EXISTENCE AND MULTIPLICITY RESULTS FOR NONLINEAR SECOND ORDER DIFFERENCE EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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Cordially dedicated to Jaroslav Kurzweil for his 80th birthday anniversary

Abstract. We use Brouwer degree to prove existence and multiplicity results for the solutions of some nonlinear second order difference equations with Dirichlet boundary conditions. We obtain in particular upper and lower solutions theorems, Ambrosetti-Prodi type results, and sharp existence conditions for nonlinearities which are bounded from below or from above.

Keywords: nonlinear difference equations, Ambrosetti-Prodi problem, Brouwer degree

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1. Introduction

Some existence and multiplicity results for periodic solutions of first and second order nonlinear difference equations have been proved in [1]. They correspond to Ambrosetti-Prodi and Landesman-Lazer problems for differential equations. The purpose of this article is to show that the corresponding existence and multiplicity results for solutions of second order ordinary differential equations with Dirichlet boundary conditions also hold for second order difference equations.

In Section 2, we extend the classical method of upper and lower solutions to second order difference equations of the form

$$D^2x_m + f_m(x_m) = 0 \quad (1 \leqslant m \leqslant n-1)$$

with Dirichlet boundary conditions

$$x_0 = 0 = x_n. \tag{1}$$

The methodology of the proof is inspired by the one introduced in [12].

In Section 3, combining the method of upper and lower solutions and Brouwer degree theory, we prove an Ambrosetti-Prodi result for solutions of the Dirichlet problem for second order difference equations of the type

$$D^2 x_m + \lambda_1 x_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n,$$

where $\lambda_1 = 4\sin^2(\pi/2n)$ is the first eigenvalue of $-D^2$ with Dirichlet boundary conditions (1), $\varphi_m^1 = \sin(m\pi/n)$ $(1 \leqslant m \leqslant n-1)$ are the components of the corresponding eigenvector φ^1 and $f_m(x) \to +\infty$ or to $-\infty$ if $|x| \to \infty$. We adapt the ideas of [5] and [13]. One should notice that, in contrast to the results of [5] for the ordinary differential equation case with g continuous only, where a weaker Ambrosetti-Prodi conclusion was proved, we obtain here the classical Ambrosetti-Prodi statement.

In Section 4 we prove an existence result for solutions of the Dirichlet problem for second order difference equations having a nonlinearity bounded from below or from above. Brouwer degree theory is used, and in particular a special case of a result in [10]. Landesman-Lazer-type existence conditions are obtained, and an example shows that the assumptions are sharp.

For other multiplicity results for nonlinear second order difference equations using upper-lower solutions and/or degree theory, see for example [2], [3], [4], [7], [8], [9], [16], [17].

2. The method of upper and lower solutions for second order difference equations

Let $n \in \mathbb{N}$ fixed and $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$. Define $(Dx_0, \ldots, Dx_{n-1}) \in \mathbb{R}^n$ and $(D^2x_1, \ldots, D^2x_{n-1}) \in \mathbb{R}^{n-1}$ by

$$\begin{aligned} Dx_m &= x_{m+1} - x_m \quad (0 \leqslant m \leqslant n-1), \\ D^2x_m &= x_{m+1} - 2x_m + x_{m-1} \quad (1 \leqslant m \leqslant n-1). \end{aligned}$$

Let $f_m \colon \mathbb{R} \to \mathbb{R}$ $(1 \leqslant m \leqslant n-1)$ be continuous functions. We study the existence of solutions for the Dirichlet boundary value problem

$$D^2 x_m + f_m(x_m) = 0 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n. \tag{2}$$

If $\alpha, \beta \in \mathbb{R}^p$, we write $\alpha \leqslant \beta$ $(\alpha < \beta)$ if $\alpha_i \leqslant \beta_i$ for all $1 \leqslant i \leqslant p$ $(\alpha_i < \beta_i$ for all $1 \leqslant i \leqslant p)$.

Definition 1. $\alpha = (\alpha_0, \ldots, \alpha_n)$ ($\beta = (\beta_0, \ldots, \beta_n)$) is called a *lower solution* (*upper solution*) for (2) if

$$\alpha_0 \leqslant 0,\ \alpha_n \leqslant 0 \quad (\beta_0 \geqslant 0,\ \beta_n \geqslant 0) \tag{3}$$

and the inequalities

$$\begin{gathered} D^2\alpha_m + f_m(\alpha_m) \geqslant 0 \\ (D^2\beta_m + f_m(\beta_m) \leqslant 0,\ \text{respectively}) \quad (1 \leqslant m \leqslant n-1) \end{gathered} \tag{4}$$

hold. Such a lower or upper solution will be called *strict* if the inequalities (4) are strict.

