COORDINATE DESCRIPTION OF ANALYTIC RELATIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. In this paper we present an algebraic approach that describes the structure of analytic objects in a unified manner in the case when their transformations satisfy certain conditions of categorical character. We demonstrate this approach on examples of functional, differential, and functional differential equations.

Keywords: canonical form, Brandt groupoid, Ehresmann groupoid, transformation, differential equation, Abel functional equation, functional differential equation

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I. Motivation

When dealing with certain sets of similar objects, one appreciates to know how they are connected, what kind of relations are among them, whether one can handle them by considering only some representatives of them while the others may be obtained by deriving them from those special objects.

Connections among objects in certain classes of functional, differential and functional differential equations can sometimes be described by algebraic means. Ehresmann and Brand groupoids play an important role in this kind of considerations [5] and a selection of suitable canonical forms together with the introduction of a coordinate system into classes of equivalence often enables us to describe even the qualitative behavior of analytic objects, like functional and differential equations, see [19].

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E x a m p l e 1. Matrices. Consider the set $\mathbb{M}$ of all square matrices with real entries and the decomposition of $\mathbb{M}$ into classes of similar matrices. Let $\mathbf{M}_A$ denote those matrices from $\mathbb{M}$ that are similar to A:

$$\mathbf{M}_A := \{B = P^{-1}AP;\ \text{for all regular } P \in \mathbb{M}\}.$$

Each such class $\mathbf{M}_A$ contains a matrix in a special, Jordan form J_A. This form need not be unique, but it is important that each class of the decomposition admits at least one matrix of this form.

It leads to the following understanding of the whole set $\mathbb{M}$ and their classes: each matrix $A \in \mathbb{M}$ can be viewed as the *Jordan form*, J_A, determining a particular class $\mathbf{M}_A$ of similar matrices, and the *similarity transformation*

$$A = P_A^{-1} J_A P_A$$

with a particular P_A giving exactly the matrix A.

Might be that this approach makes better our understanding of the structure of matrices through their Jordan forms. Of course, we may ask in what respect the Jordan form is simpler than others.

E x a m p l e 2. Parameterization of functions. Let $\mathbb{C}$ denote the set of all real continuous functions defined on open intervals of the reals and let $n \in \mathbb{N}$ be a positive integer. We say that functions f_1, f_2 from $\mathbb{C}$ are n-equivalent if there exists a diffeomorphism φ of order n that maps the definition domain of f_2 onto the definition domain of f_1 such that

$$f_1 \circ \varphi = f_2.$$

Consider the decomposition of $\mathbb{C}$ into classes with respect to this equivalence relation. Choose a representative in each class of the decomposition. Then any function from $\mathbb{C}$ may be viewed as its *representative* from the class to which a chosen function belongs together with a corresponding *change of variable*, φ.

Now, let us generalize the situations by describing the common features of these examples.

A *category* is a class of objects, $P, Q, \ldots$, together with sets $\mathrm{Hom}(P,Q), \ldots$ of morphisms satisfying:

1. The sets $\mathrm{Hom}(P,Q)$ are disjoint for different pairs (P,Q).

2. A composition $\alpha\beta \in \mathrm{Hom}(P,T)$ is defined for each $\alpha \in \mathrm{Hom}(P,Q)$ and $\beta \in \mathrm{Hom}(Q,T)$ such that

a) the associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds whenever one side is defined, and

b) there exists an identity, $\iota(Q)$, for each object Q: $\alpha\iota(Q) = \alpha$, $\iota(Q)\beta = \beta$.

A category is an *Ehresmann groupoid* if each morphism has an inverse.

Moreover, an Ehresmann groupoid is called a *Brandt groupoid* if $\mathrm{Hom}(P,Q) \neq \emptyset$ for any pair (P,Q) of its objects.

An Ehresmann groupoid is a collection of connected components, Brandt groupoids, also called classes of *equivalent* objects. The set $\mathrm{Hom}(P,P)$ is a group, which is called a *stationary group* $\mathfrak{g}(P)$ of the object P.

Property 1. Let $\alpha \in \mathrm{Hom}(P,Q)$. Then $\mathfrak{g}(Q) = \alpha^{-1}\mathfrak{g}(P)\alpha$.

