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SOME CHARACTERIZATIONS OF THE PRIMITIVE OF STRONG HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

Guoju Ye, Nanjing

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. In this paper we give some complete characterizations of the primitive of strongly Henstock-Kurzweil integrable functions which are defined on $\mathbb{R}^m$ with values in a Banach space.

Keywords: strong Henstock-Kurzweil integral, inner variation, SL condition

MSC 2000: 26A39

1. Introduction

It is well-known that the primitive F of a real-valued Henstock-Kurzweil integrable function f defined on a compact interval $[a,b] \subset \mathbb{R}$ is ACG*, and F is differentiable almost everywhere on $[a,b]$ and $F'(x) = f(x)$. The reverse implication also holds. That is, if a function F is ACG* and $F'(x) = f(x)$ almost everywhere on $[a,b]$, then f is Henstock-Kurzweil integrable on $[a,b]$ and F is the primitive of f. This fact is also valid for the strong Henstock-Kurzweil integral of Banach-space valued functions defined on 1-dimensional interval $[a,b]$, see [1, Theorem 4.5 in Chapter 7]. The question how to describe the primitive of a Banach-space valued Henstock-Kurzweil integrable function defined on a multidimensional interval $I_0 \subset \mathbb{R}^m$ arises naturally. However, the above well-known characterization of the 1-dimensional Henstock-Kurzweil integral in [1] relies heavily on the order structure of the real line, so it does not permit direct extension to the multidimensional Henstock-Kurzweil integral. Since the main

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tool in the proof of the above characterization on the real line is the Vitali covering theorem which requires regularity, we cannot succeed in higher-dimensional spaces. For the strong McShane integral of Banach-valued function defined on a higher-dimensional Euclidean space some full characterizations were given using variational measure in [1], [2]. In this paper, we first use the methods from [3], [5], [6] to discuss the SHK derivative of strong Henstock-Kurzweil integral, based on inner variation, then we make use of the derivative, inner variation and the SL (strong Lusin) condition in [3], [5] to give some complete characterizations of the primitive of a strongly Henstock-Kurzweil integrable function mapping an interval I_0 in $\mathbb{R}^m$ into a Banach space. This work is closely related to Section 5 in Chapter 7 of [1].

2. Basic definitions and theorems

Throughout this paper X will denote a real Banach space, I_0 is a compact interval in $\mathbb{R}^m$ and Σ is the family of subintervals of I_0. Let $I \subset I_0$, its Lebesgue measure being denoted by $\mu(I)$. For $x \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m)$, the norm $\|x\|$ is defined by $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_m|\}$. Given $\delta > 0$, $B(x, \delta)$ denotes the set $\{y \in \mathbb{R}^m \colon \ \|y - x\| < \delta\}$.

A partial partition D of I_0 is a finite family of interval-point pairs

$$D = \{(I_i, x_i);\ x_i \in I_i, i = 1, 2, \ldots, m\}$$

with the intervals non-overlapping, and their union a subset of I_0. If a partial partition D is such that the union of the intervals in D is I_0, then we call D a partition of I_0.

Given a positive function $\delta\colon\ I_0 \to (0, +\infty)$ (a gauge) an interval-point pair (I, x) is said to be δ-fine if $I \subset B(x, \delta(x))$. A partition D of I_0 is said to be δ-fine if each interval-point pair in D is δ-fine.

Let $f\colon\ I_0 \to X$ and $\delta\colon\ I_0 \to (0, +\infty)$. Let $D = \{(I_i, x_i)\}_{i=1}^m$ be a δ-fine partition of I_0. The Riemann sum corresponding to f and D is written as $\sum\limits_{i=1}^{m} f(x_i)\mu(I_i)$.

In the sequel, a partition $D = \{(I_i, x_i)\}_{i=1}^m$ will be often written as $D = \{(I, x)\}$ in which (I, x) represents the typical interval-point pair in D. The corresponding Riemann sum will be written shortly in the form $(D)\sum f(x)\mu(I)$.