The proof of the following result is modeled upon the one given in [12] for the case of second order ordinary differential equations.

Theorem 1. *If* (2) *has a lower solution* $\alpha = (\alpha_0, \ldots, \alpha_n)$ *and an upper solution* $\beta = (\beta_0, \ldots, \beta_n)$ *such that* $\alpha \leqslant \beta$, *then* (2) *has a solution* $x = (x_0, \ldots, x_n)$ *such that* $\alpha \leqslant x \leqslant \beta$. *Moreover, if* α *and* β *are strict, then* $\alpha_m < x_m < \beta_m$ $(1 \leqslant m \leqslant n-1)$.

Proof. I. A modified problem. Let $\gamma_m\colon \mathbb{R} \longrightarrow \mathbb{R}$ $(1 \leqslant m \leqslant n-1)$ be continuous functions defined by

$$\gamma_m(x) = \begin{cases} \beta_m, & x > \beta_m, \\ x, & \alpha_m \leqslant x \leqslant \beta_m, \\ \alpha_m, & x < \alpha_m, \end{cases}$$

and define $F_m = f_m \circ \gamma_m$ $(1 \leqslant m \leqslant n-1)$. We consider the modified problem

$$D^2x_m + F_m(x_m) - [x_m - \gamma_m(x_m)] = 0 \quad (1 \leqslant m \leqslant n-1),\ x_0 = 0 = x_n, \tag{5}$$

and show that if $x = (x_0, \ldots, x_n)$ is a solution of (5) then $\alpha \leqslant x \leqslant \beta$ and hence x is a solution of (2). Suppose by contradiction that there is some i, $0 \leqslant i \leqslant n$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max\limits_{0 \leqslant j \leqslant n} (\alpha_j - x_j) > 0$. Using the inequalities (3), we obtain that $1 \leqslant m \leqslant n-1$. Hence

$$D^2(\alpha_m - x_m) = (\alpha_{m+1} - x_{m+1}) - 2(\alpha_m - x_m) + (\alpha_{m-1} - x_{m-1}) \leqslant 0,$$

and

$$\begin{aligned} D^2\alpha_m \leqslant D^2x_m &= -F_m(x_m) + (x_m - \gamma_m(x_m)) \\ &= -f_m(\alpha_m) + (x_m - \alpha_m) < -f_m(\alpha_m) \leqslant D^2\alpha_m, \end{aligned}$$

a contradiction. Analogously we can show that $x \leqslant \beta$. We remark that if α, β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leqslant m \leqslant n-1)$.

II. Abstract formulation of problem (5). Let us introduce the vector space

$$V^{n-1} = \{x \in \mathbb{R}^{n+1} \colon \; x_0 = 0 = x_n\} \tag{6}$$

endowed with the orientation of $\mathbb{R}^{n+1}$. Its elements can be associated with the coordinates $(x_1, \dots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^{n+1}$ of the form $(0, x_1, \dots, x_{n-1}, 0)$, so that the restriction D^2 to V^{n-1} is well defined in terms of $(x_1, \dots, x_{n-1})$. We use the norm $\|x\| := \max\limits_{1 \leqslant j \leqslant n-1} |x_j|$ in V^{n-1} and $\max\limits_{1 \leqslant j \leqslant n-1} |x_j|$ in $\mathbb{R}^{n-1}$. We define a continuous mapping $G \colon \; V^{n-1} \to \mathbb{R}^{n-1}$ by

$$G_m(x) = D^2 x_m + F_m(x_m) - [x_m - \gamma_m(x_m)] \quad (1 \leqslant m \leqslant n-1). \tag{7}$$

It is clear that the solutions of (5) are the zeros of G in V^{n-1}. In order to use the Brouwer degree [6], [14] to study those zeros, we introduce the homotopy $\mathcal{G} \colon$ $[0,1] \times V^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$\begin{aligned} \mathcal{G}_m(\lambda, x) &= (1-\lambda)(D^2 x_m - x_m) + \lambda G_m(x) \\ &= D^2 x_m - x_m + \lambda [F_m(x_m) + \gamma_m(x_m)] \quad (1 \leqslant m \leqslant n-1). \end{aligned} \tag{8}$$

Notice that $\mathcal{G}(1, \cdot) = G$ and that $\mathcal{G}(0, \cdot)$ is linear.