Definition 1. Let $C(\mathbb{B})$ be a special, say, *canonical object* in a Brandt groupoid $\mathbb{B}$. For any object $P \in \mathbb{B}$ define $\alpha \in \mathrm{Hom}(C(\mathbb{B}),P)$ as the *coordinate of* P (with respect to $C(\mathbb{B})$).

Property 2. Let α, β be respectively the coordinates of objects P, Q from the same Brandt groupoid $\mathbb{B}$ with respect to the canonical object $C(\mathbb{B})$. Then all morphisms from P to Q are given by the formula

$$\mathrm{Hom}(P,Q) = \alpha^{-1}\mathfrak{g}(C(\mathbb{B}))\beta.$$

Proofs of these properties can be found e.g. in [5], see also [18]. $\square$

Whenever the structure of objects of our interest is an Ehresmann groupoid, the following problems may be considered:

- Criterion of equivalence: sufficient and necessary conditions under which two given objects are equivalent, i.e. when they are in the same Brandt groupoid.
- Canonical forms and their stationary groups in each Brandt groupoid of the Ehresmann groupoid. By virtue of Properties 1 and 2, it enables us to describe the structure of all transformations of our objects.
- Invariants in each Brandt groupoid are of interest in connection with a criterion of equivalence.

Of course, in each area of investigation there are special problems depending on a particular Ehresmann groupoid. E.g., zeros or boundedness of solutions may be studied if differential equations are objects, other questions may arise when matrices or geometrical objects are under consideration. Let us apply this general approach to particular situations.

III. Functional equations

Consider the Abel functional equation

$$f(\varphi(x)) = f(x) + 1, \tag{1}$$

where φ is a given continuous strictly increasing real-valued function defined on a half-open interval $[a,b) \subseteq \mathbb{R}$, $b \leqslant \infty$, mapping it onto a half-open interval $[c,b)$, and $\varphi(x) > x$ for all $x \in [a,b)$. Let $\varphi^{[n]}$ denote the n-th iterate of φ, i.e. $\varphi = \varphi^{[1]}$, $\varphi^{[n+1]} = \varphi \circ \varphi^{[n]}$. Evidently $\varphi^{[n]}$ exists for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \varphi^{[n]}(x) = b$ for any $x \in [a,b)$. The following Proposition 1 summarizes results of B. Choczewski [3] and E. Barvínek [1], see also M. Kuczma [6] and M. Kuczma, B. Choczewski, R. Ger [7].

Proposition 1. *Under the above conditions, there always exists a solution f of (1). If its values on the interval $[a,\varphi(a))$ are prescribed, this solution is unique. Moreover, if it is continuous on $[a,\varphi(a))$ and*

$$\lim_{t\to\varphi(a)_-} f(x) = f(a) + 1,$$

then f is continuous on $[a,b)$.

In addition, if $f(x)$ is chosen on $[a,\varphi(a))$ such that it is here strictly increasing, then the solution f is strictly increasing on $[a,b)$.

Let φ be n-times continuously differentiable on $[a,b)$ for some $n \geqslant 1$. If $f(x)$ is chosen on $[a,\varphi(a))$ such that it is here n-times continuously differentiable and

$$\lim_{x\to a_-} (f(\varphi(x)))^{(n)} = (f(x)+1)^{(n)}|_{x=a},$$

then the solution f is also n-times continuously differentiable on the whole $[a,b)$, i.e. $f \in C^n[a,b)$.

Moreover, if $f'(x)$ is positive on $[a,\varphi(a))$ and $\varphi'(x)$ is positive on $[a,b)$, then also $f'(x) > 0$ on $[a,b)$.

In the next proposition we describe the general solution, the solution space for the Abel equation (1).

Proposition 2. *Let f_0 denote a continuous and strictly increasing solution of* (1)*; under our suppositions such a solution always exists. The general solution f of the Abel functional equation* (1) *is of the form*

$$f(x) = f_0(x) + P(f_0(x)), \tag{2}$$

where P is a periodic function with the period 1*, $P(t+1) = P(t)$, defined on $[f_0(a), \infty)$.*

P r o o f can be found in [21]. □

Let us emphasize that (2) gives a l l solutions of the Abel equation (1), not only continuous or increasing ones. If the function P in (2) is continuous then also the solution f of (1) is continuous, and conversely, the continuity of f implies the continuity of P.