Definition 2.1. A function $f\colon\ I_0 \to X$ is said to be *Henstock-Kurzweil integrable* on I_0 if there is an additive interval function F with the following property: for every $\varepsilon > 0$, there exists a gauge δ on I_0 such that

$$\Bigl\|(D)\sum[f(x)\mu(I) - F(I)]\Bigr\| < \varepsilon$$

for every δ-fine partition $D = \{(I,x)\}$ of I_0. The function F is called the *primitive* of f on I_0.

$F(I_0) = (\mathrm{HK}) \int_{I_0} f \, \mathrm{d}t$ is the Henstock-Kurzweil integral of f over I_0.

Definition 2.2. A function $f\colon I_0 \to X$ is said to be strongly Henstock-Kurzweil integrable on I_0 if f is Henstock-Kurzweil integrable on I_0 with the primitive F such that for every $\varepsilon > 0$ there exists a gauge δ on I_0 such that

$$(D) \sum \|f(x)\mu(I) - F(I)\| < \varepsilon$$

for every δ-fine partition $D = \{(I,x)\}$ of I_0.

We denote $F(I_0) = (\mathrm{SHK}) \int_{I_0} f \, \mathrm{d}t$ in this case.

Denote further by $\mathrm{SHK} = \mathrm{SHK}(I_0; X)$ the set of functions $f\colon I_0 \to X$ which are strongly Henstock-Kurzweil integrable on I_0.

An additive interval function F and a point function correspond in a straightforward way uniquely to each other (see [1]). So, if there is no confusion, we use the same symbol F for an additive interval function on Σ and also for the corresponding point function on I_0.

Now we introduce some notations and concepts using the ideas from [3], [5], [6].

For each positive function δ on I_0 and each real number $\eta > 0$, let $\Gamma(\delta, \eta)$ be a family of δ-fine interval-point pairs (I,x) with I a subinterval of I_0 and $x \in I_0$.

Assume that for a fixed δ we have $\Gamma(\delta, \eta_1) \subset \Gamma(\delta, \eta_2)$ if $\eta_2 \leqslant \eta_1$ and for a fixed η, $\Gamma(\delta_1, \eta) \subset \Gamma(\delta_2, \eta)$ if $\delta_1(x) \leqslant \delta_2(x)$. A family $\Gamma(\delta, \eta)$ is called an inner cover of $E \subset I_0$ if for each $x \in E$, there is at least one $(I,x) \in \Gamma(\delta, \eta)$.

Assume that for a fixed δ, $\Gamma(\delta, \eta)$ is an inner cover of $E \subset I_0$ if η is small enough. Let us introduce the following concept.

Definition 2.3. Let G be a Banach space-valued function defined on the family of all interval-point pairs (I,x) with $I \subset I_0$ and let E be a subset of I_0. Then E is said to be of inner G-variation zero with respect to $\Gamma(\delta, \eta)$ as given above if for each $\varepsilon > 0$ there exists a positive function δ such that for every δ-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E$ and $D \subset \Gamma(\delta, \eta)$, we have

$$(D) \sum_{x \in E} \|G(I,x)\| < \varepsilon.$$

If $G(I,x)$ represents the volume of I ($G(I,x) = \mu(I)$), then E is said to have inner variation zero with respect to $\Gamma(\delta, \eta)$.

It is obvious that if a set E is of measure zero then it is of inner variation zero with respect to $\Gamma(\delta, \eta)$, and the following propositions hold.

Proposition 2.1. *Let E be a subset of an interval $I_0 \subset \mathbb{R}^m$. If E is of inner variation zero with respect to $\Gamma(\delta,\eta)$, then any subset E' of E is of inner variation zero with respect to $\Gamma(\delta,\eta)$.*

Proposition 2.2. *Let $E_k, k = 1,2,\dots$ be a sequence of disjoint subsets of I_0 and let each E_k be of inner variation zero with respect to $\Gamma(\delta,\eta)$. Then $E = \bigcup\limits_{k=1}^{\infty} E_k$ is of inner variation zero with respect to $\Gamma(\delta,\eta)$.*

The proofs are trivial, we omit them.