III. A priori estimates for possible zeros of $\mathcal{G}$. Let R be any number such that

$$R > \max_{1 \leqslant m \leqslant n-1} \max_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)| \tag{9}$$

and let $(\lambda, x_1, \dots, x_{n-1}) \in [0,1] \times V^{n-1}$ be a possible zero of $\mathcal{G}$. If $0 \leqslant x_m = \max\limits_{1 \leqslant j \leqslant n-1} x_j$, then $D^2 x_m \leqslant 0$. This is clear if $2 \leqslant m \leqslant n-2$ and if, say, $m = 1$, then

$$D^2 x_1 = x_2 - 2x_1 = x_2 - x_1 - x_1 \leqslant 0,$$

and similarly if $m = n-1$. Hence,

$$0 \geqslant D^2 x_m = x_m - \lambda [F_m(x_m) + \gamma_m(x_m)],$$

which implies

$$x_m \leqslant \max_{x \in \mathbb{R}} |F_m(x) + \gamma_m(x)| < R.$$

Analogously it can be shown that $-R < \min_{1 \leqslant j \leqslant n-1} x_j$, and hence

$$\|x\| = \max_{1 \leqslant j \leqslant n-1} |x_j| < R \tag{10}$$

for each possible zero (λ, x) of $\mathcal{G}$.

IV. The existence of a zero for G. Using the results of Parts II, III and the invariance under homotopy of the Brouwer degree, we see that the Brouwer degree $d[\mathcal{G}(\lambda, \cdot), B_R(0), 0]$ is well defined and independent of $\lambda \in [0, 1]$. But $\mathcal{G}(0, \cdot)$ is a linear mapping whose set of solutions is bounded, and hence equal to $\{0\}$. Consequently, $|d[\mathcal{G}(0, \cdot), B_R(0), 0]| = 1$, so that $|d[G, B_R(0), 0]| = 1$ and the existence property of the Brouwer degree implies the existence of at least one zero of G.

V. End of the proof. We have proved that there is some $x \in V^{n-1}$ such that $G(x) = 0$, so x is a solution of (4), which means that $\alpha \leqslant x \leqslant \beta$ and x is a solution of (2). Moreover, if α, β are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leqslant m \leqslant n-1)$. $\square$

Remark 1. Suppose that α, β are respectively strict lower and upper solutions of (2). As we have already seen, (2) admits at least one solution x such that $\alpha_m < x_m < \beta_m$ $(1 \leqslant m \leqslant n-1)$. Define an open set

$$\Omega_{\alpha,\beta} = \{(x_1, \ldots, x_{n-1}) \in V^{n-1}\colon\ \alpha_m < x_m < \beta_m\ (1 \leqslant m \leqslant n-1)\}.$$

If ϱ is large enough, then, using the additivity-excision property of the Brouwer degree, we have

$$|d[G, \Omega_{\alpha,\beta}, 0]| = |d[G, B_\varrho(0), 0]| = 1.$$

On the other hand, if we define a continuous mapping $\widetilde{G}\colon\ V^{n-1} \to \mathbb{R}^{n-1}$ by

$$\widetilde{G}_m(x) = D^2 x_m + f_m(x_m) \quad (1 \leqslant m \leqslant n-1), \tag{11}$$

$\widetilde{G}$ is equal to G on $\Omega_{\alpha,\beta}$, and then

$$|d[\widetilde{G}, \Omega_{\alpha,\beta}, 0]| = 1. \tag{12}$$

3. Some results on the Dirichlet problem for second order linear difference equations

Consider the Dirichlet eigenvalue problem

$$D^2 x_m + \lambda x_m = 0 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n. \tag{13}$$

The following results are classical, but we reproduce them for completion. If we look for a nontrivial solution of the form (for some $\theta \in \mathbb{R}$)

$$x_m = A \sin m\theta \quad (0 \leqslant m \leqslant n), \tag{14}$$

then one must have

$$\sin(m-1)\theta + (\lambda - 2)\sin m\theta + \sin(m+1)\theta = 0 \quad (1 \leqslant m \leqslant n-1)$$

or, equivalently,

$$(\sin m\theta)[2\cos\theta + \lambda - 2] = 0 \quad (1 \leqslant m \leqslant n-1).$$

This system of equations is satisfied if we choose

$$\lambda = 2 - 2\cos\theta \tag{15}$$

and the Dirichlet boundary conditions $x_0 = 0 = x_m$ hold if and only if $\sin n\theta = 0$, i.e. if and only if

$$\theta = \theta_k := \frac{k\pi}{n} \quad (k = 1, 2, \ldots, n-1).$$

Consequently, the eigenvalues of (13) (in the increasing order) are

$$\lambda_k = 2\Big(1 - \cos\frac{k\pi}{n}\Big) = 4\sin^2\frac{k\pi}{2n} \quad (k = 1, 2, \ldots, n-1) \tag{16}$$

and a corresponding eigenvector $\varphi^k = \big(\varphi_1^k, \ldots, \varphi_{n-1}^k\big)$ is given by

$$\varphi^k = \Big(\sin\frac{k\pi}{n}, \sin\frac{2k\pi}{n}, \ldots, \sin\frac{(n-1)k\pi}{n}\Big) \quad (k = 1, 2, \ldots, n-1). \tag{17}$$

