrivial tree of order n, then*

$$t(T) = 2n - 2 - \operatorname{diam}(T).$$

Proof. Since $h(T) = 2n-2$ for every tree T of order n, it follows by Lemma 2.5 that $t(T) \geqslant 2(n-1) - \operatorname{diam}(T)$. Furthermore, since the length of a longest path in T is $\operatorname{diam}(T)$, it follows by Theorem 2.1 that $t(T) \leqslant 2(n-1) - \operatorname{diam}(T)$, giving the desired result. □

If T is a tree of order $n \geqslant 3$, then $2 \leqslant \operatorname{diam}(T) \leqslant n-1$. Therefore, by Theorem 3.4, if T is a tree of order $n \geqslant 3$, then

$$n - 1 \leqslant t(T) \leqslant 2n - 4. \tag{1}$$

We saw that $t(T) = n-1$ if and only if $T = P_n$. Furthermore, only stars have diameter 2. So $t(T) = 2n-4$ if and only if $T = K_{1,n-1}$ by Theorem 3.4. More generality, we have the following the realization result.

Proposition 3.5. *For each pair k, n of integers with $3 \leqslant n-1 \leqslant k \leqslant 2n-4$, there exists a tree T of order n with $t(T) = k$.*

P r o o f. Let $P\colon v_1, v_2, \ldots, v_{2n-1-k}$ be a path of length $2n-2-k$. A tree T is constructed by adding $k+1-n$ new vertices $w_1, w_2, \ldots, w_{k+1-n}$ and joining all of these vertices to v_2. Since $\operatorname{diam}(T) = 2n-2-k$, it follows by Theorem 3.4 that $t(T) = 2n-2-(2n-2-k) = k$. $\square$

With the aid of Theorem 3.4, it is straightforward to determine those nontrivial trees T of order n such that $t(T) = n$.

Proposition 3.6. *Let T be a tree of order $n \geqslant 4$. Then $t(T) = n$ if and only if T is a caterpillar with maximum degree $\Delta(T) = 3$ and having exactly one vertex of degree 3.*

P r o o f. By Theorem 3.4, $t(T) = n$ if and only if $2n-2-\operatorname{diam}(T) = n$ and so $\operatorname{diam}(T) = n-2$. Hence T contains a path $P\colon v_1, v_2, \ldots, v_{n-1}$ of length $n-2$ and a vertex w not on P that is adjacent to some vertex v_i with $2 \leqslant i \leqslant n-2$. $\square$

By (1) and Observation 3.1, if G is a connected graph of order $n \geqslant 3$, then

$$n-1 \leqslant t(G) \leqslant 2n-4. \tag{2}$$

We now determine all those connected graphs G of order n such that $t(G) = 2n-4$ or $t(G) = 2n-5$.

Proposition 3.7. *Let G be a connected graph of order $n \geqslant 3$. Then*

$$t(G) = 2n-4 \text{ if and only if } G = K_3 \text{ or } G = K_{1,n-1}.$$

P r o o f. Let G be a connected graph of order $n \geqslant 3$ such that $t(G) = 2n-4$. If G contains a path of length 3 or more, then it follows by Theorem 2.1 that $t(G) \leqslant 2n-5$. Hence the length of a longest path in G is 2. This implies that $\Delta(G) = n-1$ and so $G = K_3$ or $G = K_{1,n-1}$. Furthermore, note that $t(K_3) = 2n-4 = n-1$ and $t(K_{1,n-1}) = 2n-4$. $\square$

A tree T is a *double star* if T contains exactly two vertices that are not end-vertices, necessarily these vertices are adjacent in T. For integers $a, b \geqslant 2$, let $S_{a,b}$ denote the double star whose two vertices that are not end-vertices have degrees a and b.

