# ASYMPTOTIC PROPERTIES OF AN UNSTABLE TWO-DIMENSIONAL DIFFERENTIAL SYSTEM WITH DELAY

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(Received December 20, 2005)

*Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday*

*Abstract.* The asymptotic behaviour of the solutions is studied for a real unstable two-dimensional system $x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(t-r) + h(t, x(t), x(t-r))$, where $r > 0$ is a constant delay. It is supposed that $\mathbf{A}$, $\mathbf{B}$ and $h$ are matrix functions and a vector function, respectively. Our results complement those of Kalas [Nonlinear Anal. 62(2) (2005), 207–224], where the conditions for the existence of bounded solutions or solutions tending to the origin as $t \to \infty$ are given. The method of investigation is based on the transformation of the real system considered to one equation with complex-valued coefficients. Asymptotic properties of this equation are studied by means of a suitable Lyapunov-Krasovskii functional and by virtue of the Ważewski topological principle. Stability and asymptotic behaviour of the solutions for the stable case of the equation considered were studied in Kalas and Baráková [J. Math. Anal. Appl. 269(1) (2002), 278–300].

*Keywords*: delayed differential equation, asymptotic behaviour, boundedness of solutions, two-dimensional systems, Lyapunov method, Ważewski topological principle

*MSC 2000*: 34K12

## 1. Introduction

Consider the real two-dimensional system

$$x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(t-r) + h(t, x(t), x(t-r)), \tag{0}$$

where $\mathbf{A}(t) = (a_{jk}(t))$, $\mathbf{B}(t) = (b_{jk}(t))$ $(j, k = 1, 2)$ are real square matrices and $h(t, x, y) = (h_1(t, x, y), h_2(t, x, y))$ is a real vector function, $x = (x_1, x_2)$, $y = (y_1, y_2)$. It is supposed that the functions $a_{jk}$ are locally absolutely continuous on $[t_0, \infty)$, $b_{jk}$ are locally Lebesgue integrable on $[t_0, \infty)$ and the function $h$ satisfies the

Carathéodory conditions on $[t_0,\infty)\times\mathbb{R}^4$. Moreover, we suppose the uniqueness property for solutions of (0). Stability and asymptotic properties of the solutions for the stable case of (0) are investigated in [2]. The unstable case of (0) was studied in [1]. In [2], it was shown that it is useful to investigate (0) also under different conditions, namely the conditions, when the shortened equation $x'(t)=\mathbf{A}(t)x(t)$ is closer to a "focus" than to a "node" at origin. In the present paper we examine (0) under these assumptions.

The method of investigation is based on the transformation of (0) to an equation with complex conjugate coordinates and on the use of a convenient Lyapunov-Krasovskii functional. This method allows to simplify some considerations and estimations and, in the two-dimensional case, leads to new, effective and easily applicable results. The key tool will be a Razumikhin-type version of Ważewski topological method. Similarly to [1], we shall concentrate considerable attention to the problem of existence of bounded solutions or solutions tending to the origin as $t\to\infty$. Related results for ordinary differential equations without delay can be found in [7] and [3]. Notice that the Razumikhin-type version of Ważewski principle for retarded functional differential equations was formulated by K. P. Rybakowski [8], [9]. Observe that complex differential systems were used also by other authors for the solution of different problems related to differential equations, let us mention here papers of J. Mawhin [4] and of R. Manásevich, J. Mawhin, F. Zanolin [5], [6].

Introducing complex variables $z=x_1+\mathrm{i}x_2$, $w=y_1+\mathrm{i}y_2$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t)=a(t)z(t)+b(t)\bar{z}(t)+A(t)z(t-r)+B(t)\bar{z}(t-r)+g(t,z(t),z(t-r)), \tag{1}$$