In particular, the eigenvector φ^1 associated with the first eigenvalue

$$\lambda_1 = 2\Big(1 - \cos\frac{\pi}{n}\Big) = 4\sin^2\frac{\pi}{2n}, \tag{18}$$

given by

$$\varphi^1 = \left(\sin\frac{\pi}{n}, \sin\frac{2\pi}{n}, \dots, \sin\frac{(n-1)\pi}{n}\right), \tag{19}$$

has all its components positive. Furthermore, as the φ^k constitute a system of eigenvectors of a symmetric matrix, they satisfy the orthogonality conditions $\langle\varphi^j, \varphi^k\rangle = 0$ for $j \neq k$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^{n-1}$.

We need some identities and inequalities for finite sequences satisfying the Dirichlet boundary conditions. The first is a type of summation by parts.

Lemma 1. *If $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ and $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ are such that $x_0 = 0 = x_n$ and $y_0 = 0 = y_n$, the identity*

$$\sum_{m=1}^{n-1} x_m D^2 y_m = \sum_{m=1}^{n-1} y_m D^2 x_m \tag{20}$$

holds.

Proof. We have

$$\begin{aligned}\sum_{m=1}^{n-1} & x_m(y_{m+1} - 2y_m + y_{m-1}) - \sum_{m=1}^{n-1} y_m(x_{m+1} - 2x_m + x_{m-1}) \\ &= \sum_{m=1}^{n-2} x_m y_{m+1} + \sum_{m=2}^{n-1} x_m y_{m-1} - \sum_{m=1}^{n-2} y_m x_{m+1} - \sum_{m=2}^{n-1} y_m x_{m-1} = 0.\end{aligned}$$

□

Define

$$x = (x_1, \dots, x_{n-1}), \quad \bar{x} = \langle x, \varphi^1\rangle \frac{\varphi^1}{\|\varphi^1\|^2}, \quad \tilde{x} = x - \bar{x}, \tag{21}$$

so that

$$(\bar{x})_m = \left(\sum_{m=1}^{n-1} x_m \sin\frac{m\pi}{n}\right)\frac{\varphi^1_m}{\|\varphi^1\|^2} \quad (1 \leqslant m \leqslant n-1), \quad \langle\tilde{x}, \varphi^1\rangle = 0.$$

Notice that

$$\begin{aligned}D^2 x_m + \lambda_1 x_m &= D^2(\bar{x})_m + \lambda_1(\bar{x})_m + D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m \\ &= D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m.\end{aligned} \tag{22}$$

Lemma 2. *If $x = (x_1, \dots, x_{n-1})$, then there exists a constant $c_n > 0$ which depends only on n such that*

$$\max_{1 \leqslant m \leqslant n-1} |(\tilde{x})_m| \leqslant c_n \sum_{m=1}^{n-1} |D^2(\tilde{x})_m + \lambda_1 (\tilde{x})_m| \varphi_m^1.$$

Proof. The applications

$$(x_1, \dots, x_{n-1}) \mapsto \max_{1 \leqslant m \leqslant n-1} |x_m|, \quad (x_0, \dots, x_n) \mapsto \sum_{m=1}^{n-1} |D^2 x_m + \lambda_1 x_m| \varphi_m^1$$

define two norms on the subspace $V = \{x \in \mathbb{R}^{n-1} \colon \langle x, \varphi^1 \rangle = 0\}$. They are equivalent, and the inequality above holds. $\square$

4. Ambrosetti-Prodi type results for second order difference equations

In this section we are interested in problems of the type

$$D^2 x_m + \lambda_1 x_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n, \tag{23}$$

where $n \geqslant 2$ is fixed, $f_1, \dots, f_{n-1} \colon \mathbb{R} \to \mathbb{R}$ are continuous, $s \in \mathbb{R}$, λ_1 is defined in (18), φ^1 is defined in (19) and

$$f_m(x) \to \infty \text{ as } |x| \to \infty \quad (1 \leqslant m \leqslant n-1). \tag{24}$$

We prove an Ambrosetti-Prodi type result for (23), which is reminiscent of the multiplicity theorem for second order differential equations with Dirichlet boundary conditions proved in [5].

The next lemma provides a priori bounds for possible solutions of (23).

