SEMIPERMEABLE SURFACES FOR NON-SMOOTH DIFFERENTIAL INCLUSIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. We investigate the regularity of semipermeable surfaces along barrier solutions without the assumption of smoothness of the right-hand side of the differential inclusion. We check what can be said if the assumptions concern not the right-hand side itself but the cones it generates. We examine also the properties of families of sets with semipermeable boundaries.

Keywords: differential inclusions, semipermeable surfaces, barrier solutions

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It is a honour and pleasure for both authors to have an occasion to dedicate this modest paper to Professor Jaroslav Kurzweil.

It was a great chance for me (the second author) to meet Professor Kurzweil in Prague during Equadiff 1977. This meeting and later contacts provided me with a subject for my PhD thesis which I prepared under the supervision of Professor Czeslaw Olech—it was a real luck to have such two guides.

Professor Kurzweil (together with J. Jarník and P. Krbec) wrote several papers devoted to multivalued maps, differential inclusions and in particular to the so called Scorza-Dragoni property: [8], [9], [10], [11], [12]—this enumeration does not pretend to completeness.

The problems considered by Scorza-Dragoni in [21], [22] included first of all the regularity of functions $f(t,x)$ continuous with respect to x and measurable with

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respect to t. He proved that for any $\varepsilon > 0$ there is a set A_ε with measure not greater than ε and such that f is continuous with respect to (t, x) if we eliminate $t \in A_\varepsilon$. This was later shown also for multifunctions, however, it occured that it is not possible to replace continuity by upper semicontinuity with respect to x. It was a surprising and very useful result of Jaroslav Kurzweil and Jiří Jarník given in [8] where they proved that for $F(t, x)$ upper semicontinuous in x one can replace the original multifunction $F(t, x)$ by $\widetilde{F}(t, x)$ which is already regular similarly to the Scorza-Dragoni case and the differential inclusions $\dot{x} \in F(t, x)$ and $\dot{x} \in \widetilde{F}(t, x)$ have the same solutions. This became an important tool in the theory of multifunctions and differential inclusions.

The results mentioned above were used by Scorza-Dragoni and others to the examination of sets of t for which the Carathéodory type solutions of differential equations or differential inclusions may not satisfy $\dot{x}(t) = f(t, x(t))$ or $\dot{x}(t) \in F(t, x(t))$. It occurs that for differential equations all those sets are contained in a common set of measure zero. This also was carried by Jaroslav Kurzweil and Jiří Jarník in [9] to differential inclusions upper semicontinuous with respect to x—the derivative $\dot{x}(t)$ had to be replaced by the contingent $Dx(t)$ (the set of all limit points of the differential quotient). A corollary to that theorem is used in the present paper.

A full description of influence of those results and the papers which appeared later referring to them would require a special survey and not a short one.

Let me express my personal admiration and gratitude to Professor Kurzweil. The papers [17], [19] (which contain a fundamental part of my thesis) were directly inspired by his works and advice. Much more recent [6] also refers in an essential way to the results mentioned above. Last but not least—I always appreciated his nice and friendly attitude.

Tadeusz Rzeżuchowski

1. Introduction and preliminaries

We consider in this paper autonomous differential inclusions

$$\dot{x} \in F(x) \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^d$, Ω open, $F(x) \subset \mathbb{R}^d$ and solutions are understood in the sense of Carathéodory, that is, defined on some interval absolutely continuous functions $x(\cdot)$ with values in Ω which satisfy $\dot{x}(t) \in F(x(t))$ almost everywhere.

In most considerations we assume the same set of conditions on F so it will be convenient to put them together and name them.

Definition 1.1. We say that $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ is a Lipschitz multifunction if the sets $F(x) \subset \mathbb{R}^d$ are nonempty, compact, convex and the Lipschitz condition

$$\exists L \geqslant 0,\ \forall x, y \in \Omega\colon\ F(y) \subset F(x) + L\|x - y\| B_1$$

is satisfied, where B_1 is the closed unit ball in $\mathbb{R}^d$.

The difference with the so called Marchaud maps (see [1]) is in the regularity of $F(\cdot)$—Marchaud maps are upper semicontinuous.

We use the notation $F(\cdot)$ or $x(\cdot)$ for maps, $F(x)$ or $x(t)$ denote their values at x or t.

By $\mathrm{Sol}_F(x_0, T)$, where $x_0 \in \Omega$ and $T > 0$, we mean the set of all solutions of (1) defined on $[0, T]$ and satisfying the initial condition

$$x(0) = x_0. \tag{2}$$

Semipermeable surfaces so important in differential games and control theory appeared in the pioneering work of Isaacs [7]. He considered a differential system

$$\dot{x}(t) = f(x(t), u(t), v(t)) \tag{3}$$

in $\mathbb{R}^d$, where $u(t)$ and $v(t)$ are functions controlled by two players—one of them wants to bring the object to some set, the other wants to avoid it. The victory domain for each of the players is the set of positions such that he can always find a winning strategy. Isaacs proved that if the boundary of a victory domain is smooth then it is a semipermeable surface—this means that this set is viable (from any point a solution starts which always stays in it) and invariable if we reverse the time (in other words, no solution can enter it from outside).

M. Quincampoix introduced in [16] the notion of nonsmooth semipermeable surfaces. He proved that the viability kernel contained in some set K (if nonempty) has semipermeable boundary—the viability kernel of K is the set of all $x_0 \in K$ for which there is a solution $x(\cdot)$ of (1) defined on $[0, +\infty)$ for which $x(0) = x_0$ and $x(t) \in K$ for all $t \geqslant 0$.

P. Cardaliaguet investigates in [3] and [4] sets with semipermeable boundary and pays special attention to the solutions which lie on that boundary—the so called barrier solutions. He proves there some results on the smoothness of barrier solutions.