This means that the general solution of the equation (1) with $f_0\varphi(x) = f_0(x)+1$ or $\varphi(x) = f_0^{-1}(f_0(x)+1)$ is obtained from the general solution $t \mapsto t + P(t), P(t+1) = P(t)$, of a particular Abel equation

$$\beta(t+1) = \beta(t) + 1, \tag{3}$$

by a substitution $t = f_0(x)$, where f_0 is a particular solution of (1). We may summarize the above considerations in

Theorem 1. *All solutions of an arbitrary Abel equation* (1) *can be obtained from the solutions of a particular, canonical equation* (3) *by the above substitution, transformation, morphism* (2)*, f_0 being considered as the coordinate of* (1) *with respect to* (3)*. The function φ in* (1) *is given by $\varphi(x) = f_0^{-1}(f_0(x)+1)$.*

IV. Differential equations

Now, we will apply our algebraic concept to linear differential equations. Ernst Kummer [8] considered transformations of second order linear differential equations. O. Borůvka [2] required global transformations, i.e. transformations of solutions of the corresponding equations on their whole intervals of definition. His approach to second order equations was extended to linear differential equations of an arbitrary order, [16].

Consider a linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_0(x)y = 0 \quad \text{on} \quad I, \tag{4}$$

I being an open interval of the reals, where p_i are real-valued continuous functions on I for $i = 0, 1, \ldots, n-1$, i.e. $p_i \in C^0(I), p_i \colon I \to \mathbb{R}$.

For functions $f \colon J \to \mathbb{R}$ and $h \colon J \to I$ such that $f \in C^n(J), f(t) \neq 0$ for each $t \in J$, and $h \in C^n(J), h'(t) \neq 0$ for each $t \in J$, and $h(J) = I$, the function z defined by

$$(f,h) \qquad z \colon J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J,$$

for a solution y of equation (4) satisfies again a differential equation of the same form

$$(5) \qquad z^{(n)} + q_{n-1}(t) z^{(n-1)} + \ldots + q_0(t) z = 0 \quad \text{on} \quad J.$$

Definition 2. In the above situation we say that equation (4) is globally equivalent to equation (5).

Since h is a C^n-diffeomorphism of J onto I, solutions y are transformed into solutions z on their whole intervals of definition. That is why we speak about a global transformation of equation (4) into equation (5). All equations (4) form an Ehresmann groupoid with respect to the above global transformations as morphisms, each class of equivalent equations being a Brandt groupoid.

Now we will describe constructions of global canonical forms for linear differential equations.

• Analytic construction. The following result can be shown [12], [16].

Theorem 2. *In each class of equations equivalent to equation* (5) *with* $q_{n-1} \in C^{n-1}$ *and* $q_{n-2} \in C^{n-2}$ *there exists a special equation of the form*

$$\mathbf{1} \cdot y^{(n)} + \mathbf{0} \cdot y^{(n-1)} + \mathbf{1} \cdot y^{(n-2)} + \mathbf{p}_{n-3}(x) y^{(n-3)} + \ldots + \mathbf{p}_0(x) y = 0.$$

This equation is characterized by

$$\mathbf{p}_{n-1} \equiv \mathbf{0}, \quad \mathbf{p}_{n-2} \equiv \mathbf{1}.$$

In accordance with our Definition 1, it can be called *canonical* for the whole class of equations globally equivalent to (5) forming a Brandt groupoid $\mathbb{B}$ and denoted by $C(\mathbb{B})$. The corresponding global transformation (f,h) converting this canonical equation into an equation in $\mathbb{B}$ may be considered as its *coordinate* (with respect to its canonical equation $C(\mathbb{B})$).

Let us note that neither Halphen, nor Laguerre-Forsyth canonical forms [4], [9], [22] are global, since we cannot find them in all equivalent classes, [13]. If they had taken 1 instead of 0 as a coefficient at the $(n-2)$nd derivative, they would have obtained global canonical forms.

There is also another construction of global canonical forms, this time without any restriction on smoothness of the coefficients, [10], [16].

• Geometrical construction. Let $\mathbf{y}(x) = (y_1(x), \ldots, y_n(x))$ denote an n-tuple of linearly independent solutions of equation (4) considered as a vector function, or as a curve in the n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x being its parameter, and $\|\mathbf{y}(x)\| = \sqrt{y_1^2(x) + \ldots + y_n^2(x)}$ denoting its Euclidean norm.