Lemma 3. *Let $a, b \in \mathbb{R}$. Then there is $\varrho > 0$ such that any possible solution x of (23) with $s \in [a, b]$ belongs to the open ball $B_\varrho(0)$.*

Proof. Let $s \in [a, b]$ and let $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ be a solution of (23). Multiplying each equation by φ_m^1 and adding, we obtain

$$s\Big[\sum_{m=1}^{n-1} (\varphi_m^1)^2\Big] = \sum_{m=1}^{n-1} [\varphi_m^1 D^2 x_m + \lambda_1 \varphi_m^1 x_m] + \sum_{m=1}^{n-1} \varphi_m^1 f_m(x_m).$$

But, using Lemma 1,

$$\sum_{m=1}^{n-1}\left[\varphi_m^1 D^2 x_m + \lambda_1 \varphi_m^1 x_m\right] = \sum_{m=1}^{n-1}\left[x_m\left(D^2\varphi_m^1 + \lambda_1\varphi_m^1\right)\right] = 0,$$

so that

$$\sum_{m=1}^{n-1}\varphi_m^1 f_m(x_m) = s\|\varphi^1\|^2. \tag{25}$$

Using (24) we deduce that there exists a constant $\alpha > 0$ such that

$$|f_m(x)| \leqslant f_m(x) + \alpha \quad (1 \leqslant m \leqslant n-1). \tag{26}$$

Using the equation (23), written in the equivalent form

$$(D^2\tilde{x})_m + \lambda_1(\tilde{x})_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leqslant m \leqslant n-1), \tag{27}$$

and the relations (25), (26) we have

$$\sum_{m=1}^{n-1}|D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m|\varphi_m^1 = \sum_{m=1}^{n-1}|s\varphi_m^1 - f_m(x_m)|\varphi_m^1$$
$$\leqslant |s|\|\varphi^1\|^2 + \sum_{m=1}^{n-1}|f_m(x_m)|\varphi_m^1 \leqslant 2|s|\|\varphi^1\|^2 + \alpha\sum_{m=1}^{n-1}\varphi_m^1,$$

which implies that there exists a constant R_1 depending only on a, b, n such that

$$\sum_{m=1}^{n-1}|D^2(\tilde{x})_m + \lambda_1(\tilde{x})_m|\varphi_m^1 \leqslant R_1. \tag{28}$$

Using the relations (27), (28) and Lemma 2 , we obtain $R_2 > 0$ such that

$$|f_m(x_m)| \leqslant R_2 \quad (1 \leqslant m \leqslant n-1).$$

Hence the assumption (24) implies the existence of $R_3 > 0$ such that $|x_m| < R_3$ for all $1 \leqslant m \leqslant n-1$. □

Theorem 2. *If the functions f_m $(1 \leqslant m \leqslant n-1)$ satisfy* (24), *then there is $s_1 \in \mathbb{R}$ such that* (23) *has no, at least one or at least two solutions provided $s < s_1$, $s = s_1$ or $s > s_1$, respectively.*

Proof. Let

$$S_j = \{s \in \mathbb{R}\colon\ (23) \text{ has at least } j \text{ solutions}\} \quad (j \geqslant 1).$$

(a) $S_1 \neq \emptyset$.

Take $s^* > \max\limits_{1 \leqslant m \leqslant n-1} (f_m(0)/\varphi_m^1)$ and use (24) to find $R_-^* < 0$ such that

$$f_m(R_-^* \varphi_m^1) > s^* \varphi_m^1 \quad (1 \leqslant m \leqslant n-1).$$

Then α with $\alpha_0 = 0 = \alpha_n$ and $\alpha_j = R_-^* \varphi_j^1 < 0$ $(1 \leqslant j \leqslant n-1)$ is a strict lower solution and β with $\beta_j = 0$ $(1 \leqslant j \leqslant n)$ is a strict upper solution for (23) with $s = s^*$. Hence, using Theorem 1, $s^* \in S_1$.

(b) If $\tilde{s} \in S_1$ and $s > \tilde{s}$ then $s \in S_1$.

Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ be a solution of (23) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{x}$ is a strict upper solution for (23). Take now $R_- < \min\limits_{1 \leqslant m \leqslant n} (\tilde{x}_m/\varphi_m^1)$ such that $f_m(R_- \varphi_m^1) > s\varphi_m^1$ $(1 \leqslant m \leqslant n-1)$. It follows that α with $\alpha_0 = 0 = \alpha_n$ and $\alpha_j = R_- \varphi_j^1$ $(1 \leqslant j \leqslant n)$ is a strict lower solution for (23), and hence, using Theorem 1, $s \in S_1$.

(c) $s_1 = \inf S_1$ is finite and $S_1 \supset \,]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (23) has a solution $(x_1, \ldots, x_n)$. Then (25) holds, where from we deduce that $s \geqslant c := \sum\limits_{m=1}^{n-1} \varphi_m^1 \min\limits_{\mathbb{R}} f_m$. To obtain the second part of claim (c) $S_1 \supset \,]s_1, \infty[$ we apply (b).

(d) $S_2 \supset \,]s_1, \infty[$.