3. Derivatives of strong Henstock-Kurzweil integrals

Definition 3.1. An interval function F on I_0 is said to be SHK differentiable at $x \in I_0$ with the SHK derivative $D_{\mathrm{SHK}}F(x)$ if for every $\varepsilon > 0$ there exists a gauge δ such that whenever (I,x) is δ-fine with $x \in I$, we have

$$\|F(I) - D_{\mathrm{SHK}}F(x)\mu(I)\| < \varepsilon\mu(I).$$

F is said to be SHK differentiable on I_0 if F is SHK differentiable at each point x in I_0.

Note that in fact the SHK derivative $D_{\mathrm{SHK}}F(x)$ is introduced by Henstock-Kurzweil interval-point pairs in the above Definitions 2.1–2.2. In order to discuss the derivatives of strong Henstock-Kurzweil integral, we need to specify $\Gamma(\delta,\eta)$ introduced above.

Let $f\colon I_0 \to X$ and let F be an X-valued interval function on I_0. For each $\delta(x) > 0$ and each $\eta > 0$, define

$$\Gamma(f,F,\delta,\eta) = \{(I,x);\ x \in I_0, \|F(I) - f(x)\mu(I)\| \geqslant \eta\mu(I) \text{ and } (I,x) \text{ is } \delta\text{-fine}\}. \tag{3.1}$$

Then $\Gamma(f,F,\delta,\eta)$ is a family of δ-fine interval-point pairs. From now on, we write $\Gamma(\delta,\eta)$ instead of $\Gamma(f,F,\delta,\eta)$ from (3.1) if it is obvious that we are discussing the case of fixed f and F; and we take the inner variation with respect to this specific family $\Gamma(\delta,\eta) = \Gamma(f,F,\delta,\eta)$, when we are discussing differentiation.

Let

$$E(f,F,\delta,\eta) = \{x \in I_0;\ \text{there exists } I \text{ such that } x \in I \text{ and } (I,x) \in \Gamma(f,F,\delta,\eta)\},$$
$$E(f,F) = \bigcup_{\eta}\bigcap_{\delta} E(f,F,\delta,\eta).$$

The set $E(f,F) \subset I_0$ consists of points x where $D_{\mathrm{SHK}}F(x) \neq f(x)$ or $D_{\mathrm{SHK}}F(x)$ does not exist, and while $\Gamma(f,F,\delta,\eta)$ need not be a Vitali cover of $E(f,F)$, but it is an inner cover and satisfies all conditions imposed on $\Gamma(\delta,\eta)$ mentioned above.

For convenience we denote $E(f,F)$ by E_0, i.e.,

$$E_0 = E(f,F) = \bigcup_{\eta}\bigcap_{\delta} E(f,F,\delta,\eta). \tag{3.2}$$

Theorem 3.1. *Let $f\colon I_0 \to X$ be a strongly Henstock-Kurzweil integrable function on I_0 with the primitive F. Then $D_{\mathrm{SHK}}F(x) = f(x)$ except at points of the set E_0 in* (3.2) *with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.*

Proof. We only need to prove that E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.

Let η be any positive real number and $\varepsilon > 0$. Since F is the primitive of the strongly Henstock-Kurzweil integrable function f, there is a gauge δ of I_0 such that for any δ-fine partition $D = \{(I,x)\}$ of I_0 we have

$$(D)\sum \|f(x)\mu(I) - F(I)\| < \varepsilon\cdot\eta.$$

Then for any δ-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta,\eta)$ we have by (3.1) the inequality

$$\eta(D)\sum \mu(I) < (D)\sum \|f(x)\mu(I) - F(I)\| < \varepsilon\eta.$$

So,

$$(D)\sum \mu(I) < \frac{1}{\eta}(D)\sum \|f(x)\mu(I) - F(I)\| < \frac{1}{\eta}\cdot\varepsilon\eta = \varepsilon.$$

The proof is complete. □

4. The primitive of strong Henstock-Kurzweil integral

In order to obtain a characterization of the primitive of a strongly Henstock-Kurzweil integrable function, we introduce the following concept.