If $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t)$ denotes an n-tuple of linearly independent solutions of equation (5), then the global transformation (f, h) can be equivalently written as

$$\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(t)).$$

Define the n-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ as

$$\mathbf{v}(x) := \mathbf{y}(x)/\|\mathbf{y}(x)\|.$$

It was shown in [10], [16] that $\mathbf{v} \in C^n(I), \mathbf{v}\colon\ I \to \mathbb{E}_n$, and the Wronski determinant of $\mathbf{v}$, $W[\mathbf{v}] := \det(\mathbf{v}, \mathbf{v}', \ldots, \mathbf{v}^{(n-1)})$, is different from zero on I. Further, $\|\mathbf{v}(x)\| = 1$, i.e. $\mathbf{v}(x)$ is the central projection of $\mathbf{y}$ onto the unit sphere $\mathbb{S}_{n-1}$ in $\mathbb{E}_n$. Evidently, the differential equation which has this $\mathbf{v}$ as its n-tuple of linearly independent solutions is globally equivalent to (4). Now, let us introduce the length parameterization into this curve $\mathbf{v}$, see again [10], [16]. Then $\|\mathbf{u}'(s)\| = 1$,

$$\mathbf{u}(s(x)) := \mathbf{v}(x), \quad s\colon\ I \to K \subseteq \mathbb{R}, \quad s(I) = K, \quad \|\mathbf{u}'(s)\| \cdot |s'| = |s'| = \|\mathbf{v}'\|.$$

Hence

$$s \in C^n(I), \quad s'(x) \neq 0 \ \text{ on } I. \tag{6}$$

The differential equation admitting $\mathbf{u}$ as its n-tuple of linearly independent solutions on K is globally equivalent to equation (4); it can be considered as a representative of the whole class of equations globally equivalent to (4), its *canonical equation* (in this geometrical presentation it is generally different from the above analytic one). However, we will again denote these canonical forms as $C(\mathbb{B})$ for each Brandt groupoid $\mathbb{B}$ of globally equivalent equations (now without any restriction on smoothness of coefficients of those equations).

Definition 3. *Canonical equations* $C(\mathbb{B})$ (in this geometrical representation) are characterized and defined as the linear differential equations admitting n-tuples of linearly independent solutions $\mathbf{u}$ satisfying

$$\|\mathbf{u}(s)\| = 1, \quad \|\mathbf{u}'(s)\| = 1. \tag{7}$$

The global transformation (f, s) converting this $C(\mathbb{B})$ into the original equation (4),

$$f(x) \cdot \mathbf{u}(s(x)) = \mathbf{y}(x),$$

can be considered as the *coordinate* of (4) with respect to this canonical $C(\mathbb{B})$ (in accordance with Definition 1).

C o n s t r u c t i o n. The explicit expression for these canonical equations can be obtained by the following procedure. The vector function $\mathbf{u}\colon K \to \mathbb{R}$ satisfies the Frenet system of differential equations, if we write $\mathbf{u}_1$ instead of $\mathbf{u}$:

$$\begin{aligned}
\mathbf{u}_1' &= \mathbf{u}_2, \\
\mathbf{u}_2' &= -\mathbf{u}_1 + k_1(s)\mathbf{u}_3, \\
\mathbf{u}_3' &= -k_1(s)\mathbf{u}_2 + k_2(s)\mathbf{u}_4, \\
&\vdots \\
\mathbf{u}_{n-1}' &= -k_{n-3}(s)\mathbf{u}_{n-2} + k_{n-2}(s)\mathbf{u}_n, \\
\mathbf{u}_n' &= -k_{n-2}(s)\mathbf{u}_{n-1},
\end{aligned} \tag{8}$$

$s \in K$, with (curvatures) $k_i \in C^{n-i-1}$, $k_i(s) \neq 0$ on K for $i = 1, \dots, n-2$.

This construction leads to the following assertion giving the global canonical forms $C(\mathbb{B})$ in our geometric representation.

Theorem 3. *The canonical differential equation corresponding to* $\mathbf{u}$ *is the* n*th order linear differential equation for* $\mathbf{u}_1$ *obtained by eliminating the other* $\mathbf{u}_i$ *from the above system* (8). *The coefficients of this equation are formed from the curvatures* k_i.