We reformulate (23) to apply Brouwer degree theory. Consider the space V^{n-1} defined in (6) and the continuous mapping $\mathcal{G}\colon\ \mathbb{R} \times V^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$\mathcal{G}_m(s, x) = D^2 x_m + \lambda_1 x_m + f_m(x_m) - s\varphi_m^1 \quad (1 \leqslant m \leqslant n-1).$$

Then $(x_0, \ldots, x_n)$ is a solution of (23) if and only if $(x_1, \ldots, x_{n-1}) \in V^{n-1}$ is a zero of $\mathcal{G}(s, \cdot)$. Let $s_3 < s_1 < s_2$. Using Lemma 3 we find $\varrho > 0$ such that each possible zero of $\mathcal{G}(s, \cdot)$ with $s \in [s_3, s_2]$ is such that $\max\limits_{1 \leqslant m \leqslant n-1} |x_m| < \varrho$. Consequently, the Brouwer degree $d[\mathcal{G}(s, \cdot), B_\varrho(0), 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $\mathcal{G}(s_3, x) \neq 0$ for all $x \in V^{n-1}$. This implies that $d[\mathcal{G}(s_3, \cdot), B_\varrho(0), 0] = 0$, so that $d[\mathcal{G}(s_2, \cdot), B_\varrho(0), 0] = 0$ and, by the excision property, $d[\mathcal{G}(s_2, \cdot), B_{\varrho'}(0), 0] = 0$ if $\varrho' > \varrho$. Let $s \in \,]s_1, s_2[$ and let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$

be a solution of (23) (using (c)). Then $\hat{x}$ is a strict upper solution of (23) with $s = s_2$. Let $R < \min\limits_{1 \leqslant m \leqslant n} (\hat{x}_m/\varphi^1_m)$ be such that $f_m(R\varphi^1_m) > s_2\varphi^1_m$ $(1 \leqslant m \leqslant n-1)$. Then $(0, R\varphi^1_1, \dots, R\varphi^1_{n-1}, 0) \in \mathbb{R}^{n+1}$ is a strict lower solution of (23) with $s = s_2$. Consequently, using Remark 1, (23) with $s = s_2$ has a solution in $\Omega_{R\varphi^1,\hat{x}}$ and

$$|d[\mathcal{G}(s_2,\cdot), \Omega_{R\varphi^1,\hat{x}}, 0]| = 1.$$

Taking ϱ' sufficiently large, we deduce from the additivity property of the Brouwer degree that

$$\begin{aligned} |d[\mathcal{G}(s_2,\cdot), B_{\varrho'}(0) \setminus \overline{\Omega}_{R\varphi^1,\hat{x}}, 0]| &= |d[\mathcal{G}(s_2,\cdot), B_{\varrho'}(0), 0] - d[\mathcal{G}(s_2,\cdot), \Omega_{R\varphi^1,\hat{x}}, 0]| \\ &= |d[\mathcal{G}(s_2,\cdot), \Omega_{R\varphi^1,\hat{x}}, 0]| = 1, \end{aligned}$$

and (23) with $s = s_2$ has a second solution in $B_{\varrho'}(0) \setminus \overline{\Omega}_{R\varphi^1,\hat{x}}$.

(e) $s_1 \in S_1$.

Taking a decreasing sequence $(\sigma_k)_{k\in\mathbb{N}}$ in $]s_1, \infty[$ converging to s_1, a corresponding sequence $(x^k_1, \dots, x^k_n)$ of solutions of (23) with $s = \sigma_k$ and using Lemma 3, we obtain a subsequence $(x^{j_k}_1, \dots, x^{j_k}_n)$ which converges to a solution $(x_1, \dots, x_n)$ of (23) with $s = s_1$. $\square$

Similar arguments allow to prove the following result.

Theorem 3. *If the functions f_m satisfy the condition*

$$f_m(x) \to -\infty \text{ as } |x| \to \infty \quad (1 \leqslant m \leqslant n-1), \tag{29}$$

then there is $s_1 \in \mathbb{R}$ such that (23) has no, at least one or at least two solutions provided $s > s_1, s = s_1$ or $s < s_1$, respectively.

Example 1. There exists $s_1 \in \mathbb{R}$ such that the problem

$$D^2 x_m + \lambda_1 x_m + |x_m|^{1/2} = s\varphi^1_m \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n$$

has no solution if $s < s_1$, at least one solution if $s = s_1$ and at least two solutions if $s > s_1$.

Example 2. There exists $s_1 \in \mathbb{R}$ such that the problem

$$D^2 x_m + \lambda_1 x_m - \exp x^2_m = s\varphi^1_m \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n$$

has no solution if $s > s_1$, at least one solution if $s = s_1$ and at least two solutions if $s < s_1$.