Definition 4.1. An interval function F is said to satisfy the SL (strong Lusin) condition with respect to $\Gamma(\delta,\eta)$ on a set $E \subset I_0$ if for every $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E$ and $D \subset \Gamma(\delta,\eta)$, we have

$$(D)\sum \|F(I)\| < \varepsilon\mu(I).$$

Theorem 4.1. *Let $f\colon I_0 \to X$ be a strongly Henstock-Kurzweil integrable function on I_0 with the primitive F. Then for every $\eta > 0$ the function F satisfies the* SL *condition with respect to* $\Gamma(\delta,\eta)$ *on* E_0 *from* (3.2).

Proof. By Theorem 3.1 we know that E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$. Let

$$E_n = \{x \in E_0 \colon\ n-1 \leqslant \|f(x)\| < n\}, n = 1, 2, \ldots$$

Then $E_0 = \bigcup\limits_n E_n$. Since for every $\eta > 0$, E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$, E_n is of inner variation zero with respect to $\Gamma(\delta,\eta)$ by Proposition 2.1. That is, for every $\varepsilon > 0$ there is a gauge δ_n on E_n such that for any δ_n-fine partial partition $D_n = \{(I,x)\}$ with $x \in E_n$ and $D_n \subset \Gamma(\delta_n,\eta)$, we have

$$(D_n)\sum \mu(I) < \frac{\varepsilon}{2^{n+1}n}. \tag{4.1}$$

Since f is a strongly Henstock-Kurzweil integrable function, for given $\varepsilon > 0$ there is a gauge δ' of I_0 such that for any δ'-fine partition $D = \{(I,x)\}$ of I_0 we have

$$(D)\sum \|f(x)\mu(I) - F(I)\| < \frac{\varepsilon}{2}. \tag{4.2}$$

Define δ on I_0 as follows: $\delta(x) = \min\{\delta_n(x), \delta'(x)\}$ if $x \in E_n$, $n = 1, 2, \ldots$ and $\delta(x) = \delta'(x)$ if $x \in I_0 \setminus E_0$. Let $D = \{(I,x)\}$ be a δ-fine partial partition of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta,\eta)$ and $D_n = \{(I,x) \in D;\ x \in E_n\}$. Then (4.1) holds for this $D_n = \{(I,x)\}$, therefore, by (4.1) and (4.2), we obtain

$$\begin{aligned}(D)\sum\|F(I)\| &\leqslant (D)\sum\|F(I) - f(x)\mu(I)\| + (D)\sum\|f(x)\mu(I)\| \\ &< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}(D_n)\sum\|f(x)\|\mu(I) \\ &< \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}\cdot n}\cdot n = \varepsilon.\end{aligned} \tag{4.3}$$

That is, F satisfies the SL condition with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ on E_0 and the proof is complete. $\square$

Theorem 4.2. *Let $f\colon I_0 \to X$ and let F be an additive interval function on Σ. If $D_{\rm SHK}F(x) = f(x)$ except at points of the set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ and F satisfies the* SL *condition with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ on E_0, then the function f is strongly Henstock-Kurzweil integrable on I_0 with the primitive F.*

Proof. Let $\varepsilon > 0$ be an arbitrary real number. Suppose $\eta < \varepsilon/2\mu(I_0)$. If $x \in I_0 \setminus E_0$ then F is differentiable at x and its derivative is $f(x)$. Hence there is a positive function $\delta_0(x)$ on $I_0 \setminus E_0$ such that

$$\|F(I) - f(x)\mu(I)\| < \eta\mu(I) \tag{4.4}$$

whenever (I,x) is δ_0-fine.

On the other hand, let

$$E_n = \{x \in E_0\colon\ n-1 \leqslant \|f(x)\| < n\}, n = 1,2,\ldots.$$

Since E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$, then for each $n \in \mathbb{N}$, E_n is of inner variation zero with respect to $\Gamma(\delta,\eta)$. So, for given $\varepsilon > 0$, there exists a positive function δ_n on E_n, $n = 1,2,\ldots$, such that

$$(D)\sum \mu(I) < \frac{\varepsilon}{2^{n+1}n} \tag{4.5}$$

for any δ_n-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_n$ and $D \subset \Gamma(\delta_n,\eta)$. On E_0, we define $\delta'(x) = \delta_n(x)$ if $x \in E_n$, $n = 1,2,\ldots$; then for any δ'-fine partial partition $D = \{(I,x)\}$ with $x \in E_0$ and $D \subset \Gamma(\delta',\eta)$, we have by (4.5)

$$(D)\sum \|f(x)\|\mu(I) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}n} n = \varepsilon. \tag{4.6}$$

