NON-SINGULAR COVERS OVER ORDERED MONOID RINGS

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Abstract. Let G be a multiplicative monoid. If RG is a non-singular ring such that the class of all non-singular RG-modules is a cover class, then the class of all non-singular R-modules is a cover class. These two conditions are equivalent whenever G is a well-ordered cancellative monoid such that for all elements $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$. For a totally ordered cancellative monoid the equalities $Z(RG) = Z(R)G$ and $\sigma(RG) = \sigma(R)G$ hold, σ being Goldie's torsion theory.

Keywords: hereditary torsion theory, torsion theory of finite type, Goldie's torsion theory, non-singular module, non-singular ring, monoid ring, precover class, cover class

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In what follows, R stands for an associative ring with the identity element and R-mod denotes the category of all unitary left R-modules. If G is a multiplicative monoid with the unit e, then RG will denote the monoid ring over R consisting of all elements of the form $\sum\limits_{i=1}^{n} r_i g_i$ with $r_i \in R$, $g_i \in G$, $i = 1, \ldots, n$, where the addition is given naturally and the multiplication is given by $\Big(\sum\limits_{i=1}^{n} r_i g_i\Big)\Big(\sum\limits_{j=1}^{m} s_j h_j\Big) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} r_i s_j g_i h_j$. Recall, that a monoid G is called *left cancellative* if for any three elements $h, g_1, g_2 \in G$ the equality $hg_1 = hg_2$ implies that $g_1 = g_2$. The *right cancellative* monoid is defined similarly and G is called *cancellative* if it is both left and right cancellative. The basic properties of rings and modules can be found in [1].

A class $\mathscr{G}$ of modules is called *abstract*, if it is closed under isomorphic copies. Recall that a *hereditary torsion theory* $\tau_R = (\mathscr{T}_\tau, \mathscr{F}_\tau)$, or simply $\tau = (\mathscr{T}, \mathscr{F})$, for

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the category R-mod consists of two abstract classes $\mathscr{T}$ and $\mathscr{F}$, the *τ-torsion class* and the *τ-torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$, the class $\mathscr{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathscr{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists a short exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$. It is easy to see that every module M contains the unique largest τ-torsion submodule (isomorphic to T), which is called the *τ-torsion part of the module M* and it is usually denoted by $\tau(M)$. A submodule K of a module M is *τ-dense* in M if the factor-module M/K is τ-torsion. Associated to each hereditary torsion theory τ is the *Gabriel filter $\mathscr{L}_\tau$* (or simply $\mathscr{L}$) of left ideals of R consisting of all the left ideals $I \leqslant R$ such that $R/I \in \mathscr{T}$. Recall that τ is said to be of *finite type*, if $\mathscr{L}$ contains a cofinal subset of finitely generated left ideals, i.e. if every element of $\mathscr{L}$ contains a finitely generated left ideal of R lying in $\mathscr{L}$.

For a module M, a submodule K is called *essential* in M, $K \leqslant' M$ in short, if $K \cap L \neq 0$ for each non-zero submodule L of M and the *singular submodule* $Z(M)$ consists of all elements $a \in M$, the *annihilator left ideal* $(0 : a)_R = \{r \in R;\ ra = 0\}$, or simply $(0 : a)$, of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (\mathscr{T}, \mathscr{F})$, where $\mathscr{T} = \{M \in R\text{-mod};\ Z(M/Z(M)) = M/Z(M)\}$ and $\mathscr{F} = \{M \in R\text{-mod};\ Z(M) = 0\}$. Note, that throughout this paper the letter σ will always denote Goldie's torsion theory and that the modules from the class $\mathscr{F}_\sigma$ are usually called *non-singular modules*. A ring R is said to be (*left*) *non-singular* if it is non-singular as a left R-module. For more details on torsion theories we refer to [10] or [9].

If $\mathscr{G}$ is an abstract class of modules, then a homomorphism $\varphi\colon G \to M$ with $G \in \mathscr{G}$ is called a *$\mathscr{G}$-precover* of the module M, if for each homomorphism $f\colon F \to M$ with $F \in \mathscr{G}$ there exists a homomorphism $g\colon F \to G$ such that $\varphi g = f$. A $\mathscr{G}$-precover φ of M is said to be a *$\mathscr{G}$-cover*, if every endomorphism f of G with $\varphi f = \varphi$ is an automorphism of the module G. An abstract class $\mathscr{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathscr{G}$-precover ($\mathscr{G}$-cover). A more detailed study of precovers and covers can be found in [14].