5. Second order difference equations with a nonlinearity bounded from below or from above

Let $n \in \mathbb{N}$ and let f_m be continuous functions $(1 \leqslant m \leqslant n-1)$. Consider the problem

$$D^2 x_m + \lambda_1 x_m + f_m(x_m) = 0 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n. \tag{30}$$

Using the notation of Section 2, we define a continuous mapping $G\colon V^{n-1} \to \mathbb{R}^{n-1}$ by

$$G_m(x) = D^2(x_m) + \lambda_1 x_m + f_m(x_m) \quad (1 \leqslant m \leqslant n-1), \tag{31}$$

so that $(0, x_1, \ldots, x_{n-1}, 0)$ is a solution of (30) if and only if $(x_1, \ldots, x_{n-1}) \in V^{n-1}$ is a zero of G. We also define $F\colon V^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(x) = (f_1(x_1), \ldots, f_{n-1}(x_{n-1})),$$

and call $L\colon V^{n-1} \to \mathbb{R}^{n-1}$ the restriction of $D^2 + \lambda_1 I$ to V^{n-1}. We have

$$\ker L = \{c\varphi^1 \colon c \in \mathbb{R}\}$$

and, by the properties of symmetric matrices,

$$\operatorname{Im} L = \{y \in \mathbb{R}^{n-1} \colon \langle y, \varphi^1 \rangle = 0\}.$$

Consider projectors $P\colon V^{n-1} \to V^{n-1}$ and $Q\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$P(x) = \langle x, \varphi^1 \rangle \frac{\varphi^1}{\|\varphi^1\|^2}, \quad Q(y) = \langle y, \varphi^1 \rangle \frac{\varphi^1}{\|\varphi^1\|^2},$$

so that $\ker Q = \operatorname{Im} L, \operatorname{Im} P = \ker L$.

In order to study the existence of zeros of G using the Brouwer degree, we introduce a homotopy $\mathcal{G}\colon [0,1] \times V^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$\mathcal{G}(\lambda, x) = (1-\lambda)(L + QF)(x) + \lambda G(x) = Lx + (1-\lambda)QF(x) + \lambda F(x),$$

which, for $\lambda = 1$, reduces to G. We first obtain a priori estimates for possible zeros of $\mathcal{G}$.

Lemma 4. *If each function f_m $(1 \leqslant m \leqslant n-1)$ is bounded from below or from above, and if, for some $R > 0$, one has*

$$\sum_{m=1}^{n-1} f_m(x_m)\varphi_m^1 \neq 0 \quad \textit{whenever} \quad \min_{1\leqslant j\leqslant n-1} x_j \geqslant R \quad \textit{or} \quad \max_{1\leqslant j\leqslant n-1} x_j \leqslant -R \tag{32}$$

then there exists $\varrho > R$ such that each possible zero (λ, x) of $\mathcal{G}$ is such that $\|x\| < \varrho$.

P r o o f. Assume first that each f_m is bounded from below. Let $(\lambda, x) \in [0,1] \times V^{n-1}$ be a possible zero of $\mathcal{G}$. Applying Q and $(I - Q)$ to the equation, we obtain

$$QF(x) = 0, \quad Lx + \lambda(I - Q)F(x) = 0$$

or, equivalently,

$$\sum_{m=1}^{n-1} f_m(x_m)\varphi_m^1 = 0, \tag{33}$$

$$(D^2\tilde{x})_m + \lambda_1(\tilde{x})_m + \lambda f_m(x_m) = 0 \quad (1 \leqslant m \leqslant n-1). \tag{34}$$

We can then repeat the reasoning of the proof of Lemma 3 to obtain that

$$\max_{1\leqslant m\leqslant n-1} |(\tilde{x})_m| \leqslant R_2.$$

Then, by (32) and (33), there exists $1 \leqslant k \leqslant n-1$ and $1 \leqslant l \leqslant n-1$ such that $x_k < R$ and $x_l > -R$. Consequently, $(\bar{x})_k = x_k - (\tilde{x})_k < R + R_2$ and $(\bar{x})_l = x_l - (\tilde{x})_k > -R - R_2$. Therefore, for each $1 \leqslant m \leqslant n-1$,

$$(\bar{x})_m = \frac{(\bar{x})_k}{\varphi_k^1}\varphi_m^1 < (R + R_2) \max_{1\leqslant m\leqslant n-1} \frac{\varphi_m^1}{\varphi_k^1} := R_3,$$

$$(\bar{x})_m = \frac{(\bar{x})_l}{\varphi_l^1}\varphi_m^1 > -(R + R_2) \max_{1\leqslant m\leqslant n-1} \frac{\varphi_m^1}{\varphi_k^1} := -R_3.$$

Consequently, $\|x\| < \varrho$ for some $\varrho > 0$.