E.g., if we write u instead of $\mathbf{u}_1$ we get

$$\begin{aligned}
&u'' + u = 0 \quad \text{for} \quad n = 2, \\
&u''' - \frac{k_1'(s)}{k_1(s)}u'' + (1 + k_1^2(s))u' - \frac{k_1'(s)}{k_1(s)}u = 0 \qquad \text{for} \ \ n = 3, \\
&\dots
\end{aligned} \tag{9}$$

on (different) $K \subset \mathbb{R}$, since the equivalent classes in general depend on K, cf. [2], [16]. Even for $u'' + u = 0$, which looks as the only one for the second order equations, different K can give different classes of equivalence, [2].

Let us mention that for the second order equations both the types (analytical and geometrical) of our suggested canonical forms coincide with Borůvka's one. The important notion of the (first) *phases* for the second order linear differential equations $y'' = p(x)y$ in the Jacobi form was introduced by O. Borůvka, see again [2]. In fact, these phases can be viewed as *coordinates* in our sense with respect to his canonical second order equation $u'' + u = 0$.

• Applications to qualitative behavior. Qualitative behavior of solutions of equations (4) can be expressed by their coordinates and, conversely, equations (4) admitting prescribed properties of their solutions may be constructed in this manner, [19].

Bounded solutions. In accordance with Definition 3 consider a class $\mathbb{B}$ of equivalent equations (4), its canonical equation $C(\mathbb{B})$ from (9) and its n-tuple of linearly independent solutions $\mathbf{u}$ satisfying (7).

Proposition 3. *All linear differential equations* (4) *in the class* $\mathbb{B}$ *with only bounded solutions are exactly those having coordinates* (f, s) *with bounded* f.

Proof. ($\Rightarrow$) Let all solutions of an equation (4) be bounded on its definition interval I. Then also each component y_i of its n-tuple of linearly independent solutions $\mathbf{y}$ is bounded. This means that

$$\|\mathbf{y}(x)\| = \|f(x) \cdot \mathbf{u}(s(x))\| = |f(x)| \cdot \|\mathbf{u}(s(x))\| = |f(x)| \tag{10}$$

is bounded.

($\Leftarrow$) If f in (f, s) is bounded, then each solution y of (4) is bounded, because

$$|y| = \left|\sum_{i=1}^{n} c_i y_i\right| = |f| \cdot \left|\sum_{i=1}^{i=n} c_i u_i\right| \leqslant |f| \cdot \sum_{i=1}^{n} |c_i u_i| \leqslant |f| \cdot \sum_{i=1}^{n} |c_i|. \tag{11}$$

$\square$

In Proposition 3 all linear differential equations with only bounded solution are constructed when all canonical equations are considered, since each equation belongs to a certain class of equivalent equations having always their canonical form.

Solutions tending to zero.

Proposition 4. *All solutions of an equation* (4) *in* $\mathbb{B}$ *tend to zero as they approach the right end of the interval of definition* I *exactly when the function* f *in its coordinate* (f, s) *tends to zero as* x *approaches the right end of* I.

Proof. Analogously to the proof of Proposition 3, the sufficiency and the necessity follow from the relations (10) and (11). □

All equations admitting only solutions tending to zero can be constructed in a similar way from the canonical equation as mentioned above.

Solutions in $\mathbf{L}^p$.

Proposition 5. *An equation* (4) *admits only solutions in the class* L^p *if and only if its coordinate* (f, h) *has* f *in* L^p.

Proof is analogous to the above ones, see also [19]. □

• Applications to nonsmooth functions—generalized equations.

In [20] we have seen that even for not sufficiently smooth functions we may consider relations analogous to differential equations. E.g., for $y_1, \ldots, y_n \in C^{n-1}$, but $\notin C^n$, with a nonvanishing Wronskian; they are linearly independent. How can we describe their linear combinations as the solution space of a relation?

In some cases we may handle the situation in the following manner. For simplicity, let us demonstrate it for $n = 2$.