Let $\mathcal{A}(x_0, t) = \{x(t)\colon x(\cdot) \in \mathrm{Sol}_F(x_0, t)\}$. Under certain conditions on $F(\cdot)$ the surface $(\partial\,\mathrm{Graph}\,\mathcal{A}(x_0, \cdot)) \cap ([0, +\infty) \times \Omega)$ is semipermeable with respect to the differential inclusion $(\dot{\sigma}, \dot{y}) \in \{-1\} \times (-F(y))$. This permits to prove that under some additional assumptions it is possible to recover the initial point x_0 if we have some information on $\mathcal{A}(x_0, t)$ (see [18]).

H. Frankowska proved in [5] that the graph of the value function $V(t,x)$ for a control problem is a semipermeable surface for some differential inclusion (although she did not use explicitly this notion) which permitted her to prove that V is a solution in the sense of viscosity of the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t} + H\Big(t, x, -\frac{\partial V}{\partial x}(t,x)\Big) = 0.$$

S. Plaskacz and M. Quincampoix gave in [14] a similar result for the graph of the value function in differential games.

In this paper we treat problems similar to those considered by Cardaliaguet in [3] and [4]—the properties of semipermeable surfaces and especially the regularity of barrier solutions. The main assumption there was, apart from Lipschitz regularity, the smoothness of the boundary $\partial F(x)$ for all x. We show that some of these results can be proved even if this assumption is not satisfied. Next we treat the families of sets with semipermeable boundary. The last section is devoted to the application of equivalent differential inclusions to weakening of assumptions on $F(\cdot)$ (such as regularity and smoothness of the boundary $\partial F(x)$).

We recall now a few notions from set-valued analysis to be used in the sequel. Let $K \subset \mathbb{R}^d$ be closed.

The contingent (or Bouligand) cone to $K \subset \mathbb{R}^d$ at any $x \in \overline{K}$ is defined as

$$\mathrm{T}_K(x) = \Big\{ v \in \mathbb{R}^d\,;\ \liminf_{h \to 0+} \frac{\operatorname{dist}(x+hv, K)}{h} = 0 \Big\}$$

where $\operatorname{dist}(y,K) = \inf\limits_{z \in K} \|z-y\|$.

The normal cone to K at x is defined by

$$K^- = \big\{ v \in \mathbb{R}^d\,;\ \forall u \in K\colon\ \langle u, v\rangle \leqslant 0 \big\}.$$

The Dubovitski-Miliutin tangent cone is defined as

$$\mathrm{D}_K(x) = \big\{ v \in \mathbb{R}^d\,;\ \exists \alpha > 0\colon\ x + (0,\alpha](v + \alpha B_1) \subset K \big\}$$

and the hypertangent cone (it is the interior of Clarke's tangent cone) as

$$\mathrm{C}^\circ_K(x) = \big\{ v \in \mathbb{R}^d\,;\ \exists \varepsilon > 0, \exists \delta > 0, \forall y \in B(x,\delta) \cap K\colon\ y + (0,\varepsilon](v + \varepsilon B_1) \subset K \big\}.$$

The cone generated by a set K is denoted by S_K.

The contingent derivative of a function $x\colon [a,b] \to \mathbb{R}^d$ at $t \in [a,b]$ is defined as the set of all limit points of the differential quotient $(x(t+h) - x(t))/h$ as $h \to 0$, and is denoted by $\mathcal{D}x(t)$.

We pass now to the definition of the notion fundamental to our considerations.

Definition 2.1. The boundary ∂M of a closed set $M \subset \mathbb{R}^d$ is semipermeable in an open set U with respect to the differential inclusion (1) if the following conditions are fulfilled:

(i) $\forall \xi \in M \cap U,\ \exists T > 0,\ \exists x(\cdot) \in \mathrm{Sol}_F(\xi, T),\ \forall t \in [0, T]\colon\ x(t) \in M$,

(ii) $\forall \xi \in M \cap U,\ \exists T > 0,\ \forall x(\cdot) \in \mathrm{Sol}_{-F}(\xi, T),\ \forall t \in [0, T]\colon\ x(t) \in M$.

Remark that in (ii) the set of solutions of the differential inclusion $\dot{x} \in -F(x)$ is used.

Condition (ii) in this definition is equivalent to the following one:

(ii)′ $\forall \xi \in M \cap U$, $\forall T > 0$ if $x(\cdot) \in \mathrm{Sol}_{-F}(\xi, T)$ and $x(t) \in U$ for all $t \in [0, T]$ then $x(t) \in M$ for all $t \in [0, T]$.

This last property means that any solution of $\dot{x} \in -F(x)$ starting at some point belonging to $M \cap U$ does not leave M as long as it stays in U.

The implication (ii)′ $\Rightarrow$ (ii) is obvious. To prove the inverse suppose (ii)′ is not true. This means that for some $\xi \in M \cap U$ and $T > 0$ there is a solution $x(\cdot) \in \mathrm{Sol}_{-F}(\xi, T)$ such that for all $t \in [0, T]$ we have $x(t) \in U$ but $x(t) \notin M$ for some t. Let $\tau = \max\{t > 0\colon\ x(s) \in M \text{ for all } s \leqslant t\}$. Consider now the function $y(t) = x(t + \tau)$. For every $\widetilde{T} > 0$ there are $t \in (0, \widetilde{T}]$ for which $y(t) \notin M$. The negation of (ii) is thus true, which finally proves the implication (ii) $\Rightarrow$ (ii)′.

Cardaliaguet defines in [3] semipermeable surfaces using the notion of lower Hamiltonian

$$h_F(x, p) = \inf\{\langle v, p\rangle\colon\ v \in F(x)\}$$

—according to this definition a closed set M has semipermeable boundary with respect to (1) in a neighborhood of $x_0 \in \partial M$ if there is $r > 0$ such that

$$\forall x \in M \cap B(x_0, r),\ \forall p \in \mathrm{T}_M(x)^-\colon\ h_F(x, p) = 0 \tag{4}$$

—$B(x_0, r)$ is the closed ball centered at x_0 with radius r. He proves for Lipschitz regular multifunctions that (4) is equivalent to the existence of two open neighborhoods O and O' of x_0, with $O \subset O'$, and the existence of $T > 0$ for which the following conditions hold:

(Ci) $\forall x \in M \cap O,\ \exists x(\cdot) \in \mathrm{Sol}_F(x, T),\ \forall t \in [0, T]\colon\ x(t) \in M \cap O'$,

(Cii) $\forall x \in M \cap O,\ \forall x(\cdot) \in \mathrm{Sol}_{-F}(x, T),\ \forall t \in [0, T]\colon\ x(t) \in M \cap O'$.