In the case when the f_m are bounded from above, if suffices to write the problem

$$\widetilde{L}x + \widetilde{F}(x) = 0$$

with $\widetilde{L} = -L$ and $\widetilde{F} = -F$ to reduce it to a problem with $\widetilde{F}$ bounded from below, noticing that L and $\widetilde{L}$ have the same kernel and the same range. $\square$

Lemma 5. *Let $\varphi\colon \mathbb{R}\to\mathbb{R}$ be the continuous function defined by*

$$\varphi(u)=\sum_{m=1}^{n-2} f_m(u\varphi_m^1)\varphi_m^1. \tag{35}$$

Then, under the assumptions from Lemma 4, we have

$$|d[L+QF,B_\varrho(0),0]|=|d[\varphi,]-\varrho,\varrho[,0]|. \tag{36}$$

P r o o f. It is a special case of Proposition II.12 in [10]. We prove it for completeness. If $J\colon \ker L\to \operatorname{Im} Q$ is any isomorphism, then it is easy to check that $L+JP\colon V^{n-1}\to\mathbb{R}^{n-1}$ is an isomorphism and that $(L+JP)^{-1}h=J^{-1}h$ for every $h\in\operatorname{Im} Q$. Consequently,

$$\begin{aligned}L+QF=L+JP-JP+QF&=(L+JP)[I+(L+JP)^{-1}(-JP+QF)]\\&=(L+JP)(I-P+J^{-1}QF).\end{aligned}$$

Consequently, by the product formula of the Brouwer degree,

$$\begin{aligned}|d[L+QF,B_\varrho(0),0]|&=|\operatorname{ind}[L+JP,0]d[I-P+J^{-1}QF,B_\varrho(0),0]|\\&=|d[I-P+J^{-1}QF,B_\varrho(0),0]|.\end{aligned}$$

Now, by the Leray-Schauder reduction theorem for the Brouwer degree, we have

$$\begin{aligned}|d[I-P+J^{-1}QF,B_\varrho(0),0]|&=|d[(I-P+J^{-1}QF)|_{\ker L},B_\varrho(0)\cap\ker L,0]|\\&=|d[\varphi,]-\varrho,\varrho[,0]|.\end{aligned}$$

□

Theorem 4. *If the functions f_m $(1\leqslant m\leqslant n-1)$ satisfy the conditions of Lemma 4 and if*

$$\varphi(-R)\varphi(R)<0, \tag{37}$$

then problem (30) has at least one solution.

P r o o f. It follows from Lemma 4, Lemma 5 and from the invariance of the Brouwer degree under a homotopy that

$$\begin{aligned}|d[G,B_\varrho(0),0]|&=|d[\mathcal{G}(1,\cdot),B_\varrho(0),0]|=|d[\mathcal{G}(0,\cdot),B_\varrho(0),0]|\\&=|d[L+QF,B_\varrho(0),0]|=|d[\varphi,]-\varrho,\varrho[,0]|=1,\end{aligned} \tag{38}$$

the last equality coming from (37). □

E x a m p l e 3. The problem

$$D^2x_m + \lambda_1 x_m + \exp x_m - t_m = 0 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n,$$

has at least one solution if and only if $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ is such that

$$\sum_{m=1}^{n-1} t_m \varphi_m^1 > 0.$$

Necessity follows from summing both members of the equation from 1 to $n-1$ after multiplication by φ_m^1, and sufficiency from Theorem 4, if we observe that there exists $R > 0$ such that the function φ defined by

$$\varphi(u) = \sum_{m=1}^{n-1} \left[\exp(u\varphi_m^1) - t_m\right] \varphi_m^1$$

is such that $\varphi(u) > 0$ for $u \geqslant R$ and $\varphi(u) < 0$ for $u \leqslant -R$.

E x a m p l e 4. If $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function bounded from below or from above and $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ is such that

$$-\infty < \limsup_{x \to -\infty} g(x) < \frac{\sum\limits_{m=1}^{n-1} t_m \varphi_m^1}{\sum\limits_{m=1}^{n-1} \varphi_m^1} < \liminf_{x \to +\infty} g(x) < +\infty, \tag{39}$$

then the problem

$$D^2x_m + \lambda_1 x_m + g(x_m) - t_m = 0 \quad (1 \leqslant m \leqslant n-1), \quad x_0 = 0 = x_n,$$

has at least one solution.

Condition (39) is a Landesman-Lazer-type condition for difference equations. It is easily shown to be necessary if $\limsup\limits_{x \to -\infty} g(x) < g(x) < \liminf\limits_{x \to +\infty} g(x)$ for all $x \in \mathbb{R}$.

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