Proposition 3.8. *Let G be a connected graph of order $n \geqslant 4$. Then $t(G) = 2n-5$ if and only if* (1) $n = 4$ *and* $G \neq K_{1,3}$ *and* (2) $n \geqslant 5$ *and* $G = K_{1,n-1} + e$ *or* $G = S_{a,b}$ *for some positive integers a and b with $a + b = n$.*

Proof. Let G be a connected graph of order $n \geqslant 4$ such that $t(G) = 2n-5$. From Theorem 2.1, it follows that the length of a longest path in G is 3. This implies that (1) $n = 4$ and $G \neq K_{1,3}$, (2) $n \geqslant 5$, $\Delta(G) = n-1$, and $G = K_{1,n-1} + e$, or (3) $n \geqslant 5$, $\Delta(G) \leqslant n-2$ and G is a double star. The converse is straightforward. □

4. Traceable numbers of vertices

Let G be a connected graph of order n. For $v \in V(G)$, the *traceable number* $t(v)$ of v is defined by

$$t(v) = \min\{d(s)\},$$

where the minimum is taken over all linear orderings s of the vertices of G whose first term is v. Thus $t(v) \geqslant n-1$ for every vertex v of G. Furthermore, $t(v) = n-1$ if and only if G contains a Hamiltonian path with initial vertex v. Observe that

$$t(G) = \min\{t(v)\colon\ v \in V(G)\}.$$

Using an argument similar to that used in the proof of Proposition 1.1, we have the following.

Proposition 4.1. *Let G be a nontrivial connected graph and let $v \in V(G)$. Then $t(v)$ is the minimum length of a spanning walk in G whose initial vertex is v.*

We present a result concerning the traceable number of adjacent vertices in a connected graph.

Proposition 4.2. *Let G be a connected graph and let u and v be adjacent vertices of G. Then*

$$|t(u) - t(v)| \leqslant 1.$$

Proof. Let $s\colon\ v = v_1, v_2, \ldots, v_n$ be a linear ordering of the vertices of G such that $d(s) = t(v)$. Thus $u = v_i$ for some integer i with $2 \leqslant i \leqslant n$. We consider two cases.

Case 1. $u = v_i$, where $2 \leqslant i \leqslant n-1$. Let

$$s'\colon\ u = v_i, v_{i-1}, \ldots, v_2, v_1 = v, v_{i+1}, v_{i+2}, \ldots, v_n.$$

Then

$$\begin{aligned} t(u) &\leqslant d(s') = d(s) - d(u, v_{i+1}) + d(v, v_{i+1}) \\ &\leqslant d(s) - d(u, v_{i+1}) + d(v, u) + d(u, v_{i+1}) = d(s) + 1 = t(v) + 1. \end{aligned}$$

Thus $t(u) - t(v) \leqslant 1$.

C a s e 2. $u = v_n$. Consider the sequence

$$s''\colon\ u = v_n, v_{n-1}, \ldots, v_2, v_1 = v.$$

Then $t(u) \leqslant d(s'') = d(s) = t(v)$ and so $t(u) - t(v) \leqslant 0$.

In either case, $t(u) - t(v) \leqslant 1$. Applying a similar argument to that given above, we have $t(v) - t(u) \leqslant 1$ as well and so $|t(u) - t(v)| \leqslant 1$. □

For a connected graph G, let

$$t^+(G) = \max\{t(v)\colon\ v \in V(G)\}.$$

Obviously, $t(G) \leqslant t^+(G)$ for every connected graph G. The following is a consequence of Proposition 4.2.

Corollary 4.3. *Let G be a connected graph and let k be an integer such that $t(G) \leqslant k \leqslant t^+(G)$. Then there exists a vertex w of G such that $t(w) = k$.*

P r o o f. The statement is obvious if $k = t(G)$ or $k = t^+(G)$. Hence we may assume that $t(G) < k < t^+(G)$. Let u be a vertex such that $t(u) = t(G)$ and let v be a vertex such that $t(v) = t^+(G)$. Since G is connected, G contains a $u - v$ path $P\colon\ u = u_1, u_2, \ldots, u_s = v$. By Proposition 4.2, $|t(u_i) - t(u_{i+1})| \leqslant 1$ for all i with $1 \leqslant i \leqslant s-1$. Let j be the largest integer with $1 \leqslant j < s$ such that $t(u_j) \leqslant k$. Then $t(u_j) = k$; for otherwise, $t(u_j) \leqslant k - 1$ and so $t(u_{j+1}) \leqslant 1 + (k-1) = k$, producing a contradiction. □

For a vertex v in a connected graph G, the *eccentricity* $e(v)$ of v is the largest distance between v and a vertex of G.