This definition and properties have a local character wheras Definition 2.1 that we use can be considered as global although they do not differ in an essential way.

Using Filippov's theorem (see for example Therorem 10.4.1 in [2]) Cardaliaguet proves in [3] (Proposition 1.1) a property of semi-permeable surfaces which in our setting has the following shape (below $\widehat{M} = \overline{\mathbb{R}^d \setminus M}$).

Lemma 2.1. *If F is a Lipschitz multifunction and (i), (ii) hold then*
(iii) $\forall \xi \in \partial M \cap U,\ \exists T > 0,\ \forall x(\cdot) \in \mathrm{Sol}_F(\xi, T), \forall t \in [0, T]\colon\ x(t) \in \widehat{M}$.

This lemma coupled with (i) gives

Corollary 2.1. *If M has semipermeable boundary then for every $\xi \in \partial M$ there are $T > 0$ and $x(\cdot) \in \mathrm{Sol}_F(\xi, T)$ such that $x(t) \in \partial M$ for $t \in [0, T]$.*

The solutions like in Corollary 2.1—staying on ∂M—are called *barrier solutions*. Their trajectories $\{x(t)\colon\ t \in [0, T]\}$ will be called *barriers*.

We give at the end of this section an example which is very simple, nonetheless explains well the situation.

Example 2.1. For every $x \in \mathbb{R}^2$ put $F(x) \equiv B((1, 1), 1)$ and

$$M_1 = \{(y_1, y_2);\ y_1 \in \mathbb{R}, y_2 \leqslant 0\} \quad M_2 = \{(y_1, y_2);\ y_1 \leqslant 0, y_2 \in \mathbb{R}\}.$$

The surfaces ∂M_1, ∂M_2 and $\partial(M_1 \cup M_2)$ are semipermeable for differential inclusion (1). The third one has one point at which it is not smooth—the origin $(0, 0)$. There are, of course, barrier solutions which start from this point (two) but no one starting from any other point reaches it. This situation is typical.

The position of the origin 0 in $\mathbb{R}^d$ with respect to the sets $F(x)$ is important. We check first that if $0 \in \mathrm{int} F(x_0)$ then x_0 may not lie on the semipermeable boundary of any M. Suppose that $x_0 \in M$. We remark first that if $0 \in \mathrm{Int} F(x_0)$ then there is $r > 0$ and a ball $B(x_0, \delta)$ such that $B(0, r) \subset -F(x)$ for any $x \in B(x_0, \delta)$. Hence for sufficiently small $\tau > 0$ all the functions $x(s) = x_0 + sv$, $s \in [0, \tau]$, where $v \in B_1$ is arbitrary, are solutions of $\dot{x} \in -F(x)$. As no such solution can exit M so there are no points from $\mathbb{R}^d \setminus M$ in a neighborhood of x_0 and thus $x_0 \notin \partial M$ contrary to the assumption.

When $0 \in \partial F(x)$ for all x in a neighborhood of x_0 the condition (i) in Definition 2.1 is automatically satisfied as constant maps are solutions. To have the property (ii) the condition $-F(x) \subset \mathrm{T}_M(x)$ for $x \in \partial M$ is necessary and sufficient (due to the Lipschitz assumption—see Chapter V in [1]).

The most interesting case for our investigation is when $0 \notin F(x)$ for all x in some neighborhood of x_0.

3. Properties of semipermeable surfaces when the boundaries $\partial F(x)$ need not be smooth

In this section we are interested in some properties of semipermeable surfaces when the sets $F(x)$ may have nonsmooth boundary. Smoothness of $\partial F(x)$ was one of the main assumptions in [3] and [4]. We try to show what can be said if it is omitted.

Let us start with an observation that an important property proved in [4] (Theorem 1.3)

$$\operatorname{Limsup} \mathrm{T}_M(x') \subset \mathrm{T}_M(x) \text{ when } x' \to x,\ x' \in \partial M \tag{5}$$

is no more true when the boundaries $\partial F(x)$ are nonsmooth.

Example 3.1. Let $x \in \mathbb{R}^3$ and

$$F(x) \equiv \Big\{(y_1, y_2, y_3);\ \sqrt{(y_1-1)^2 + y_2^2} \leqslant y_3 \leqslant 1\Big\}.$$

The set $M = \{(x_1, x_2, x_3);\ x_3 \leqslant -|x_2|\}$ has semipermeable boundary but the property (5) is not satisfied—it is sufficient to reach a point on the edge $\{(x_1, 0, 0);\ x_1 \in \mathbb{R}\}$ of M by a sequence of points not belonging to that edge.

The upper semicontinuity of $\mathrm{T}_M(x)$ expressed in (5) can be preserved but only if we approach a point of ∂M along barriers.

Lemma 3.1. *Suppose F is a Lipschitz multifunction and M has semipermeable boundary in an open set $U \subset \Omega$. If $x(\cdot)$ is a barrier solution defined on $[0, T]$ then*

$$\operatorname{Limsup}_{t\to\tau+} \mathrm{T}_M(x(t)) \subset \mathrm{T}_M(x(\tau))$$

for $\tau \in [0, T)$ and

$$\operatorname{Limsup}_{t\to\tau-} \mathrm{T}_{\widehat{M}}(x(t)) \subset \mathrm{T}_{\widehat{M}}(x(\tau))$$

for $\tau \in (0, T]$.