Recently, in [4; Corollary 3], it has been proved that for each hereditary torsion theory τ with $\tau \geqslant \sigma$ in the usual sense that $\mathscr{T}_\sigma \subseteq \mathscr{T}_\tau$ the class of all τ-torsionfree modules is a precover class if and only if it is a cover class and these conditions are satisfied exactly when the torsion theory τ is of finite type. Moreover, one of the main results in [5] states that these conditions are equivalent for Goldie's torsion theory for all members of the countable set $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\sigma(R[x_1, \dots, x_n]), n < \omega\}$ of rings whenever they are equivalent for an arbitrary member of this set.

The purpose of this note is to study some relations between the class of non-singular modules in the category R-mod and that in the category RG-mod, G being a multiplicative monoid. Especially, in Corollary 15 we shall obtain a direct generalization of [5; Theorem 16] dealing with the polynomial rings. The main result may be concentrate in the following Theorem.

Theorem. *Let G be a monoid and let R be an arbitrary ring. Then*
(i) *if RG is a non-singular ring and the class of all non-singular RG-modules is a cover class, then the class of all non-singular R-modules is a cover class;*
(ii) *if G is a totally ordered and cancellative monoid, then the equalities $Z(RG) = Z(R)G$ and $\sigma(RG) = \sigma(R)G$ hold;*
(iii) *if G is a well-ordered cancellative monoid such that for all elements $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$, then the class of all non-singular R-modules is a cover class if and only if the class of all non-singular RG-modules is a cover class.*

P r o o f. With respect to [4; Corollary 3] it suffices to apply the following Theorems 8, 11 and 14, respectively. □

Now we are ready to start our investigations.

Lemma 1. *If every essential left ideal of the ring R contains a σ-dense finitely generated left ideal, then every left ideal of R contains a σ-dense finitely generated left ideal.*

P r o o f. Let $0 \neq I \leqslant R$ be an arbitrary non-essential left ideal of the ring R and let $J \leqslant R$ be a left ideal of R maximal with respect to $I \cap J = 0$. Then $I \oplus J$ is essential in R and consequently the hypothesis yields the existence of a finitely generated left ideal $K = \sum\limits_{i=1}^{n} Ra_i$ which is σ-dense in $I \oplus J$ and hence in R. Now $a_i = b_i + c_i$, $b_i \in I$, $c_i \in J$, $i = 1, \ldots, n$, and it remains to show that the left ideal $L = \sum\limits_{i=1}^{n} Rb_i$ is σ-dense in I. If $s \in I$ and $r \in (K : s)$ are arbitrary elements then $rs = \sum\limits_{i=1}^{n} r_i b_i + \sum\limits_{i=1}^{n} r_i c_i$ for suitable elements $r_1, \ldots, r_n, r \in R$ and consequently $rs = \sum\limits_{i=1}^{n} r_i b_i \in L$. Thus $(K : s) \subseteq (L : s)$ and so $(L : s) \in \mathscr{L}$, showing that L is σ-dense in I. □

Lemma 2. *The following conditions are equivalent for Goldie's torsion theory σ for the category R-mod:*
(i) *σ is of finite type;*

(ii) *every left ideal of R contains a σ-dense finitely generated left ideal;*
(iii) *every essential left ideal of R contains a σ-dense finitely generated left ideal;*
(iv) *every non-singular left ideal of R essentially contains a finitely generated left ideal.*

Proof. Obviously, (iii) follows from (ii) trivially, while the converse follows from Lemma 1. Further, if (i) holds then especially (iii) holds and (i) follows from (ii) trivially.

(ii) implies (iv). Let $I \leqslant R$ be a non-singular left ideal of R. By the hypothesis there is a finitely generated left ideal K of R which is σ-dense in I. Let $J \leqslant I$ be a left ideal maximal with respect to $K \cap J = 0$. Then $J \cong (J \oplus K)/K \in \mathscr{T} \cap \mathscr{F} = 0$ and so K is essential in I.