Theorem 4.4. *If T is a nontrivial tree of order n and let v be a vertex of T, then*

$$t(v) = 2(n-1) - e(v).$$

P r o o f. First, we show that $t(v) \geqslant 2(n-1) - e(v)$. Let $s\colon\ v = v_1, v_2, \ldots, v_n$ be a linear ordering of the vertices of T such that $d(s) = t(v)$, and let

$$s'\colon\ v = v_1, v_2, \ldots, v_n, v_1$$

be the cyclic ordering of the vertices of T obtained by adding $v_1 = v$ at the end of s. Then

$$2(n-1) = h(T) \leqslant d(s') = d(s) + d(v_n, v_1) \leqslant t(v) + e(v)$$

and so $t(v) \geqslant 2(n-1) - e(v)$.

Next, we show that $t(v) \leqslant 2(n-1) - e(v)$ for each vertex v in a nontrivial tree of order n. We proceed by induction on n. This is certainly true for a tree of order 2. Assume, for every tree T' of order $n-1$, where $n-1 \geqslant 2$, and every vertex u of T', that $t(u) \leqslant 2(n-2) - e(u)$. We show that if T is a nontrivial tree of order n and v is a vertex of T, then

$$t(v) \leqslant 2(n-1) - e(v).$$

This is certainly the case if T is the path P_n and v is an end-vertex of P_n. Hence we may assume that this is not the case. Let P be a longest path in T with initial vertex v, say P is a $v-w$ path. Then $d(v,w) = e(v)$. Hence there exists an end-vertex x of T such that x does not lies on P. Let y be the vertex of T that is adjacent to x. Thus $T - x$ is a tree of order $n-1$ such that $v \in V(T-x)$ and $e_{T-x}(v) = e_T(v)$. By the induction hypothesis,

$$t_{T-x}(v) \leqslant 2(n-2) - e_{T-x}(v) = 2(n-2) - e_T(v).$$

Let $s_1\colon\ v = u_1, u_2, \ldots, u_{n-1}$ be a linear ordering of the vertices of $T - x$ such that $d(s_1) = t_{T-x}(v)$. Then $y = u_i$ for some i with $2 \leqslant i \leqslant n-1$. Let z be the vertex of $T - x$ that immediately follows or immediately precedes y in s_1, say z immediately follows y in s_1. Thus $z = u_{i+1}$. Let s be the linear ordering of the vertices of T obtained by inserting x between y and z. Then

$$\begin{aligned} d(s) &= d(s_1) - d(y,z) + d(y,x) + d(x,z) \leqslant d(s_1) - d(y,z) + 1 + 1 + d(y,z) \\ &= d(s_1) + 2 = t_{T-x}(v) + 2 \leqslant 2(n-2) - e_T(v) + 2. \end{aligned}$$

Therefore, $t_T(v) \leqslant d(s) \leqslant 2(n-1) - e_T(v)$. Hence $t(v) = 2(n-1) - e(v)$. $\square$

By Theorem 4.4,

$$t(v) = h(T) - e(v)$$

for every tree T and every vertex v of T. Since $t(T) = \min\{t(v)\colon\ v \in V(G)\}$, it follows that

$$t(T) = h(T) - \max\{e(v)\colon\ v \in V(T)\} = 2n - 2 - \operatorname{diam}(T),$$

which provides us with an alternative proof of Theorem 3.4.

Observe that Theorem 4.4 is not true in general for connected graphs that are not trees. Consider the graphs G and H in Figure 5. Each vertex of G and H is labeled with its traceable number. The Hamiltonian number of graph G is $h(G) = 7$. Since $e(u) = e(y) = 3$ and $e(v) = e(w) = e(x) = 3$, it follows that $t(z) = h(G) - e(z)$ for every vertex z of G. On the other hand, for the graph H, $h(H) = 6$. While $t(z) = h(H) - e(z)$ for $z = w$ and $z = x$, this is not true otherwise.