This is in fact Lemma 2.1 from [3] expressed in other terms. Let us justify the first inclusion. Take $v \in \operatorname{Limsup}_{t\to\tau+} \mathrm{T}_M(x(t))$. Then for some $t_n \to \tau+$ and $v_n \in \mathrm{T}_M(x(t_n))$ we have $v_n \to v$,

$$\operatorname{dist}(v, \mathrm{T}_M(x(\tau)) \leqslant \|v - v_n\| + \operatorname{dist}(v_n, \mathrm{T}_M(x(\tau))$$

and in view of the lemma mentioned above $\operatorname{dist}(v_n, \mathrm{T}_M(x(\tau)) \leqslant C\|v_n\|(t_n - \tau)$, which implies $v \in \mathrm{T}_M(x(\tau))$. The justification of the second inclusion is similar.

Let us note one of the implications of Lemma 3.1. If at a point $x(\tau)$ of the trajectory of a barrier solution $x(\cdot)$ the contingent cone to M does not contain any half-space then for t in an interval $(\tau, \tau+\varepsilon)$ the contingent cone $\mathrm{T}_M(x(t))$ cannot contain any half-space, either. If we consider the situation described in Example 3.1 then we see that no barrier can exit the edge. One could comment that the only way to exit an edge of this kind is that the edge disappears while the solution follows it. Similar interpretation can be given for the second semicontinuity described in Lemma 3.1.

It is shown in [3], under the assumption that $\partial F(x)$ is of class C^1 for all $x \in \Omega$, that for each barrier solution $x(\cdot)$ there exists an absolutely continous nonzero function $p\colon [0,T] \to \mathbb{R}^d$ (adjoint function) for which

$$\mathrm{T}_M(x(t)) = (p(t))^- \quad \text{and} \quad h_F(x(t), p(t)) = \langle \dot{x}(t), p(t) \rangle = 0.$$

This implies in particular that at those points where the derivative exists the inclusion $\dot{x}(t) \in \partial \mathrm{S}_{F(x(t))}$ holds. We show a bit further that this last inclusion is valid without the assumption of smoothness of $\partial F(x)$ and for all t and all elements of the contingent $\mathcal{D}x(t)$ instead of $\dot{x}(t)$ only.

We start with another theorem to be used in the proof of the above mentioned property and which may present an interest by itself. In [3] (Lemma 2.2) the equalities

$$-\mathrm{Int}\,\mathrm{T}_{F(x(t))}(\dot{x}(t)) = \mathrm{D}_M(x(t)), \quad \mathrm{Int}\,\mathrm{T}_{F(x(t))}(\dot{x}(t)) = \mathrm{D}_{\widehat{M}}(x(t))$$

were proved, where $x(\cdot)$ is a barrier solution, F a Lipschitz multifunction and the boundaries $\partial F(x)$ are smooth. It was shown first, even without the smoothnes mentioned, that the left-hand terms are included in the right-hand terms. The theorem below gives a stronger version of this result. First, we consider not only the derivatives $\dot{x}(t)$ (where they exist) but any $v \in \mathcal{D}x(t)$ for any t. Next, the left-hand side terms will be essentially larger and at the right-hand side we put the hypertangent cones which on the contrary are usually essentially smaller than the Dubovitsky-Miliutin cone.

Theorem 3.1. *Let $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ be a Lipschitz multifuction and let $0 \notin F(x)$ in an open set $U \subset \Omega$. If the boundary ∂M is semipermeable in U and $x(\cdot)$ is a barrier solution defined on $[0,T]$ then for every $t \in (0,T]$ and $v \in \mathcal{D}x(t)$*

$$-\mathrm{Int}\,\mathrm{T}_{\mathrm{S}_{F(x(t))}}(v) \subset \mathrm{C}^{\circ}_M(x(t)), \quad \mathrm{Int}\,\mathrm{T}_{\mathrm{S}_{F(x(t))}}(v) \subset \mathrm{C}^{\circ}_{\widehat{M}}(x(t)). \tag{6}$$

P r o o f. The idea is similar to that in Lemma 1.2 in [3], however, some technical difficulties have to be overcome as the interiors of $F(x)$ may be empty. If

$\operatorname{Int} \mathrm{S}_{F(x(t))} \neq \emptyset$ then the same is true for s near to t. So for the proof we may assume it to be nonempty.

Let $w \in -\operatorname{Int} \mathrm{T}_{\mathrm{S}_{F(x(t))}}(v)$, $w \neq 0$. In view of Proposition 4.2.3 [2] there is $h > 0$ for which

$$v - hw \in \operatorname{Int} \mathrm{S}_{F(x(t))}.$$

Due the continuity of F we have

$$\exists \varrho > 0,\ \exists \gamma > 0,\ \forall y \in B(x(t), \varrho)\colon\ \mathrm{S}_{v-hw+\gamma B_1} \subset \mathrm{S}_{F(y)}.$$

We fix ϱ and γ as above and define

$$\alpha = \inf\left\{\operatorname{dist}(0, F(y) \cap \mathrm{S}_{v-hw+\gamma B_1});\ y \in B(x(t), \varrho)\right\}$$

which is greater than 0 due to the continuity of F and compactness of $B(x(t), \varrho)$.

Take $z \in M \cap B(x(t), \varrho/2)$ and $u \in \mathrm{S}_{v-hw+\gamma B_1}$. The intersection $F(y) \cap \mathrm{S}_{\{u\}}$ defines on $B(x(t), \varrho)$ an upper semicontinuous map with nonempty values and $\operatorname{dist}(0, F(y) \cap \mathrm{S}_{\{u\}}) \geqslant \alpha$. Thus on some interval $[0, \tau_{z,u}]$ there exists an absolutely continuous, real valued function $\xi(\cdot)$ satisfying the conditions

$$\dot{\xi}(s) \geqslant \alpha,\ \xi(0) = 0,\ \xi(\tau_{z,u}) = \frac{\varrho}{2}$$

and such that the function

$$y(s) = z - \xi(s)u$$

is a solution of the differential inclusion $\dot{y} \in -F(y)$. The condition (ii) from Definition 2.1 implies now that the segment $[z, z + \varrho u/2]$ is contained in M.

