ON THE VOLTERRA INTEGRAL EQUATION WITH WEAKLY SINGULAR KERNEL

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(Received July 11, 2005)

Dedicated to Professor Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. We give sufficient conditions for the existence of at least one integrable solution of equation $x(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s$. Our assumptions and proofs are expressed in terms of measures of noncompactness.

Keywords: integral equation, integrable solution, measure of noncompactness

MSC 2000: 45N05

Let E be a Banach space and let $J=[0,d]$ be a compact interval in $\mathbb{R}$. Denote by $L^1(J,E)$ the space of all Bochner integrable functions $x\colon J\to E$ equipped with the norm $\|x\|_1=\int_J\|x(t)\|\,\mathrm{d}t$.

In this paper we give sufficient conditions for the existence of a solution $x\in L^1(J,E)$ of the integral equation

$$x(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s \tag{1}$$

with the kernel

$$K(t,s)=\frac{A(t,s)}{|t-s|^r}\quad (t,s\in J,\ t\neq s),$$

where $0<r<1$ and A is a bounded strongly measurable function from $J\times J$ into the space of continuous linear mappings $E\to E$.

Throughout this paper we shall assume that

1. $f\in L^1(J,E)$;
2. $(s,x)\mapsto g(s,x)$ is a function from $J\times E$ into E such that

(i) g is strongly measurable in s and continuous in x;

(ii) $\|g(s,x)\| \leqslant a(s) + b\|x\|$ for $s \in J$ and $x \in E$, where $a \in L^1(J,\mathbb{R})$ and $b \geqslant 0$.

Since $\int_0^t (t-s)^{-r}\,\mathrm{d}s = (1-r)^{-1}t^{1-r}$, we have

$$\int_0^d \frac{\mathrm{d}s}{|t-s|^r} \leqslant Q \quad \text{for all } t \in J, \text{ where } Q = \frac{2d^{1-r}}{1-r}. \tag{2}$$

Put $c = \max\{\|A(t,s)\|:\ s,t \in J\}$, $L^1 = L^1(J,E)$ and

$$(Sx)(t) = \int_J K(t,s)x(s)\,\mathrm{d}s \quad (x \in L^1,\ t \in J).$$

Lemma 1. *S is a continuous linear mapping of L^1 into itself and $\|S\| \leqslant cQ$.*

Proof. By (2) for each $z \in L^1(J,\mathbb{R})$ we have

$$\iint_{J\times J} \frac{|z(s)|}{|t-s|^r}\,\mathrm{d}s\,\mathrm{d}t = \int_J \left(\int_J \frac{\mathrm{d}t}{|t-s|^r}\right)|z(s)|\,\mathrm{d}s \leqslant Q\int_J |z(s)|\,\mathrm{d}s, \tag{3}$$

and therefore for almost every $t \in J$ the integral

$$\int_J \frac{|z(s)|}{|t-s|^r}\,\mathrm{d}s$$

exists. This shows that S is well defined. Moreover, if $x \in L^1$, then

$$\|(Sx)(t)\| \leqslant \int_J \frac{\|A(t,s)\|\|x(s)\|}{|t-s|^r}\,\mathrm{d}s \leqslant c\int_J \frac{\|x(s)\|}{|t-s|^r}\,\mathrm{d}s.$$

Thus

$$\begin{aligned}\int_J \|(Sx)(t)\|\,\mathrm{d}t &\leqslant c\int_J \left(\int_J \frac{\|x(s)\|}{|t-s|^r}\,\mathrm{d}s\right)\mathrm{d}t\\ &= c\int_J\left(\int_J \frac{\mathrm{d}t}{|t-s|^r}\right)\|x(s)\|\,\mathrm{d}s \leqslant cQ\int_J \|x(s)\|\,\mathrm{d}s,\end{aligned}$$

so that $\|Sx\|_1 \leqslant cQ\|x\|_1$.

Lemma 2. *Put $\tilde{g}(x)(s) = g(s, x(s))$ for $x \in L^1$ and $s \in J$. Then $\tilde{g}$ is a continuous mapping of L^1 into itself.*

P r o o f. Let $x_n, x_0 \in L^1$ and $\lim\limits_{n\to\infty} \|x_n - x_0\|_1 = 0$. Suppose that $\|\tilde{g}(x_n) - \tilde{g}(x_0)\|_1$ does not converge to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence $\{x_{n_j}\}$ such that

$$\|\tilde{g}(x_{n_j}) - \tilde{g}(x_0)\|_1 > \varepsilon \quad \text{for } j = 1, 2, 3, \ldots, \tag{4}$$

and $\lim\limits_{j\to\infty} x_{n_j}(s) = x_0(s)$ for a.e. $s \in J$. By 2(i) we have

$$\lim_{j\to\infty} \|g(s, x_{n_j}(s)) - g(s, x_0(s))\| = 0 \quad \text{for a.e. } s \in J.$$