Recall that

$$\|F(I) - f(x)\mu(I)\| \geqslant \eta\mu(I)$$

for all $(I,x) \in D \subset \Gamma(\delta',\eta)$ with $x \in E_0$. Suppose (I,x) is $\delta'(x)$-fine with $x \in E_0$ and $(I,x) \notin \Gamma(\delta',\eta)$, then

$$\|F(I) - f(x)\mu(I)\| < \eta\mu(I). \tag{4.7}$$

Since F satisfies the SL condition with respect to $\Gamma(\delta,\eta)$ on E_0, there exists a gauge δ'' on E_0 such that

$$(D)\sum \|F(I)\| < \varepsilon \tag{4.8}$$

for any δ''-fine partial partition $D = \{(I,x)\}$ of I_0 with $x \in E_0$ and $D \subset \Gamma(\delta'',\eta)$.

Now we define a $\delta(x)$ on I_0 as follows: $\delta(x)=\delta_0(x)$ if $x\in I_0\setminus E_0$, and $\delta(x)=\min\{\delta'(x),\delta''(x)\}$ if $x\in E_0$. Then for any δ-fine partition $D=\{(I,x)\}$ of I_0, by (4.4),

$$(D)\sum_{x\in I_0\setminus E_0}\|f(x)\mu(I)-F(I)\|<\eta\cdot(D)\sum_{x\in I_0\setminus E_0}\mu(I)\leqslant\eta|I_0|. \tag{4.9}$$

On the other hand, a δ-fine partial partition $D=\{(I,x)\}$ of I_0 with all $x\in E_0$ can be decomposed into D' and D'', where

$$D'=\{(I,x)\in D;\ x\in E_0,(I,x)\notin\Gamma(\delta,\eta)\},$$
$$D''=\{(I,x)\in D;\ x\in E_0,(I,x)\in\Gamma(\delta,\eta)\}.$$

Then D' satisfies (4.7) and D'' satisfies (4.6) and (4.8). Thus

$$(D')\sum\|f(x)\mu(I)-F(I)\|<\eta|I_0|, \tag{4.10}$$

$$(D'')\sum\|f(x)\|\mu(I)<\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}n}\,n=\varepsilon \tag{4.11}$$

and

$$(D'')\sum\|F(I)\|<\varepsilon. \tag{4.12}$$

Hence, by (4.9)–(4.12), we have

$$\begin{aligned}(D)\sum\|f(x)\mu(I)-F(I)\| &= (D)\sum_{x\in I_0\setminus E_0}\|f(x)\mu(I)-F(I)\|+(D')\sum_{x\in E_0}\|f(x)\mu(I)-F(I)\|\\ &\quad+(D'')\sum_{x\in E_0}\|f(x)\mu(I)-F(I)\|\\ &\leqslant 2\eta|I_0|+(D'')\sum_{x\in E_0}\|f(x)\|\mu(I)+(D'')\sum_{x\in E_0}\|F(I)\|\\ &<\varepsilon+\varepsilon+\varepsilon=3\varepsilon.\end{aligned} \tag{4.13}$$

Therefore f is a strongly Henstock-Kurzweil integrable function on I_0 and F is its primitive. □

By Theorems 3.1, 4.1 and 4.2 we obtain the following complete characterization of the primitive of a strong Henstock-Kurzweil integrable function mapping an interval I_0 in $\mathbb{R}^m$ into a Banach space X.

Theorem 4.3. *A function $f\colon I_0 \to X$ is strongly Henstock-Kurzweil integrable on I_0 if and only if there exists an additive interval function F such that $D_{\rm SHK}F(x) = f(x)$ except at points of the set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$, and F satisfies the* SL *condition on E_0 with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$.*

To establish another characterization of the primitive of a strongly Henstock-Kurzweil integrable function, we recall the concept ACG_δ^{**} in [7], [8], which can also be found in [1].