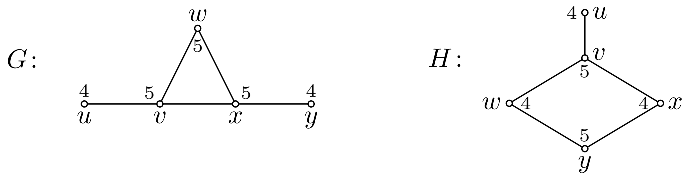

Figure 5. The graphs G and H

5. Graphs with prescribed Hamiltonian and traceable numbers

We have seen in Lemma 2.5 that for every nontrivial connected graph G,

$$1 \leqslant h(G) - t(G) \leqslant \operatorname{diam}(G).$$

Furthermore, by Proposition 2.6, Hamiltonian graphs are the only connected graphs G for which $h(G) - t(G) = 1$. By Theorems A and 3.4, if T is a tree then $h(T) - t(T) = \operatorname{diam}(T)$. However, trees are not the only connected graphs with this property. In fact, there are other classes of connected graphs with this property. For example, if $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph, where $k \geqslant 2$, $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$, and $n_1 + n_2 + \ldots + n_{k-1} < n_k$, then $h(G) - t(G) = 2 = \operatorname{diam}(G)$. Next, we show that for each pair k, d of integers with $1 \leqslant k \leqslant d$, there exists a connected graph G with $\operatorname{diam}(G) = d$ such that $h(G) - t(G) = k$. In order to do this, we first state a useful lemma that appeared in [5].

Lemma B. *Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then*

$$h(G) = \sum_{i=1}^{k} h(B_i).$$

Proposition 5.1. *For each pair k, d of integers with $1 \leqslant k \leqslant d$, there exists a connected graph G with diameter d such that $h(G) - t(G) = k$.*

P r o o f. If $k = d$, let G be a tree with $\operatorname{diam}(G) = d$. It then follows by Theorem A and Theorem 3.4 that $h(G) - t(G) = (2n-2) - (2n-d-2) = d$. Thus, we may assume that $k < d$. For $k = 1$, the cycle C_{2d} of order $2d$ has the desired property. For $k \geqslant 2$, let G be the graph obtained from the cycle $C_{2(d-k+1)}\colon\ u_1, u_2, \ldots, u_{2(d-k+1)}, u_1$ and the path $P_{k-1}\colon\ v_1, v_2, \ldots, v_{k-1}$ by joining u_{d-k+1} and v_{k-1}. Then the order of G is $n = 2d - k + 1$ and its diameter is $\operatorname{diam}(G) = d$. By Lemma B,

$$h(G) = h(C_{2(d-k+1)}) + (k-1)h(P_2) = 2(d-k+1) + 2(k-1) = 2d.$$

Since G is traceable, $t(G) = n - 1 = 2d - k$. Therefore, $h(G) - t(G) = k$. $\square$

Since $h(G) \leqslant t(G) + \operatorname{diam}(G)$ for every nontrivial connected graph G and, trivially, $t(G) \geqslant \operatorname{diam}(G)$, it follows that $t(G) < h(G) \leqslant 2t(G)$. Thus if G is a connected graph with $t(G) = a$ and $h(G) = b$, then $a < b \leqslant 2a$. Next, we show that every pair a, b of positive integers with $a < b \leqslant 2a$ is realizable as the traceable number and the Hamiltonian number of some connected graph, respectively.

Proposition 5.2. *For every pair a, b of positive integers with $a < b \leqslant 2a$, there is a connected graph G with $t(G) = a$ and $h(G) = b$.*

P r o o f. If $b = 2a$, then $G = P_{a+1}$ has the desired properties. Hence we may assume that $a < b < 2a$. Let $k = b - a$. Thus $k < a$. Let G be the graph obtained from the path $P\colon\ u_1, u_2, \ldots, u_a, u_{a+1}$ by joining u_{a+1} and u_k. By Lemma B,

$$h(G) = h(C_{a-k+2}) + (k-1)h(P_2) = (a-k+2) + 2(k-1) = b.$$

Since G is traceable, $t(G) = (a+1) - 1 = a$. $\square$

By Theorem A, Lemma 2.5, and (2), if G is a connected graph of order $n \geqslant 3$ with $t(G) = a$ and $h(G) = b$, then

$$1 \leqslant n - 1 \leqslant a < b \leqslant 2n - 2. \tag{3}$$

Next we determine all triples (a, b, n) of positive integers satisfying (3) that can be realized as the traceable number, Hamiltonian number, and order, respectively, of some connected graph.

Theorem 5.3. *For each triple (a,b,n) of positive integers with $1 \leqslant n-1 \leqslant a < b \leqslant 2n-2$ and $n \geqslant 3$, there is a connected graph G of order n such that $t(G)=a$ and $h(G)=b$ if and only if* (1) $b=a+1=n$ *or* (2) $b \geqslant a+2$.

Proof. Let G be a connected graph of order n such that $t(G)=a$ and $h(G)=b$. If $b=a+1$, then $h(G)-t(G)=1$. By Proposition 2.6, G is Hamiltonian. Thus $t(G)=n-1$ and $h(G)=n$. Thus $b=a+1=n$. If $b \neq a+1$, then $b \geqslant a+2$ by Lemma 2.5.