We have thus proved that $w \in \mathrm{C}^{\circ}_{M}(x(t))$ and so the first inclusion from the assertion is true.

The second inclusion is an immediate consequence of the first due to the equality $\mathrm{C}^{\circ}_{\widehat{M}}(x(t)) = -\mathrm{C}^{\circ}_{M}(x(t))$ ([2], Proposition 4.5.9.).

To see an example where the hypertangent cone is a proper subset of the Dubovitsky-Miliutin cone at points belonging to a barrier, one can use F from Example 3.1, $M = \{(x_1, x_2, x_3);\ x_3 \leqslant |x_2|\}$ and a barrier solution defined by $x(t) = (t, 0, 0)$. Then we have

$$\mathrm{C}^{\circ}_{M}(x(t)) = \{(y_1, y_2, y_3)\colon\ y_3 < -|y_2|\},\ \mathrm{D}_M(x(t)) = \{(y_1, y_2, y_3)\colon\ y_3 < |y_2|\}.$$

Theorem 3.1 in connection with the equality

$$\mathrm{T}_{\mathrm{S}_{F(x(t))}}(v) = \operatorname{cl}\left(\mathrm{S}_{F(x(t))} + \mathbb{R} \cdot v\right)$$

(Lemma 4.2.5 [2]) implies

Corollary 3.1. *Under the assumptions of Theorem* 3.1 *the following inclusions hold for* $t \in (0,T]$*:*

$$-\operatorname{Int} \mathrm{S}_{F(x(t))} \subset \mathrm{C}^{\circ}_{M}(x(t)), \ \operatorname{Int} \mathrm{S}_{F(x(t))} \subset \mathrm{C}^{\circ}_{\widehat{M}}(x(t)). \tag{7}$$

It was mentioned before that for $\partial F(x)$ smooth the inclusion $\dot{x}(t) \in \partial\, \mathrm{S}_{F(x(t))}$ holds (for those t for which the derivative $\dot{x}(t)$ exists). Another corollary to Theorem 3.1 is the following generalization of this property—we do not use any smoothness assumption.

Theorem 3.2. *Let* $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ *be a Lipschitz multifunction,* $0 \notin F(x)$ *for* x *in an open set* $U \subset \Omega$ *and let* M *have semipermeable boundary in* U*. If* $x(\cdot)$ *is a barrier solution defined on* $[0,T]$ *then*

$$\mathcal{D}x(t) \subset \partial \mathrm{S}_{F(x(t))} \tag{8}$$

for all $t \in (0,T]$.

P r o o f. The solution $x(\cdot)$ does not exit M and so $\mathcal{D}x(t)) \subset \mathrm{T}_M(x(t))$ for all t. Suppose that for some $t > 0$ we have

$$v \in \mathcal{D}x(t) \setminus \partial \mathrm{S}_{F(x(t))}.$$

As $v \in F(x(t))$ (due to the inclusion $\mathcal{D}x(t) \subset F(x(t))$ true for all t—see for example [19] or [9]) so v must belong to $\operatorname{Int} \mathrm{S}_{F(x(t))}$. Taking into account the inclusion $\mathrm{C}^{\circ}_{\widehat{M}}(x(t)) \subset \mathrm{D}_{\widehat{M}}(x(t))$, the equality $\mathrm{D}_{\widehat{M}}(x(t)) = \mathbb{R}^d \setminus \mathrm{T}_M(x(t))$ and (7) we get $v \notin \mathrm{T}_M(x(t))$—this contradicts the previous statement and completes the proof. □

Corollary 3.2. *If* $v_1, v_2 \in \mathcal{D}x(t)$ *then*

$$(-\operatorname{Int} \mathrm{T}_{\mathrm{S}_{F(x(t))}}(v_1)) \cap \operatorname{Int} \mathrm{T}_{\mathrm{S}_{F(x(t))}}(v_2) = \emptyset.$$

It is implied by the equality

$$\mathrm{D}_M(x(t)) \cap \mathrm{D}_{\widehat{M}}(x(t))) = \emptyset$$

and the inclusions

$$-\operatorname{Int} \mathrm{T}_{\mathrm{S}_{F(x(t))}}(v_1) \subset \mathrm{D}_M(x(t)), \ \operatorname{Int} \mathrm{T}_{\mathrm{S}_{F(x(t))}}(v_2) \subset \mathrm{D}_{\widehat{M}}(x(t))$$

provided by Theorem 3.1.

A consequence of this corollary is the following property:

Corollary 3.3. *If some $v \in \mathcal{D}x(t)$ lies in the relative interior of an extremal face of $\mathrm{T}_{\mathrm{S}_{F(x(t))}}(v)$ of dimension $d-1$ then the whole contingent $\mathcal{D}x(t)$ is contained in that extremal face.*

4. Families of sets with semipermeable boundary.

We start with the unions of sets with semipermeable boundaries.

Proposition 4.1. *Let $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ be a Lipschitz multifunction and $\{M_\alpha\colon \alpha \in I\}$ a family of sets with semipermeable boundaries in an open set $U \subset \Omega$. Then $\partial \operatorname{cl}\big(\bigcup M_\alpha\big)$ is semipermeable in U.*

Proof. Remark first that if for some $\varepsilon > 0$ the inclusion $V + \varepsilon B_1 \subset U$ holds then there is $T > 0$ such that for every $\xi \in V$ and every solution $x(\cdot)$ of (1) with $x(0) = \xi$ this solution can be extended to $[0, T]$ (if it was not already defined on that interval) and every such solution on $[0, T]$ stays in U.