Moreover, as $\lim\limits_{n\to\infty} \|x_n - x_0\|_1 = 0$ implies that the sequence (x_n) has equi-absolutely continuous norms in L^1, it follows from 2(ii) that the functions $\|g(\cdot, x_n) - g(\cdot, x_0)\|$ $(n = 1, 2, \ldots)$ are equi-integrable on J. Hence, by the Vitali convergence theorem, $\lim\limits_{j\to\infty} \|g(\cdot, x_{n_j}) - g(\cdot, x_0)\|_1 = 0$. This contradicts (4).

Denote by α and α_1 the Kuratowski measures of noncompactness in E and $L^1(J, E)$, respectively. The next lemma clarifies the relation between α and α_1. For any set V of functions belonging to $L^1(J, E)$ denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in J$ (under the convention that $\alpha(X) = \infty$ if X is unbounded), where $V(t) = \{x(t) \colon x \in V\}$.

Lemma 3. *Assume that V is a countable set of strongly measurable functions $J \to E$ and there exists an integrable function μ such that $\|x(t)\| \leqslant \mu(t)$ for all $x \in V$ and $t \in J$. Then the corresponding function v is integrable on J and*

$$\alpha\bigg(\bigg\{\int_J x(t)\,\mathrm{d}t \colon x \in V\bigg\}\bigg) \leqslant 2\int_J v(t)\,\mathrm{d}t.$$

If, in addition, $\lim\limits_{h\to\infty} \sup\limits_{x\in V} \int_J \|x(t+h) - x(t)\|\,\mathrm{d}t = 0$, then

$$\alpha_1(V) \leqslant 2\int_J v(t)\,\mathrm{d}t.$$

(See [3], Th. 2.1 and [8], Th. 1).

The main result of this paper is the following

Theorem. *Let $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $\omega(0)=0$, $\omega(t)>0$ for $t>0$ and*

$$\int_0^\delta \frac{1}{s}\Big[\frac{s}{\omega(s)}\Big]^{\frac{1}{1-r}}\,\mathrm{d}s=\infty \quad (\delta>0). \tag{5}$$

If 1–2 *hold and*

$$\alpha(g(s,X))\leqslant\omega(\alpha(X)) \tag{6}$$

for any $s\in J$ and for any bounded subset X of E, then there exists a solution $x\in L^1(J,E)$ of (1).

Proof. It is known that there exists a nonnegative solution $u\in L^1(J,\mathbb{R})$ of the integral equation

$$u(t)=\|f(t)\|+\int_0^t\|K(t,s)\|a(s)\,\mathrm{d}s+b\int_0^t\|K(t,s)\|u(s)\,\mathrm{d}s.$$

Put $B=\{x\in L^1\colon\ \|x(t)\|\leqslant u(t)$ for a.e. $t\in J\}$ and

$$G(x)(t)=f(t)+\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s \quad \text{for } x\in L^1 \text{ and } t\in J.$$

Since

$$\begin{aligned}\|G(x)(t)\|\leqslant\|f(t)\|+\int_0^t\|K(t,s)\|(a(s)+b\|x(s)\|)\,\mathrm{d}s\\ \leqslant\|f(t)\|+\int_0^t\|K(t,s)\|a(s)\,\mathrm{d}s+b\int_0^t\|K(t,s)\|u(s)\,\mathrm{d}s=u(t)\end{aligned}$$

for $x\in B$ and $t\in J$, Lemmas 1 and 2 prove that G is a continuous mapping $B\to B$.

Putting

$$\overline{K}(t,s)=\begin{cases}K(t,s) & \text{for } 0\leqslant s\leqslant t\leqslant d\\ 0 & \text{for } s>t,\end{cases}$$

we see that

$$G(x)(t)=f(t)+\int_J\overline{K}(t,s)g(s,x(s))\,\mathrm{d}s \qquad \text{for } x\in L^1 \text{ and } t\in J.$$

Without loss of generality we shall always assume that all functions from L^1 are extended to $\mathbb{R}$ by putting $x(t)=0$ outside J.