Definition 4.2 [1], [7], [8]**.** Let I_0 be an interval in $\mathbb{R}^m$ and $M \subset I_0$. An interval function F defined on Σ is said to be $AC_\delta^{**}(M)$ if for every $\varepsilon > 0$ there exist a gauge $\delta\colon I_0 \to (0,\infty)$ and $\eta > 0$ such that for any two δ-fine partitions $D_1 = \{(t_i, I_i)\}$, $D_2 = \{(s_j, J_j)\}$ with tags $t_i, s_j \in M$ such that any interval J_j lies in some interval I_i, we have

$$\sum_{D_1 \setminus D_2} \mu(I) < \eta \Longrightarrow \sum_{D_1 \setminus D_2} \|F(I)\|_X < \varepsilon$$

where $D_1 \setminus D_2 = \{(t_i, I_i \setminus \bigcup\limits_{j, J_j \subset I_i} J_j)\}$. If $I = I_i \setminus \bigcup\limits_{j, J_j \subset I_i} J_j$ then $F(I) = F(I_i \setminus \bigcup\limits_{j, J_j \subset I_i} J_j) = F(I_i) - \sum\limits_{j, J_j \subset I_i} F(J_j)$ and $\mu(I) = \mu(I_i \setminus \bigcup\limits_{j, J_j \subset I_i} J_j) = \mu(I_i) - \sum\limits_{j, J_j \subset I_i} \mu(J_j)$.

Furthermore, F is $\mathrm{ACG}_\delta^{**}(I_0)$ if $I_0 = \bigcup\limits_{i=1}^{\infty} M_i$ and F is $AC_\delta^{**}(M_i)$ for each $i \in \mathbb{N}$.

It is known that the primitive F of a strongly Henstock-Kurzweil integrable function f is ACG_δ^{**} (see [7]). Further, we prove the following theorem.

Theorem 4.4. *Let $f\colon I_0 \to X$ and let an additive $\mathrm{ACG}_\delta^{**}(I_0)$ interval function F be given.*

If $D_{\rm SHK}F(x) = f(x)$ except at points of a set E_0 with inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$, then f is a strongly Henstock-Kurzweil integrable function on I_0 and F is its primitive.

Proof. Let $\varepsilon > 0$ be an arbitrary real number. Suppose $\eta < \varepsilon/2\mu(I_0)$.

Since $D_{\rm SHK}F(x) = f(x)$ for each $x \in I_0 \setminus E_0$, there is a gauge $\delta_{I_0\setminus E_0}$ on $I_0 \setminus E_0$ such that for any $\delta_{I_0\setminus E_0}$-fine partial partition $D_{I_0\setminus E_0} = \{(I,x)\}$ of I_0 with $x \in I_0 \setminus E_0$, we have

$$(D_{I_0\setminus E_0}) \sum \|f(x)\mu(I) - F(I)\| < \eta |I_0|. \tag{4.14}$$

Since F is $\mathrm{ACG}_\delta^{**}(I_0)$, $I_0 = \bigcup\limits_{i=1}^{\infty} E_i$ and F is $AC_\delta^{**}(E_i)$. We assume that $E_i \cap E_j = \emptyset$ for any $i \neq j$. Then for any given $\varepsilon > 0$ there is a gauge $\tilde{\delta}_i$ on each E_i and

$0 < \eta_i \leqslant \varepsilon 2^{-i}$ $\left(\sum\limits_{i=1}^{\infty} \eta_i \leqslant \varepsilon\right)$, such that for any $\tilde{\delta}_i$-fine partial partition $D_i = \{(I,x)\}$ of I_0 with $x \in E_i$, we have

$$(4.15) \qquad (D_i)\sum \mu(I) < \eta_i \Rightarrow (D_i)\sum \|F(I)\| < \frac{\varepsilon}{2^i}.$$