where

$$\begin{aligned}
a(t)&=\frac{1}{2}(a_{11}(t)+a_{22}(t))+\frac{\mathrm{i}}{2}(a_{21}(t)-a_{12}(t)),\\
b(t)&=\frac{1}{2}(a_{11}(t)-a_{22}(t))+\frac{\mathrm{i}}{2}(a_{21}(t)+a_{12}(t)),\\
A(t)&=\frac{1}{2}(b_{11}(t)+b_{22}(t))+\frac{\mathrm{i}}{2}(b_{21}(t)-b_{12}(t)),\\
B(t)&=\frac{1}{2}(b_{11}(t)-b_{22}(t))+\frac{\mathrm{i}}{2}(b_{21}(t)+b_{12}(t)),\\
g(t,z,w)&=h_1\Big(t,\frac{1}{2}(z+\bar{z}),\frac{1}{2\mathrm{i}}(z-\bar{z}),\frac{1}{2}(w+\overline{w}),\frac{1}{2\mathrm{i}}(w-\overline{w})\Big)\\
&\quad+\mathrm{i}h_2\Big(t,\frac{1}{2}(z+\bar{z}),\frac{1}{2\mathrm{i}}(z-\bar{z}),\frac{1}{2}(w+\overline{w}),\frac{1}{2\mathrm{i}}(w-\overline{w})\Big).
\end{aligned}$$

Conversely, putting $a_{11}(t)=\operatorname{Re}[a(t)+b(t)]$, $a_{12}(t)=\operatorname{Im}[b(t)-a(t)]$, $a_{21}(t)=\operatorname{Im}[a(t)+b(t)]$, $a_{22}(t)=\operatorname{Re}[a(t)-b(t)]$, $b_{11}(t)=\operatorname{Re}[A(t)+B(t)]$, $b_{12}(t)=\operatorname{Im}[B(t)-$

$A(t)]$, $b_{21}(t) = \operatorname{Im}[A(t)+B(t)]$, $b_{22}(t) = \operatorname{Re}[A(t)-B(t)]$, $h_1(t,x,y) = \operatorname{Re} g(t, x_1 + \mathrm{i}x_2, y_1 + \mathrm{i}y_2)$, $h_2(t,x,y) = \operatorname{Im} g(t, x_1 + \mathrm{i}x_2, y_1 + \mathrm{i}y_2)$, $\mathbf{A}(t) = (a_{ij}(t))$, $\mathbf{B}(t) = (b_{ij}(t))$, the equation (1) can be written in the real form (0).

We shall use the following notation:

$\mathbb{R}$ set of all real numbers,
$\mathbb{R}_+$ set of all positive real numbers,
$\mathbb{R}_+^0$ set of all non-negative real numbers,
$\mathbb{R}_-$ set of all negative real numbers,
$\mathbb{R}_-^0$ set of all non-positive real numbers,
$\mathbb{C}$ set of all complex numbers,
$\mathcal{C}$ class of all continuous functions $[-r,0] \to \mathbb{C}$,
$AC_{\mathrm{loc}}(I,M)$ class of all locally absolutely continuous functions $I \to M$,
$L_{\mathrm{loc}}(I,M)$ class of all locally Lebesgue integrable functions $I \to M$,
$K(I \times \Omega, M)$ class of all functions $I \times \Omega \to M$ satisfying the Carathéodory conditions on $I \times \Omega$,
$\operatorname{Re} z$ real part of $z$,
$\operatorname{Im} z$ imaginary part of $z$,
$\bar{z}$ complex conjugate of $z$.

## 2. Results

Consider the equation

$$(1) \qquad z'(t) = a(t)z(t) + b(t)\bar{z}(t) + A(t)z(t-r) + B(t)\bar{z}(t-r) + g(t, z(t), z(t-r)),$$

where $r > 0$ is a constant, $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $A, B \in L_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \mathbb{C}^2, \mathbb{C})$, $J = [t_0, \infty)$. Throughout the paper we shall suppose that (1) satisfies the uniqueness property of solutions. The equation (1) can be written in the form

$$(1') \qquad z' = F(t, z_t),$$

where $F\colon J \times \mathcal{C} \to \mathbb{C}$ is defined by

$$F(t,\psi) = a(t)\psi(0) + b(t)\overline{\psi}(0) + A(t)\psi(-r) + B(t)\overline{\psi}(-r) + g(t, \psi(0), \psi(-r))$$

and $z_t$ is an element of $\mathcal{C}$ defined by the relation $z_t(\theta) = z(t+\theta)$, $\theta \in [-r, 0]$. Instead of the case $\liminf\limits_{t\to\infty}(|a(t)| - |b(t)|) > 0$ investigated in [1], we will consider a case

$$\liminf_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0.$$

The last inequality is equivalent to the existence of $T \geqslant t_0 + r$ and $\mu > 0$ such that

$$|\operatorname{Im} a(t)| > |b(t)| + \mu \quad \text{for } t \geqslant T - r. \tag{2}$$