For the converse, let (a,b,n) be a triple of positive integers with $1 \leqslant n-1 \leqslant a < b \leqslant 2n-2$ such that $b=a+1=n$ or $b \geqslant a+2$. If $b=a+1=n$, then any Hamiltonian graph of order n has the desired property. Thus, we may assume that $b \geqslant a+2$. Observe that $b-a-1 \geqslant 1$ and $2n-b \geqslant 2$. We consider two cases.

Case 1. $a=n-1$. Let G_1 be the graph obtained from the path P_{b-a-1}: $u_1,u_2,\ldots,u_{b-a-1}$ of order $b-a-1$ and the complete graph K_{2n-b} with $V(K_{2n-b})=\{v_1,v_2,\ldots,v_{2n-b}\}$ by joining u_{b-a-1} to v_1. Then the order of G_1 is $n=(b-a-1)+(2n-b)=n$. By Lemma B,

$$h(G_1)=(b-a-1)h(P_2)+h(K_{2n-b})=2(b-a-1)+(2n-b)=b.$$

Since G_1 is traceable, $t(G_1)=n-1=a$.

Case 2. $a \geqslant n$. Let G_2 be the graph obtained from the graph G_1 in Case 1 by adding $a-n+1$ new vertices $w_1,w_2,\ldots,w_{a-n+1}$ and joining w_i to v_1 for $1 \leqslant i \leqslant a-n+1$. Then the order of G_2 is $n=(b-a-1)+(2n-b)+(a-n+1)=n$ and $\operatorname{diam}(G_2)=b-a$. By Lemma B,

$$\begin{aligned} h(G_2) &= (b-a-1)h(P_2)+h(K_{2n-b})+(a-n+1)h(P_2) \\ &= 2(b-a-1)+(2n-b)+2(a-n+1)=b. \end{aligned}$$

It remains to show that $t(G_2)=a$. By Lemma 2.5,

$$t(G_2) \leqslant h(G_2)-\operatorname{diam}(G_2)=b-(b-a)=a.$$

Since the maximum size of a spanning linear forest in G_2 is $p=2n-a-2$, it follows by Proposition 2.2 that $t(G_2) \geqslant 2n-2-p=a$. Thus $t(G_2)=a$. $\square$

For a connected graph G of order n, the *Hamiltonian-connected number* $\mathrm{hcon}(G)$ of G is defined by

$$\mathrm{hcon}(G) = \sum_{v\in V(G)} t(v).$$

Since $t(v) \geqslant n-1$ for every vertex v of G, it follows that $\mathrm{hcon}(G) \geqslant n(n-1)$. Furthermore, $\mathrm{hcon}(G) = n(n-1)$ if and only if G is Hamiltonian-connected. Therefore, the Hamiltonian-connected number of a connected graph G of order n can be considered as a measure of how close G is to being Hamiltonian-connected—the closer $\mathrm{hcon}(G)$ is to $n(n-1)$, the closer G is to being Hamiltonian-connected.

Consider the graphs H_1 and H_2 in Figure 6, where H_1 is obtained from the complete graph K_{n-1} by adding a pendant edge and $H_2 \cong 2K_1 + (K_{n-4} \cup 2K_1)$. For the graph H_1, every vertex of H_1 has traceable number $n-1$, except for the vertex v which has traceable number n. Thus $\mathrm{hcon}(H_1) = n(n-1)+1$. Every vertex of the graph H_2 has traceable number $n-1$, except for v_1 and v_2, which have traceable number n. Thus $\mathrm{hcon}(H_2) = n(n-1)+2$.