(iv) implies (iii). Let I be an essential left ideal of R and let $J \leqslant I$ be maximal with respect to $(I \cap \sigma(R)) \cap J = 0$. For $J = 0$ we see that $I \cap \sigma(R)$ and hence $\sigma(R)$ is essential in R and the assertion is trivial. In the opposite case J is a non-singular left ideal of R and consequently there is a finitely generated left ideal K of R which is essential in J. Summarizing we have $K \subseteq J \subseteq J \oplus (I \cap \sigma(R)) \subseteq I$ where all the inclusions are obviously σ-dense and we are through. $\square$

Lemma 3. *Let G be a monoid and let $0 \neq a \in R$ be an arbitrary element. Then $(0 : a)_{RG} = RG(0 : a)_R = (0 : a)_R G$.*

Proof. For the sake of simplicity we shall denote by I the left annihilator ideal $(0 : a)_R$ of a in R and by J the left annihilator ideal $(0 : a)_{RG}$ of a in RG. For any element $u = \sum_{i=1}^{n} r_i g_i \in RG$ and any $r \in I$ we have $ra = 0$, hence $0 = ura = \sum_{i=1}^{n} (r_i ra) g_i$, which proves the inclusion $RGI \subseteq J$. Conversely, let $u = \sum_{i=1}^{n} r_i g_i \in J$ be an arbitrary element. Then $0 = ua = \sum_{i=1}^{n} (r_i a) g_i$ yields $r_i a = 0$ and consequently $r_i \in I$ for each $i = 1, \ldots, n$. But this means that $u = \sum_{i=1}^{n} r_i g_i \in RGI$ and the proof is complete, the rest being obvious. $\square$

Lemma 4. *Let G be a monoid and let $I \leqslant R$ be an essential left ideal of the ring R. Then $J = IG = RGI$ is an essential left ideal of the ring RG. Especially, if the left annihilator ideal $(0 : a)_R$ of an element $0 \neq a \in R$ is essential in R, then the left annihilator ideal $(0 : a)_{RG}$ of a is essential in RG.*

Proof. Let $u = \sum_{i=1}^{n} r_i g_i$ be an arbitrary element of the ring RG with $r_i \neq 0$, $i = 1, \ldots, n$, which does not belong to J. If $r_1 \in I$ then we put $s_1 = 1$, while in the opposite case there is an element $s_1 \in R$ such that $0 \neq s_1 r_1 \in I$. Continuing by the induction let us assume that the elements $s_1, \ldots, s_m \in R$, $1 \leqslant m < n$, such that $s_m \ldots s_1 r_i \in I$ for all $i = 1, \ldots, m$, and such that at least one of these elements is non-zero, have been already constructed. If $s_m \ldots s_1 r_{m+1} \in I$ then we put $s_{m+1} = 1$ and we shall find $s_{m+1} \in R$ such that $0 \neq s_{m+1} s_m \ldots s_1 r_{m+1} \in I$ in the opposite case. It is clear now, that after n steps we obtain a non-zero multiple ru which lies in J. The special statement now immediately follows from Lemma 3. □

Lemma 5. *Let G be a monoid. If I is a left ideal of the ring R such that the left ideal $J = RGI$ is essential in RG, then I is essential in R.*

P r o o f. Let $0 \neq r \in R$ be an arbitrary element. Then $r = re \in RG$ and consequently there is an element $u = \sum_{i=1}^{n} r_i g_i \in RG$ such that $0 \neq ur \in J$. Thus there is a non-zero coefficient $r_i r$ of ur, which obviously lies in I and the proof is complete. □

Proposition 6. *If G is a monoid, then the inclusions $Z(R)G \subseteq Z(RG)$ and $\sigma(R)G \subseteq \sigma(RG)$ hold. Especially, if the ring RG is non-singular, then so is R.*

P r o o f. If $u = \sum_{i=1}^{n} r_i g_i$ is an element of $Z(R)G$, then $(0 : r_i)$ is essential in R for each $i = 1, \ldots, n$ and consequently the intersection $I = \bigcap_{i=1}^{n} (0 : r_i)$ is essential in R. By Lemma 4 the left ideal IG is essential in RG and the obvious inclusion $IG \subseteq (0 : u)$ yields that $u \in Z(RG)$, as we wished to show. So, let $u = \sum_{i=1}^{n} r_i g_i \in \sigma(R)G$ be arbitrary. Then $(Z(R) : r_i) \leqslant' R$ for each $i = 1, \ldots, n$ and consequently $I = \bigcap_{i=1}^{n} (Z(R) : r_i)$ is essential in R. By Lemma 4 the left ideal IG is essential in RG. For an arbitrary element $v = \sum_{j=1}^{m} s_j h_j \in IG$ we have $s_j \in I$ and consequently $s_j r_i \in Z(R)$ for all relevant indices i and j. Thus $vu \in Z(R)G$ and so $v \in (Z(R)G : u)$. This means that $IG \subseteq (Z(R)G : u) \subseteq (Z(RG) : u)$, consequently $u \in \sigma(RG)$ and the inclusion $\sigma(R)G \subseteq \sigma(RG)$ is verified. The rest is now clear. □