Therefore

$$\|G(x)(t+h) - G(x)(t)\| \leqslant d(t,h) \quad \text{for } x \in B, t \in J \text{ and small } |h|, \tag{7}$$

where

$$d(t,h) = \begin{cases} u(t) & \text{if } t \in J \text{ and } t+h \notin J \\ \|f(t+h) - f(t)\| + \int_J \|\overline{K}(t+h,s) - \overline{K}(t,s)\|(a(s) + bu(s))\,\mathrm{d}s \\ & \text{if } t, t+h \in J. \end{cases}$$

In view of (3) the function $(t,s) \mapsto W(t,s) = \overline{K}(t,s)(a(s) + bu(s))$ is integrable on $J \times J$. Hence

$$\lim_{h\to 0} \int_J \left(\int_J \|\overline{K}(t+h,s) - \overline{K}(t,s)\|(a(s) + bu(s))\,\mathrm{d}s \right) \mathrm{d}t \\ = \lim_{h\to 0} \iint_{J\times J} \|W(t+h,s) - W(t,s)\|\,\mathrm{d}s\,\mathrm{d}t = 0,$$

and consequently

$$\lim_{h\to 0} \int_J d(t,h)\,\mathrm{d}t = 0 \quad \text{for } t \in J. \tag{8}$$

This fact, plus (7), implies that

$$\lim_{h\to 0} \sup_{x\in B} \int_J \|G(x)(t+h) - G(x)(t)\|\,\mathrm{d}t = 0. \tag{9}$$

Let V be a countable subset of B such that

$$V \subset \overline{\mathrm{conv}}(G(V) \cup \{0\}). \tag{10}$$

Then $V(t) \subset \overline{\mathrm{conv}(G(V)(t) \cup \{0\})}$ for a.e. $t \in J$, so that

$$\alpha(V(t)) \leqslant \alpha(G(V)(t)) \quad \text{for a.e. } t \in J. \tag{11}$$

Put $v(t) = \alpha(V(t))$ for $t \in J$. From (9) and (10) it is clear that

$$\lim_{h\to 0} \sup_{x\in V} \int_J \|x(t+h) - x(t)\|\,\mathrm{d}t = 0.$$

Moreover, $\|x(t)\| \leqslant u(t)$ for all $x \in V$ and a.e. $t \in J$. Hence, by Lemma 3, $v \in L^1(J, \mathbb{R})$ and

$$\alpha_1(V) \leqslant 2 \int_J v(t)\,\mathrm{d}t. \tag{12}$$

From (3) it follows that

$$\int_J \frac{a(s)+bu(s)}{|t-s|^r}\,\mathrm{d}s < \infty \quad \text{for a.e. } t \in J. \tag{13}$$

Fix now $t \in J$ such that the integral (13) is finite.

Since

$$\|\overline{K}(t,s)g(s,x(s))\| \leqslant c\frac{a(s)+bu(s)}{|t-s|^r} \quad \text{for } x \in B \text{ and } s \in J,$$

owing to (11), (6) and Lemma 3 we get

$$\begin{aligned}
\alpha(V(t)) \leqslant \alpha(G(V)(t)) &\leqslant \alpha\Bigl(\Bigl\{\int_0^t K(t,s)g(s,x(s))\,\mathrm{d}s\colon\ x \in V\Bigr\}\Bigr) \\
&\leqslant 2\int_0^t \alpha(\{K(t,s)g(s,x(s))\colon\ x \in V\})\,\mathrm{d}s \\
&\leqslant 2\int_0^t \|K(t,s)\|\alpha(g(s,V(s))\,\mathrm{d}s \\
&\leqslant 2\int_0^t \|K(t,s)\|\omega(\alpha(V(s)))\,\mathrm{d}s,
\end{aligned}$$

i.e.

$$v(t) \leqslant 2c\int_0^t \frac{\omega(v(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J.$$

Putting $w(t) = 2c\int_0^t \omega(v(s))(t-s)^{-r}\,\mathrm{d}s$ for $t \in J$ we see that w is a continuous function such that $v(t) \leqslant w(t)$ for $t \in J$. Hence

$$w(t) \leqslant 2c\int_0^t \frac{\omega(w(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J. \tag{14}$$

By the Mydlarczyk-Gripenberg theorem ([7], Th. 3.1) and assumption (5), the integral equation

$$z(t) = 2c\int_0^t \frac{\omega(z(s))}{(t-s)^r}\,\mathrm{d}s \quad \text{for } t \in J$$

has the unique continuous solution $z(t) \equiv 0$. Applying now the theorem on integral inequalities ([1], Th. 2), from (14) we deduce that $w(t) \equiv 0$. Thus $v(t) = 0$ for $t \in J$ and consequently, by (12), $\alpha_1(V) = 0$. Hence the set V is relatively compact in L^1. Thus we can apply the Mönch fixed point theorem ([6], Th. 2.1) which yields the existence of a function $x \in L^1$ such that $x = G(x)$. Clearly x is a solution of (1).

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