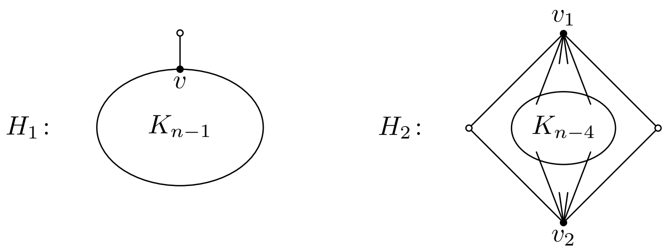

Figure 6. The graphs H_1 and H_2

Next consider the graphs G_1 and G_2 in Figure 7, where G_1 is obtained from the complete graph K_{n-2} $(n \geqslant 5)$ by adding two pendant edges and G_2 is obtained from the cycle C_{n-1} $(n \geqslant 4)$ by adding a pendant edge. The graph G_1 of order n in Figure 7 contains exactly two vertices with traceable number $n-1$, namely $t(u) = t(v) = n-1$. All other vertices of G_1 have traceable number n. Thus $\mathrm{hcon}(G_1) = n(n-1)+(n-2)$. The graph G_2 of order n in Figure 7 contains exactly three vertices with traceable number $n-1$, namely $t(u) = t(v) = t(w) = n-1$. All other vertices of G_2 have traceable number n. Thus $\mathrm{hcon}(G_2) = n(n-1)+(n-3)$. Therefore, the graphs H_1 and H_2 in Figure 6 are closer to being Hamiltonian-connected than are the graphs G_1 and G_2 of Figure 7.

The minimum eccentricity among the vertices of G is its *radius*, which is denoted by $\mathrm{rad}(G)$. A vertex v in G is a *central vertex* if $e(v) = \mathrm{rad}(G)$ and the subgraph

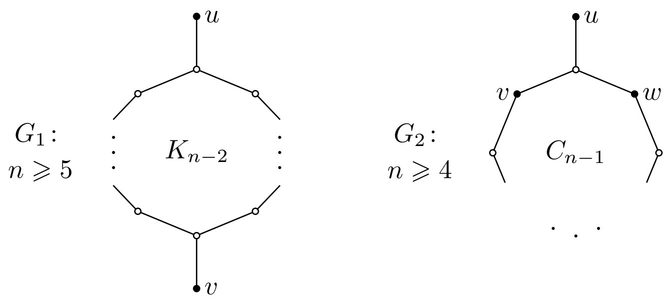

Figure 7. The graphs G_1 and G_2

induced by the central vertices of G is the *center* of G. Next, we establish upper and lower bounds for the Hamiltonian-connected number of a connected graph in terms of its order, beginning with trees.

Theorem 6.1. *For every tree T of order $n \geqslant 3$,*

$$n(n-1) + \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \leqslant \operatorname{hcon}(T) \leqslant n(n-1) + (n^2 - 3n + 1).$$

Proof. For a tree T, it is known (see [10]) that there exists at least one vertex v with $e(v) = \operatorname{rad}(T)$ and there exist at least two vertices v with $e(v) = k$ for every integer k with $\operatorname{rad}(T) < k \leqslant \operatorname{diam}(T)$. Furthermore, it is well-known that for every tree T, either

$$\operatorname{diam}(T) = 2\operatorname{rad}(T) \text{ or } \operatorname{diam}(T) = 2\operatorname{rad}(T) - 1$$

where the center of T contains exactly one vertex in the first case and exactly two vertices in the second case. Since $\operatorname{diam}(T) \leqslant n-1$ for every tree T of order n, the largest possible radius of a tree T having odd order is $(n-1)/2$, while the largest possible radius of a tree T having even order is $n/2$. We consider the cases when n is odd or n is even separately.

Case 1. n is odd. In this case,

$$\begin{aligned}\sum_{v \in V(T)} e(v) &\leqslant \frac{n-1}{2} + 2\left[\frac{n+1}{2} + \frac{n+3}{2} + \ldots + (n-1)\right] \\ &= \frac{n-1}{2} + (n+1) + (n+3) + \ldots + 2(n-1) \\ &= \frac{n-1}{2} + \frac{n(n-1)}{2} + \left(\frac{n-1}{2}\right)^2 = \frac{n^2-1}{2} + \left(\frac{n-1}{2}\right)^2.\end{aligned}$$

It then follows by Theorem 4.4 that

$$\begin{aligned}\operatorname{hcon}(T) &= \sum_{v\in V(T)} t(v) = \sum_{v\in V(T)} (2n-2-e(v)) = n(2n-2) - \sum_{v\in V(T)} e(v)\\ &\geqslant n(2n-2) - \left[\frac{n^2-1}{2} + \left(\frac{n-1}{2}\right)^2\right] = n(n-1) + \left(\frac{n-1}{2}\right)^2.\end{aligned}$$