Lemma 7. *Let G be a monoid. If I is a left ideal of the ring R such that the left ideal $J = RGI$ essentially contains a finitely generated left ideal of RG, then I essentially contains a finitely generated left ideal of the ring R.*

P r o o f. By the hypothesis the left ideal J contains a finitely generated left ideal $K = \sum_{i=1}^{n} RGu_i$, which is essential in J. So, we can write $u_i = \sum_{j=1}^{m} r_{ij} g_j$ for each

$i = 1, \ldots, n$, where some coefficients may be zero. Now we put $L = \sum_{i=1}^{n} \sum_{j=1}^{m} Rr_{ij}$ and we are going to verify that L is essential in I. If $r \in I \setminus L$ is an arbitrary element, then especially $r \in J = RGI$ and consequently $0 \neq ur \in K$ for some element $u \in RG$. Thus $ur = \sum_{i=1}^{n} v_i u_i$ for suitable elements $v_i \in RG$, $i = 1, \ldots, n$. Now it is clear that any non-zero coefficient of ur is a left multiple of r and it lies in L. $\square$

Theorem 8. *Let G be a monoid and let RG be a non-singular ring. If Goldie's torsion theory for the category RG-*mod *is of finite type, then Goldie's torsion theory for the category R-*mod *is of finite type, too.*

P r o o f. Since R is non-singular by Proposition 6, the Gabriel filter of Goldie's torsion theory for the category R-mod consists of essential left ideals, only. So, let I be an arbitrary essential left ideal of the ring R. Then the left ideal $J = RGI$ is essential in RG by Lemma 4 and consequently it essentially contains a finitely generated left ideal of the ring RG by the hypothesis. An application of Lemma 7 yields the existence of a finitely generated left ideal of R, which is essential in I and the proof is therefore complete. $\square$

N o t a t i o n. Let G be a totally ordered monoid. If J is a left ideal of the ring RG then we denote by $J[g]$ the set of all coefficients at the element $g \in G$ of all elements $u \in J$ of the form $u = \sum_{i=1}^{n} r_i g_i$, where $g = g_1 > \ldots > g_n$. Note, that this notation is the same as that in [5] which works with "leading" coefficients of the polynomials.

Lemma 9. *Let G be a totally ordered monoid, let J be a left ideal of the ring RG and let $g, h \in G$ be arbitrary elements. Then $J[g]$ is a left ideal of R and if G satisfies the left cancellation law, then $J[g] \subseteq J[hg]$.*

P r o o f. If $a, b \in J[g]$ and $r \in R$ are arbitrary elements, then there are elements $u = ag + \sum_{i=1}^{n} a_i g_i$ and $v = bg + \sum_{j=1}^{m} b_j h_j$ from J such that $g > g_1 > \ldots > g_n$ and $g > h_1 > \ldots > h_m$. So, $u - v = (a - b)g + \sum_{i=1}^{n} a_i g_i - \sum_{j=1}^{m} b_j h_j \in J$, $ru = rag + \sum_{i=1}^{n} ra_i g_i \in J$ and consequently $a - b, ra \in J[g]$, showing that $J[g]$ is a left ideal of R. Further, $hu = ahg + \sum_{i=1}^{n} a_i hg_i$, which yields that $a \in J[hg]$ in view of the fact that $hg > hg_1 > \ldots > hg_n$ by the left cancellation law for the monoid G. $\square$

Lemma 10. *Let G be a totally ordered cancellative monoid and let $u = \sum_{k=1}^{n} r_k g_k$ be a non-zero element of the ring RG such that $g_1 > \ldots > g_n$ and $r_k \neq 0$ for each $k = 1, \ldots, n$. If K is a left ideal of the ring R such that the left ideal $J = (RGK : u)$ is essential in RG, then the left ideal $I = (K : r_1)$ is essential in R.*