Let $(y_1, y_2) =: \mathbf{y}$ be linearly independent of the class C^1 but $\notin C^2$, and let their Wronskian be nonvanishing. Still, and it is important from our point of view, we may look at (y_1, y_2) as "solutions" obtained from solutions of the canonical $u'' + u = 0$ by a transformation (f, h), nevertheless we cannot write a differential equation for (y_1, y_2). However, some properties of solution spaces may remain valid. E.g.,

Theorem 4 (Separation Theorem)**.** *Between any two consecutive zeros of one function of the* 2*-dimensional space* $c_1y_1 + c_2y_2$ *there exists exactly one zero of any other linearly independent function of the same space.*

Proof. In the Euclidean plane we may consider this couple as a curve and its central projection to the unit circle $\mathbb{S}_1$. Then we may introduce the length parameterization in it as described above. Necessarily we get $(\sin, \cos)$ on some interval of $\mathbb{R}$. Evidently we obtain (y_1, y_2) by a backward procedure as

$$(y_1, y_2) = f \cdot (\sin(s), \cos(s)) \tag{12}$$

for $f = \|\mathbf{y}\|$. Moreover,

$$\begin{aligned} 0 &\neq \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \\ &= \det\begin{pmatrix} f\cdot\sin s & f\cdot\cos s \\ f'\cdot\sin s + fs'\cdot\cos s & f'\cdot\cos s - fs'\cdot\sin s \end{pmatrix} = f^2\cdot s' \in C^1. \end{aligned}$$

Hence f is in C^1 and also $s \in C^1$ with $f.h' \neq 0$. But this is still enough for asserting that formula (12) guarantees that any two linear combinations $c_{11}y_1 + c_{12}y_2$ and $c_{21}y_1 + c_{22}y_2$ of y_1, y_2 have the same ordering of zeros as the corresponding linear combinations of $\sin, \cos$. □

V. Functional differential equations

Consider linear functional differential equations of the first order with several deviating arguments:

$$y'(x) = p_0(x)y(x) + p_1(x)y(\tau_1(x)) + \ldots + p_n(x)y(\tau_n(x)). \tag{13}$$

Again the transformation (f, h) converts equation (13) into a functional differential equation of the same type. It can be shown [11], [14], [17] that the following assertion holds.

Proposition 6. *If a simultaneous solution h of a system of the Abel functional equations*

$$h(\tau_i(x)) = h(x) - c_i,\ c_i = \mathit{const.},\ i = 1, \ldots, n$$

exists then equation (13) *can be converted into an equation with constant deviations*

$$z'(t) = q_0(t)z(t) + q_1(t)z(t - c_1) + \ldots + q_n(t)z(t - c_n).$$

Moreover, by choosing a suitable f we can achieve $q_0 \equiv 0$.

Hence we have

Theorem 5. *Equations*

$$z'(t) = q_1(t)z(t-c_1) + \ldots + q_n(t)z(t-c_n) \tag{14}$$

may serve as canonical ones for (13) *and the transformation that converts* (14) *into* (13) *can be viewed as the coordinate of* (13) *with respect to* (14).

E.g.

$$z'(t) = q(t)z(t-1)$$

is canonical for

$$y'(x) = p_0(x)y(x) + p_1(x)y(\tau_1(x))$$

provided $n = 1$.

Due to the form of the transformation it is clear that every possible behavior of zeros of solutions, i.e. oscillatory, nonoscillatory, disconjugacy, number of zeros, etc. of all these equations can be described by considering only equations in their canonical form, [14].

VI. Comments, open problem

This coordinate approach can be applied in many situations and may serve as a source of intensive research describing the detailed structure of other objects under consideration.

There exist several open problems concerning the selection of suitable canonical forms in various areas and interesting questions connected with their effective constructions.

Another problem consists in the selection of the morphisms. In some cases they are historically given, another way is to choose properties that should be satisfied and to derive the most general form of morphisms that keep them unchanged.

One of these problems is also the selection of suitable canonical forms in particular situations. Consider again Example 2 on parameterization of functions.

• Problem. For a nonnegative integer n we define the n-equivalence between functions f_1 and f_2 by the existence of a homomorphism φ of the class C^n between the definition domain of f_2 and that of f_1, such that

$$f_1 \circ \varphi = f_2$$

holds.

The problem consists in finding a representative, a canonical form in each class of this equivalence. Partial answers have been known only for rather restrictive subsets of the set $\mathbb{C}$ of continuous functions, see [15]. Results for larger sets of continuous functions would have important consequences e.g. in construction of u n i q u e canonical forms for linear differential equations and their invariants, see [16].

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