Case 2. n is even. In this case,

$$\begin{aligned}\sum_{v\in V(T)} e(v) &\leqslant 2\left[\frac{n}{2} + \frac{n+2}{2} + \ldots + (n-1)\right]\\ &= n + (n+2) + \ldots + 2(n-1) = \frac{n^2}{2} + \frac{n^2-2n}{4}.\end{aligned}$$

It then follows by Theorem 4.4 that

$$\begin{aligned}\operatorname{hcon}(T) &= \sum_{v\in V(T)} t(v) = \sum_{v\in V(T)} (2n-2-e(v)) = n(2n-2) - \sum_{v\in V(T)} e(v)\\ &\geqslant n(2n-2) - \left(\frac{n^2}{2} + \frac{n^2-2n}{4}\right) = n(n-1) + \frac{n^2-2n}{4}.\end{aligned}$$

Therefore, $\operatorname{hcon}(T) \geqslant n(n-1) + \lfloor(\frac{n-1}{2})^2\rfloor$ for every tree T of order $n \geqslant 3$.

If a tree T of order $n \geqslant 3$ contains a vertex with eccentricity 1, then T is a star and all other vertices have eccentricity 2. If the minimum eccentricity of a vertex of T is 2, then at most two vertices of T have eccentricity 2, with all other vertices have eccentricity 3 or 4. In any case,

$$\sum_{v\in V(T)} e(v) \geqslant 1 + (n-1)\cdot 2 = 2n-1.$$

Consequently,

$$\begin{aligned}\operatorname{hcon}(T) &= \sum_{v\in V(T)} t(v) = \sum_{v\in V(T)} (2n-2-e(v)) = n(2n-2) - \sum_{v\in V(T)} e(v)\\ &\leqslant n(2n-2) - (2n-1) = n(n-1) + (n^2-3n+1).\end{aligned}$$

Therefore, $\operatorname{hcon}(T) \leqslant n(n-1) + (n^2-3n+1)$ for every tree T of order $n \geqslant 3$. $\square$

Since $\operatorname{hcon}(P_n) = n(n-1) + \lfloor(\frac{n-1}{2})^2\rfloor$ and $\operatorname{hcon}(K_{1,n-1}) = n(n-1) + (n^2-3n+1)$, the lower and upper bounds in Theorem 6.1 are both sharp.

Corollary 6.2. *For a nontrivial connected graph G of order n,*

$$n(n-1) \leqslant \mathrm{hcon}(G) \leqslant n(n-1) + (n^2 - 3n + 1).$$

P r o o f. We have already noted that $\mathrm{hcon}(G) \geqslant n(n-1)$, so it remains only to show that $\mathrm{hcon}(G) \leqslant n(n-1)+(n^2-3n+1)$. For every connected spanning subgraph H of G and every two vertices x and y of G, $d_G(x,y) \leqslant d_H(x,y)$. Therefore, for every vertex v of G, $t_G(v) \leqslant t_H(v)$. Hence if T is a spanning tree of G, then $t_G(v) \leqslant t_T(v)$ for every vertex v of G. This implies that among all connected graphs G of order n, the maximum value of $\mathrm{hcon}(G)$ occurs when G is a tree. The result then follows by Theorem 6.1. □

We now show that for every integer $n \geqslant 3$ and integer k with $2 \leqslant k \leqslant n$, there exists a connected graph G of order n containing k vertices v with $t(v) = n-1$ such that $\mathrm{hcon}(G) = n(n-1) + (n-k)$.

Proposition 6.3. *For every integer $n \geqslant 3$ and integer k with $2 \leqslant k \leqslant n$, there exists a connected graph of order n containing k vertices with traceable number $n-1$ and $n-k$ vertices with traceable number n.*

P r o o f. Since every Hamiltonian-connected graph has the desired properties for $k = n$, we restrict our attention to those integers k for which $2 \leqslant k \leqslant n-1$. For $3 \leqslant n \leqslant 5$, the graphs $G_{k,n}$ of Figure 8 have the desired properties.