Proof. Proving indirectly let us suppose that there exists a non-zero left ideal L of R such that $L \cap I = 0$. Now RGL is a non-zero left ideal of RG and we are going to show that $RGL \cap J = 0$. Assume, on the contrary, that $v = \sum_{l=1}^{m} s_l h_l$ is a non-zero element of $RGL \cap J$ such that $h_1 > \ldots > h_m$ and $s_l \neq 0$ for each $l = 1, \ldots, m$. Note that $h_j g_1 > h_j g_i$ by the left cancellation law and $h_1 g_i > h_j g_i$ by the right cancellation law. Thus $h_1 g_1 > h_j g_i$ for all $j = 1, \ldots, m$ and $i = 1, \ldots, n$, where at least one of the indices i, j is different from 1. Now $v \in J$ yields $vu = \sum_{k=1}^{n} \sum_{l=1}^{m} s_l r_k h_l g_k \in RGK$ and consequently $s_1 r_1 \in K$. On the other hand, $0 \neq s_1 \in L$ means that $s_1 \notin I$, hence $s_1 r_1 \notin K$, which is a contradiction finishing the proof. $\square$

Theorem 11. *If G is a totally ordered cancellative monoid, then the equalities $Z(RG) = Z(R)G$ and $\sigma(RG) = \sigma(R)G$ hold. Especially, a ring R is non-singular if and only if the ring RG is so.*

Proof. We start with the equality $Z(RG) = Z(R)G$. The inclusion $Z(R)G \subseteq Z(RG)$ holds by Proposition 6. In order to prove the converse let $u = \sum_{k=1}^{n} r_k g_k \in Z(RG)$ be an arbitrary non-zero element such that $g_1 > \ldots > g_n$ and $r_k \neq 0$ for each $k = 1, \ldots, n$. Then $(0 : u) \leqslant' RG$ and so $(0 : r_1) \leqslant' R$ by Lemma 10. Hence $r_1 \in Z(R)$ yields that $r_1 g_1 \in Z(R)G \subseteq Z(RG)$. Thus $u - r_1 g_1 \in Z(RG)$ and continuing by the induction we finally obtain that $u = \sum_{k=1}^{n} r_k g_k \in Z(R)G$, as we wished to show.

Now we are going to prove the second equality in the similar way. By Proposition 6 we know that $\sigma(R)G \subseteq \sigma(RG)$. Thus, let $0 \neq u = \sum_{k=1}^{n} r_k g_k$ be an arbitrary element of $\sigma(RG)$ such that $g_1 > \ldots > g_n$ and $r_k \neq 0$ for each $k = 1, \ldots, n$. Then $(Z(RG) : u)$ is essential in RG and so the left annihilator ideal $(Z(R) : r_1)$ is essential in R by Lemma 10 in view of the equality $Z(RG) = Z(R)G$ proved in the first part of the proof. Thus $r_1 \in \sigma(R)$ gives that $r_1 g_1 \in \sigma(R)G \subseteq \sigma(RG)$. From this we infer that $u - r_1 g_1 \in \sigma(RG)$ and we can proceed by the induction. Finally we obtain that $u = \sum_{k=1}^{n} r_k g_k \in \sigma(R)G$, as required. The rest is now obvious. $\square$

Corollary 12. *Let G be a totally ordered cancellative monoid and let $u = \sum_{k=1}^{n} r_k g_k$ be a non-zero element of the ring RG. If the left annihilator ideal $(0 : u)$ is essential in RG then the intersection $\bigcap_{k=1}^{n} (0 : r_k)$ is essential in R.*

Proof. Without loss of generality we may assume that $g_1 > \ldots > g_n$. In the proof of Theorem 11 we have shown that $(0 : r_1) \leqslant' R$ and that $u - r_1 g_1 \in Z(RG)$. Continuing by the induction we shall obtain that $(0 : r_k) \leqslant' R$ for each $k = 1, \ldots, n$, from which the assertion follows immediately. □

Lemma 13. *Let G be a well-ordered cancellative monoid such that for all $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$. If every essential left ideal of a non-singular ring R essentially contains a finitely generated left ideal, then every essential left ideal of the ring RG essentially contains a finitely generated left ideal.*