Take $\xi \in U \cap \partial \operatorname{cl}\big(\bigcup M_\alpha\big)$. There is a sequence $\xi_n \to \xi$ with $\xi_n \in M_{\alpha_n}$. Due to the semipermeability of ∂M_α and the previous remark we get for some $T > 0$ a family of solutions $x_n(\cdot) \in \mathrm{Sol}_F(\xi_n, T)$ such that $x_n(t) \in M_{\alpha_n}$. The sequence $x_n(\cdot)$ has a subsequence which is convergent to a solution $x(\cdot) \in \mathrm{Sol}_F(\xi, T)$ which satisfies $x(t) \in \operatorname{cl}\big(\bigcup M_\alpha\big)$—so (i) is justified.

Suppose now that (ii) is not true for $M = \operatorname{cl}\big(\bigcup M_\alpha\big)$. For some $\xi \in U \cap \operatorname{cl}\big(\bigcup M_\alpha\big)$ and $T > 0$ we would have then $x(\cdot) \in \mathrm{Sol}_{-F}(\xi, T)$ satisfying $x(t) \in U$ on $[0, T]$ and $x(s) \notin \operatorname{cl}\big(\bigcup M_\alpha\big)$ for some $s \in (0, T]$. We take a sequence $\xi_n \in \bigcup M_\alpha$ converging to ξ. In view of Filippov's theorem ([2], Theorem 10.4.1) there is a sequence $x_n(\cdot) \in \mathrm{Sol}_{-F}(\xi_n, T)$ convergent uniformly to $x(\cdot)$. By semipermeability of ∂M_{α_n} we have $x_n(s) \in M_{\alpha_n}$, which contradicts $x(s) \notin \operatorname{cl}\big(\bigcup M_\alpha\big)$ and completes the proof. □

For an arbitraty family M_α of sets with boundary semipermeable in U the intersection $U \cap \partial \operatorname{cl}\big(\bigcup M_\alpha\big)$ may be empty. If we consider families for which this does not happen it is possible to prove the existence of greatest set with semipermeable boundary like in the following theorem.

Theorem 4.1. *Suppose $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ is a Lipschitz multifunction, $U \subset \Omega$ open, $\xi \in U$ and $\mathcal{M}_\xi$ the family of sets M with semipermeable boundary in $U \subset \Omega$ for which $\xi \in \widehat{M}$. Then $\mathcal{M}_\xi$ has the greatest element with respect to inclusion.*

Using the proof of Lemma 1.1 in [3] one can show that for arbitrary nonempty subfamily of $\mathcal{M}_\xi$, if M is its union then $\xi \in \widehat{M}$. The remaining part of proof consists in applying in a standard way the Kuratowski-Zorn lemma.

We pass now to the discussion of semipermeability of the boundary of the intersection of sets with semipermeable boundaries. Remark first that the boundary of the intersection of even only two such sets may not be semipermeable—for example, the intersection $M_1 \cap M_2$ of sets defined in Example 2.1. In order to get the semipermeability of intersection we need the condition of monotonicity.

Proposition 4.2. *Assume $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ is a Lipschitz multifunction. Let I be a linearly ordered set and $\{M_\alpha\colon \alpha \in I\}$ a monotone family of sets with semipermeable boundaries in an open set $U \subset \Omega$. Then the intersection $\bigcap\limits_{\alpha\in I} M_\alpha$ has semipermeable boundary in U.*

P r o o f. Let $\xi \in \bigcap\limits_{\alpha\in I} M_\alpha$. There is $T > 0$ such that for every $\alpha \in I$ there is a solution $x_\alpha(\cdot) \in \mathrm{Sol}_F(\xi, T)$ such that $x_\alpha(t) \in M_\alpha$ for $t \in [0, T]$.

By Lindelöf's theorem there is a sequence $\alpha_n \in I$ such that

$$\bigcap_{\alpha\in I} M_\alpha = \bigcap_{n=1}^{\infty} M_{\alpha_n}.$$

Consider the sequence $x_{\alpha_n}(\cdot)$. It has a subsequence convergent to a solution $x(\cdot)$ which of course satisfies $x(t) \in \bigcap\limits_{n=1}^{\infty} M_{\alpha_n}$ on $[0, T]$—the condition (i) is thus proved.

It is obvious that $\bigcap\limits_{\alpha\in I} M_\alpha$ satisfies (ii)$'$ which is equivalent to (ii)—this completes the proof.

The intersection of a family of sets with semipermeable boundary in U may have no common points with U. But even if we restrict ourselves to a subfamily where this does not happen there may not exist the smallest with respect to inclusion set with semipermeable boundary. Take Example 2.1 mentioned above and the family of M such that the origin belongs to $\widehat{M}$. Then both M_1 and M_2 are minimal sets in this family and none of them is of course the smallest. The existence of minimal elements can be proved—Proposition 4.2 permits to prove by a standard reasoning, again using the Kuratowski-Zorn lemma, the following property.

Theorem 4.2. *Assume that $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ is a Lipschitz multifunction and $\mathcal{N}_\xi$ the family of sets M with semipermeable boundary in U for which $\xi \in M$. Then $\mathcal{N}_\xi$ has at least one minimal element with respet to inclusion.*

The topic of this section is the C^1-regularity of barrier solutions and barriers. Obviously, these closely connected notions are not equivalent—the trajectory of a C^1 function need not be a C^1 curve and also a C^1 curve may be a trajectory of a function which is not C^1. The section consists of two parts. First, we show how the domain of applicability of results in [3] and [4] can be enlarged using the notion of equivalent differential inclusions. Next, using a different approach, we prove a result which cannot be deduced from those papers.

5.1. Application of equivalent differential inclusions. Condition (4), equivalent to semipermeability under Lipschitz condition, suggests that semipermeability of surfaces should not be altered if we replace $F(\cdot)$ by another multifunction $G(\cdot)$ such that the sets $F(x)$ and $G(x)$ generate the same cones. As we define semipermeability through (i), (ii) in Definition 2.1 so we must have existence of solutions—some additional assumptions on G will be thus required.