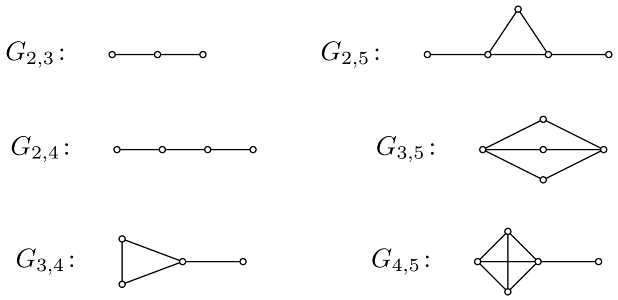

Figure 8. Graphs $G_{k,n}$ where $2 \leqslant k \leqslant n-1 = 4$

For $n \geqslant 6$, the graphs $G_{k,n}$ of Figure 9 have the appropriate properties. □

There is no graph of order n containing exactly one vertex with traceable number $n-1$. We know of no example of a nontrivial connected graph of order n, every vertex of which has traceable number n, that is, of a non-traceable graph G of order n for which $\mathrm{hcon}(G) = n^2$.

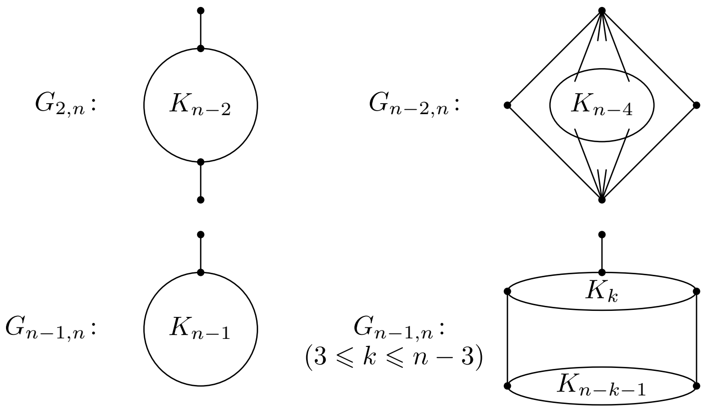

Figure 9. Graphs $G_{k,n}$ where $2 \leqslant k \leqslant n-1$ and $n \geqslant 6$

A c k n o w l e d g m e n t s. We are grateful to Professor Gary Chartrand for suggesting the concepts of traceable number and Hamiltonian-connected number to us and kindly providing useful information on this topic. Also, we are grateful to Professor Ladislav Nebeský whose valuable suggestions resulted in an improved paper.

References

[1] *T. Asano, T. Nishizeki, T. Watanabe*: An upper bound on the length of a Hamiltonian walk of a maximal planar graph. J. Graph Theory *4* (1980), 315–336. Zbl 0433.05037
[2] *T. Asano, T. Nishizeki, T. Watanabe*: An approximation algorithm for the Hamiltonian walk problem on maximal planar graphs. Discrete Appl. Math. *5* (1983), 211–222. Zbl 0511.05042
[3] *J. C. Bermond*: On Hamiltonian walks. Congr. Numerantium *15* (1976), 41–51. Zbl 0329.05113
[4] *G. Chartrand, T. Thomas, V. Saenpholphat, P. Zhang*: On the Hamiltonian number of a graph. Congr. Numerantium *165* (2003), 51–64. Zbl 1043.05041
[5] *G. Chartrand, T. Thomas, V. Saenpholphat, P. Zhang*: A new look at Hamiltonian walks. Bull. Inst. Combin. Appl. *42* (2004), 37–52. Zbl 1056.05093
[6] *G. Chartrand, P. Zhang*: Introduction to Graph Theory. McGraw-Hill, Boston, 2005. Zbl pre02141600
[7] *S. E. Goodman, S. T. Hedetniemi*: On Hamiltonian walks in graphs. Congr. Numerantium (1973), 335–342. Zbl 0321.05133
[8] *S. E. Goodman, S. T. Hedetniemi*: On Hamiltonian walks in graphs. SIAM J. Comput. *3* (1974), 214–221. Zbl 0269.05113
[9] *L. Nebeský*: A generalization of Hamiltonian cycles for trees. Czech. Math. J. *26* (1976), 596–603. Zbl 0365.05030
[10] *L. Lesniak*: Eccentric sequences in graphs. Period. Math. Hungar *6* (1975), 287–293. Zbl 0363.05053

[11] *P. Vacek*: On open Hamiltonian walks in graphs. Arch. Math., Brno *27A* (1991), 105–111. Zbl 0758.05067

Authors' addresses: *Varaporn Saenpholphat*, Department of Mathematics, Srinakharinwirot University, Sukhumvit Soi 23, Bangkok, 10110, Thailand; *Futaba Okamoto*, *Ping Zhang*, Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA, e-mail: ping.zhang@wmich.edu.