Proof. Let J be an essential left ideal of the ring RG. It follows immediately from Lemma 9 that the set $\{J[g];\ g \in G\}$ is ordered by the inclusion. Note, that G has not the largest element. Clearly, if $h \in G$ is such that $g \leqslant h$ for each $g \in G$, then for $g < h$ there is $l \in G$ with $lg = h$. But then the left cancellation law gives $h = lg < lh \leqslant h$, which is impossible. Let $g_0 \in G$ be the smallest element in the well-order on G. If $J[g_0]$ is not essential in $J[h]$ for each $h \in G$, then let $g_1 \in G$ be the first element of G such that $J[g_0]$ is not essential in $J[g_1]$. Continuing by the induction, after a finite number of steps we shall come to $J[g_k]$ which is essential in $J[h]$ for each $h \geqslant g_k$. Clearly, in the opposite case we shall construct an infinite sequence $g_0 < g_1 < \ldots$ of elements of G such that $J[g_i]$ is not essential in $J[g_{i+1}]$ for each $i < \omega$. Thus there is a non-zero left ideal $L_i \leqslant J[g_{i+1}]$ such that $J[g_i] \cap L_i = 0$ and so we obtain an infinite direct sum $\bigoplus_{i<\omega} L_i$ of σ-torsionfree left ideals of the ring R. Since Goldie's torsion theory σ for the category R-mod is of finite type by the hypothesis and Lemma 2, we shall come to a contradiction with [13; Theorem 2.1] stating that σ is of finite type if and only if R contains no infinite direct sum of σ-torsionfree left ideals. We have proved the existence of an element $g \in G$ such that the left ideal $J[g]$ is essential in R and consequently it essentially contains a finitely generated left ideal $K = \sum_{i=1}^{r} Ra_i$. Now for each a_i there is an element $u_i = a_i g + \sum_{j=1}^{s} b_{ij} g_j \in J$ such that $g > g_1 > \ldots > g_s$ and some coefficients may be zero. Now we put $L = \sum_{i=1}^{r} RGu_i$ and we are going to show that L is essential in J. So, let $u = \sum_{i=1}^{s} b_i h_i$ be an element of J such that $h_1 < \ldots < h_s$. If $h_s < g$ and if $l' \in G$ is such that $g = l' h_s$ then we can take $l' u$ instead of u and so we may assume that $h_s \geqslant g$. Then there is $l \in G$ such that $h_s = lg$ and taking an element $t \in (K : b_s) \setminus (0 : b_s)$ we have $0 \neq tb_s = \sum_{i=1}^{r} t_i a_i$ and consequently the coefficient of the element $tu - l \sum_{i=1}^{r} t_i u_i$ at h_s is equal to $tb_s - \sum_{i=1}^{r} t_i a_i = 0$. Thus we have shown that $tu + u_s = \sum_{j=1}^{m} c_j k_j$, where $k_1 < \ldots < k_m < h_s$, $u_s \in L$ and tu is non-zero. Now we can proceed by the induction. Since G is well-ordered, after a finite number of steps we have to come to zero, from which it immediately follows the existence of a non-zero multiple of u belonging to L and the proof is complete. $\square$

Theorem 14. *Let G be a well-ordered cancellative monoid such that for all elements $g, h \in G$ with $g < h$ there is $l \in G$ such that $lg = h$. Then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

P r o o f. By [5; Theorem 5] Goldie's torsion theory σ for the category R-mod is of finite type if and only if Goldie's torsion theory for the category $R/\sigma(R)$-mod is of finite type. By Theorem 11 we have $\sigma(RG) = \sigma(R)G$ from which we easily obtain the ring isomorphism $(R/\sigma(R))G \cong RG/\sigma(RG)$. So, if Goldie's torsion theory for the category RG-mod is of finite type, then so is that for the category $RG/\sigma(RG)$-mod and consequently that for the category $R/\sigma(R)$-mod by Theorem 8. Conversely, if Goldie's torsion theory for the category R-mod is of finite type then so is that for the category $R/\sigma(R)$-mod by [5; Theorem 5] and so that for the category $RG/\sigma(RG)$-mod by Lemma 13. Now it remains to use [5; Theorem 5] again. $\square$

Corollary 15. *If G is an infinite cyclic monoid then Goldie's torsion theory for the category R-*mod *is of finite type if and only if Goldie's torsion theory for the category RG-*mod *is of finite type.*

P r o o f. Obvious. $\square$

N o t e. It is clear that Theorem 14 holds once we replace the well-order on G into an inverse well-order on G, i.e. into a total order satisfying the maximum condition. Moreover, Corollary 15 says that Goldie's torsion theory for the category R-mod is of finite type if and only if Goldie's torsion theory for the category $R[x]$-mod is of finite type and consequently Theorem 14 generalizes [5; Theorem 16].

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