Conditions (i), (ii), although formulated in terms of solutions, depend in fact only on their trajectories by which we mean images $\{x(t)\colon\ t \in [0,T]\}$ of solutions. We call two differential inclusions equivalent if they have the same families of trajectories. The following proposition is an obvious consequence

Proposition 5.1. *If two differential inclusions are equivalent then a surface is semipermeable with respect to one of them if and only if it is semipermeable with respect to the other.*

A condition on equivalence of differential inclusions given in [13] can be applied here. It requires the function

$$w(K,p) = \sup\Big(\Big\{\lambda > 0\colon\ \lambda \cdot \frac{p}{\|p\|} \in K\Big\} \cup \{0\}\Big)$$

defined for all closed sets $K \subset \mathbb{R}^d$ and $0 \neq p \in \mathbb{R}^d$. The following theorem is proved in [13].

Theorem 5.1. *Suppose $F, G\colon\ \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ are Borel measurable and there are positive constants c_1, c_2 such that if $w(F(x),p)$ and $w(G(x),p) < +\infty$ then*

$$w(F(x),p) \leqslant c_1 \cdot w(G(x),p),\ \ w(G(x),p) \leqslant c_2 \cdot w(F(x),p). \tag{9}$$

Assume also existence of $\eta > 0$ such that

$$\text{if } w(F(x),p) = +\infty \text{ then } w(G(x),p) \geqslant \eta,$$
$$\text{if } w(G(x),p) = +\infty \text{ then } w(F(x),p) \geqslant \eta.$$

Differential inclusions $\dot{x} \in F(x)$ and $\dot{x} \in G(x)$ are then equivalent.

It can be applied to prove the following theorem which is of interest for us.

Theorem 5.2. *Let a multifunction $F\colon \Omega \rightsquigarrow \mathbb{R}^d$ be Borel measurable and have closed convex values. We assume that for some $r > 0$ and all $x \in \Omega$ the intersection $rB_1 \cap F(x)$ is empty, $\partial \mathrm{S}_{F(x)} \setminus \{0\}$ is a C^1 manifold for all x and all faces of $\mathrm{S}_{F(x)}$ are half-lines (apart from $\mathrm{S}_{F(x)}$ itself). If in addition the multifunction $H(\cdot)$ defined as $H(x) = \mathrm{S}_{F(x)} \cap B_1$ is Lipschitzian then every barrier on an arbitrary semipermeable surface for (1) is a C^1 curve in $\mathbb{R}^d$.*

We underline that barrier solutions themselves need not be C^1—they are merely absolutely continuous.

We sketch now the proof of Theorem 5.2. It is done by showing that there is another differential inclusion $\dot{x} \in G(x)$ equivalent to (1) to which we can apply Corollary 2.2 in [3]. This corollary states that if $F(\cdot)$ is Lipschitzian, the sets $F(x)$ are strictly convex, compact and the boundaries $\partial F(x)$ are C^1 manifolds then any barrier solution is C^1. This and the fact that the derivative of barrier solutions here cannot be zero implies the regularity of the trajectory itself, i.e. of the barrier.

We shall need the notion of a Steiner point of a convex compact set $K \subset \mathbb{R}^d$—one of the possible definitions is by the integral formula [20]

$$s(K) = \frac{1}{\mathrm{vol}(B_1)} \int\limits_{S^{d-1}} p\sigma(p,K)\,\mathrm{d}\omega(p)$$

where ω is the Lebesgue measure on the unit sphere S^{d-1} in $\mathbb{R}^d$ and

$$\sigma(p,K) = \max\{\langle p,u\rangle\colon\ u \in K\}$$

is the support function of the set K. The important feature of a Steiner point is that its dependence on K is Lipschitzian with respect to the Hausdorff metric (see for example [15]).

For $0 \neq p \in \mathbb{R}^d$ put

$$\Pi_p = \{u \in \mathbb{R}^d\colon\ \langle p,u\rangle = \|p\|\}.$$

We introduce an auxiliary Lipschitzean multifunction

$$\Gamma(x) = \mathrm{S}_{F(x)} \cap \Pi_{s(H(x))}.$$

Note that for every $v \in \partial \mathrm{S}_{F(x)}$, $v \neq 0$, the intersection $\mathrm{S}_{\{v\}} \cap \Gamma(x)$ is a singleton. Moreover, $\Gamma(x)$ is strictly convex in $\Pi_{s(H(x))}$.

Using exactly the same method as in Lemma 4.1 in [18] one can define for every x a set $G(x)$ with boundaries $\partial G(x)$ being C^1, $\Gamma(x) \cap \partial \mathrm{S}_{F(x)} = G(x) \cap \partial \mathrm{S}_{F(x)}$ and $0 \notin G(x)$. Moreover, $G(\cdot)$ is a Lipschitzian multifunction equivalent to $F(\cdot)$.

5.2. Regularity of barrier solutions for multifunctions whose values have nonsmooth boundaries. We are now interested in the regularity of barrier solutions themselves, not only of their trajectories. The result given in the first part of this section made use of Corollary 2.2 in [3] by constructing equivalent differential inclusions to which it could be applied. The second part will be devoted to the case where the recourse to this theorem is not possible as no equivalent differential inclusion may in general satisfy its assumptions.

We sketch very shortly the two steps which lead to the proof of Corollary 2.2 in [3] and next discuss what and how can be done in a more general setting. So let $x(\cdot)$ be a barrier solution for M with semipermeable boundary.

First Step. (We mentioned it already before.) It is shown that when $F(\cdot)$ is a Lipschitz multifunction and $\partial F(x)$ are C^1 manifolds then to each barrier solution there corresponds an absolutely continuous map $p\colon [0,T] \to \mathbb{R}^d$, called the adjoint function, for which

$$\mathrm{T}_M(x(t)) = p(t)^- \tag{10}$$

holds for all $t \in (0,T)$ and $p(0)^- \subset \mathrm{T}_M(x(0))$. Moreover,

$$\mathrm{h}_F(x(t), p(t)) = \langle \dot{x}(t), p(t) \rangle = 0 \quad \text{a.e. in } (0,T). \tag{11}$$

Equality (10) shows a kind of regularity of ∂M along barriers—the contingent cone to M at every point which is crossed by a barrier solution is a half-space.