Let $X_i = E_0 \cap E_i$, $Y_n = \{x \in I_0\colon\ n-1 \leqslant \|f(x)\| < n\}$ and $X_{in} = X_i \cap Y_n$. Then

$$(4.16) \qquad E_0 = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty} X_{in} \text{ and also } E_0 = \bigcup_{n=1}^{\infty}(E_0 \cap Y_n).$$

Since E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$, each X_{in} is of inner variation zero with respect to $\Gamma(\delta,\eta)$. So, for $\eta_{in} = \eta_i/n2^n$, there is a $\delta_{in} \leqslant \tilde{\delta}_i$ on X_{in} such that for any δ_{in}-fine partial partition $D_{in} = \{(I,x)\}$ of I_0 with $x \in X_{in}$ and $D_{in} \subset \Gamma(\delta_{in},\eta)$, we have

$$(4.17) \qquad (D_{in})\sum \mu(I) < \frac{\eta_i}{n2^n}.$$

Now define a gauge δ_{E_0} on E_0 as follows: $\delta_{E_0}(x) = \delta_{in}(x)$ if $x \in X_{in}$, $i,n = 1,2,\ldots$. Then for any δ_{E_0}-fine partial partition $D_i = \{(I,x)\}$ of I_0 with $x \in X_i$ and $D_i \subset \Gamma(\delta_{E_0},\eta)$, by (4.16) and (4.17), we have

$$(4.18) \qquad (D_i)\sum \mu(I) = \sum_{n=1}^{\infty}(D_i)\sum_{x\in X_{in}} \mu(I) < \sum_{n=1}^{\infty}\frac{\eta_i}{2^n} = \eta_i$$

and by (4.15) and (4.18), we obtain

$$(4.19) \qquad (D_i)\sum \|F(I)\| < \frac{\varepsilon}{2^i}.$$

Let us now define a gauge $\delta(x)$ on I_0 in the following way: $\delta(x) = \delta_{E_0}(x)$ if $x \in E_0$ and $\delta(x) = \min\{\delta_{I_0\setminus E_0}(x), \tilde{\delta}_i(x)\}$ if $x \in (I_0 \setminus E_0) \cap E_i$, $i = 1,2,\ldots$. Then for any δ-fine partition $D = \{(I,x)\}$ of I_0 (similarly to the proof of (4.13)), by (4.14), (4.17) and (4.19), we conclude

$$
\begin{aligned}
&(D)\sum \|f(x)\mu(I)-F(I)\|\\
&\quad = (D)\sum_{x\in I_0\setminus E_0}\|f(x)\mu(I)-F(I)\| + (D)\sum_{x\in E_0}\|f(x)\mu(I)-F(I)\|\\
&\quad \leqslant 2(D)\sum_{x\in I_0\setminus E_0}\eta\mu(I) + (D)\sum_{x\in E_0,(I,x)\in\Gamma(\delta,\eta)}\|f(x)\|\mu(I)\\
&\qquad + (D)\sum_{x\in E_0,(I,x)\in\Gamma(\delta,\eta)}\|F(I)\|\\
&\quad \leqslant 2\eta|I_0| + \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}(D)\sum_{x\in X_{in},(I,x)\in\Gamma(\delta,\eta)}\|f(x)\|\mu(I) + \sum_{i=1}^{\infty}(D)\sum_{x\in X_i}\|F(I)\|\\
&\quad < \varepsilon + \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} n\frac{\eta_i}{n2^n} + \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}\\
&\quad < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon.
\end{aligned}
$$

The proof is complete. $\square$

By Theorem 4.4 and Theorem 4.1 of [7], it is easy to obtain the following theorem giving another complete characterization of the primitive of strongly Henstock-Kurzweil integrable functions.

Theorem 4.5. *Let $f\colon I_0 \to X$, and let F be an additive interval function defined on Σ. Let E_0 be as in (3.2). Then f is strongly Henstock-Kurzweil integrable on I_0 with the primitive F if and only if E_0 is of inner variation zero with respect to $\Gamma(\delta,\eta)$ for every $\eta > 0$ and F is ACG_δ^{**} on I_0.*

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Author's address: *Guoju Ye*, College of Science, Hohai University, Nanjing, 210098, P. R. China, e-mail: yegj@hhu.edu.cn.