Second Step. Under the additional assumption of strict convexity of $F(x)$ the equality (11) implies that

$$\dot{x}(t) = \operatorname{Arg} \min_{v \in F(x(t))} \langle v, p(t) \rangle \quad \text{a.e. in } [0,T]. \tag{12}$$

Due to the continuity of $F(\cdot)$ and strict convexity of $F(x)$ the single-valued map

$$(x,p) \to \operatorname{Arg} \min_{v \in F(x)} \langle v, p \rangle$$

is continuous and thus

$$t \to \operatorname{Arg} \min_{v \in F(x(t))} \langle v, p(t) \rangle$$

is continuous and, as $x(\cdot)$ is obviously continuous, looking at (12) we see that $\dot{x}(\cdot)$ is equal almost everywhere to a function continuous everywhere, so they must coincide.

Remark first that strict convexity of $F(x)$ is essential for the regularity of barrier solutions. To be exact, it is not the strict convexity of the whole $F(x)$—as $\dot{x}(t)$ lies always on the intersection $F(x(t)) \cap \partial \mathrm{S}_{F(x(t))}$, only the strict convexity of the part of $F(x)$ which touches $\partial \mathrm{S}_{F(x)}$ is important. This can be put precisely in two conditions:
(a) for every $0 \neq v \in \partial \mathrm{S}_{F(x)}$ the intersection $F(x) \cap \mathrm{S}_{\{v\}}$ is a singleton.
(b) The only extremal faces of $\mathrm{S}_{F(x)}$ (apart from $\mathrm{S}_{F(x)}$ itself and $\{0\}$) are extremal rays.

The last property is equivalent to the strict convexity of $\mathrm{S}_{F(x)} \cap \Pi_{s(H(x))}$ in $\Pi_{s(H(x))}$.

To see why $x(\cdot)$ may be not C^1 when (a) is not true it is enough to analyze Example 2.1 with slighty modified $F(x)$ composed of points $(y_1, y_2) \in B((1,1), \frac{1}{2}(1+\sqrt{2}))$ with $y_1, y_2 \geqslant 0$. There are plenty of barrier solutions which are not C^1. For example all solutions of the initial value problem

$$\dot{x}_1 = 0,\ \dot{x}_2 \in [1-\alpha, 1+\alpha],\ x(0) = (0,0)$$

where $\alpha = \frac{1}{2}\sqrt{2\sqrt{2}-1}$, are barrier solutions for the semipermeable surface ∂M_1.

To see the role of (b) one can construct a similar example in $\mathbb{R}^3$ where (a) will be satisfied but (b) will not. F can be chosen to be constant, contained in the positive orthant and such that the intersection $\partial F(x) \cap M$ will be a whole segment Z with positive length, where $M = \{(y_1, y_2, y_3)\colon\ y_2 = y_3 = 0\}$ is semipermeable. Again, all solutions to the initial value problem $\dot{x} \in Z$, $x(0) = 0$, will be barrier solutions and many of them not of class C^1.

The smoothness of $\partial F(x)$ or even $\partial \mathrm{S}_{F(x)} \setminus \{0\}$ is not crucial for the smoothness of barrier solutions. In the remaining part of this section we show a possible way to prove the theorem on smoothness of barrier solutions in this situation. We follow the plan described in two steps above, i.e. we want to use continuous adjoint functions for which (12) proves the desired continuity of $\dot{x}(\cdot)$.

The absence of smoothness of $\partial F(x)$ does not permit to use the adjoint functions on the basis of results from [3]—the method of obtaining adjoint functions there is based in an essential way on that smoothness. To get them it was necessary to know that $\mathrm{T}_M(x(t))$ is a half-space along barrier solutions and $p(t)$ was chosen then according to (10). This is no more possible without smoothness of $\partial F(x)$ as shows the following example:

Example 5.1. We use $F(\cdot)$ from Example 3.1. The set $M = \{(x_1, x_2, x_3)\colon\ x_3 \leqslant -|x_2|\}$ has semipermeable boundary, $x(t) = (t, 0, 0)$ is a barrier solution and the cone $T_M(x(t))$ does not contain any half-space. The set $M = \{(x_1, x_2, x_3)\colon\ x_3 \leqslant |x_2|\}$ has also semipermeable boundary, the same $x(\cdot)$ is a boundary solution and $T_M(x(t))$ is not contained in any half-space. In the former case we have too many directions to be chosen as $p(t)$, in the latter case there is no one.

We show below how a theorem on existence of adjoint functions for time optimal solutions can be applied to our problem.

Let ∂M be semipermeable in U and suppose $x(\cdot)$ is a barrier solution, $x(0) = x_0$. As $x(t) \in \mathcal{A}(x_0, t) \subset \widehat{M}$ and $x(t) \in \partial M$ so $x(t) \in \partial\widehat{M}$ and thus $x(t) \in \partial\mathcal{A}(x_0, t)$. This implies that $x(\cdot)$ is a boundary solution for the initial value problem (1), (2) and we can try to find the adjoint function using this fact.

If $x(\cdot)$ is a time optimal solution then it is also a boundary solution—the inverse is not always true. However, each point $\xi \in \partial\mathcal{A}(t, x_0)$ is reached by at least one time optimal solution so among barrier solutions there are many that can also be treated as time optimal. One can use then a theorem on existence of adjoint functions to time optimal solutions (see for example [23], Theorem 7.4) to justify the following proposition.

Proposition 5.2. *Let $F(\cdot)$ be a Lipschitz multifunction with $F(x)$ strictly convex for all x. If a barrier solution is at the same time a time optimal solution then it is of class C^1.*

More detailed discussion of this and other ways of proving regularity of barrier solutions without regularity of $\partial F(x)$ will be included in another paper which is under preparation by the first author.

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