Denote

$$\tilde{\gamma}(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\,\operatorname{sgn}(\operatorname{Im} a(t)), \qquad \tilde{c}(t) = -\mathrm{i}b(t). \tag{3}$$

As $|\tilde{\gamma}(t)| > |\operatorname{Im} a(t)|$ and $|\tilde{c}(t)| = |b(t)|$, the inequality

$$|\tilde{\gamma}(t)| > |\tilde{c}(t)| + \mu \tag{4}$$

holds for $t \geqslant T - r$. It can be easily verified that $\tilde{\gamma}, \tilde{c} \in AC_{\text{loc}}([T - r, \infty), \mathbb{C})$. Notice that, instead of the function $\gamma$ from [1], the function $\tilde{\gamma}$ need not be positive. A simple example ensuing Theorem 1 shows that, in some cases, our results can be applicable more often than those given in [1].

The equation (1) will be studied subject to suitable subsets of the following assumptions:

(i) The numbers $T \geqslant t_0 + r$ and $\mu > 0$ are such that (2) holds.

(ii) There exist functions $\varkappa, \kappa, \varrho\colon [T, \infty) \to \mathbb{R}$ such that

$$\begin{aligned}|\tilde{\gamma}(t)g(t, z, w) + \tilde{c}(t)\overline{g}(t, z, w)| \leqslant \varkappa(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\overline{z}|\\ + \kappa(t)|\tilde{\gamma}(t - r)w + \tilde{c}(t - r)\overline{w}| + \varrho(t)\end{aligned}$$

for $t \geqslant T$, $z, w \in \mathbb{C}$, where $\varrho$ is continuous on $[T, \infty)$.

($\mathrm{ii_n}$) There exist numbers $R_n \geqslant 0$ and functions $\varkappa_n, \kappa_n\colon [T, \infty) \to \mathbb{R}$ such that

$$|\tilde{\gamma}(t)g(t, z, w) + \tilde{c}(t)\overline{g}(t, z, w)| \leqslant \varkappa_n(t)|\tilde{\gamma}(t)z + \tilde{c}(t)\overline{z}| + \kappa_n(t)|\tilde{\gamma}(t - r)w + \tilde{c}(t - r)\overline{w}|$$

for $t \geqslant \tau_n \geqslant T$, $|z| > R_n$, $|w| > R_n$.

(iii) $\beta \in AC_{\text{loc}}([T, \infty), \mathbb{R}^0_-)$ is a function satisfying

$$\beta(t) \leqslant -\lambda(t) \quad \text{a.e. on } [T, \infty), \tag{5}$$

where $\lambda$ is defined by

$$\lambda(t) = \kappa(t) + (|A(t)| + |B(t)|)\frac{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}{|\tilde{\gamma}(t - r)| - |\tilde{c}(t - r)|} \tag{6}$$

for $t \geqslant T$.

(iii$_\mathrm{n}$) $\beta_n \in AC_{\mathrm{loc}}([T,\infty), \mathbb{R}^0_-)$ is a function satisfying

$$\beta_n(t) \leqslant -\lambda_n(t) \quad \text{a.e. on } [\tau_n, \infty), \tag{7}$$

where $\lambda_n$ is defined by

$$\lambda_n(t) = \kappa_n(t) + (|A(t)| + |B(t)|)\frac{|\tilde\gamma(t)| + |\tilde c(t)|}{|\tilde\gamma(t-r)| - |\tilde c(t-r)|} \tag{8}$$

for $t \geqslant T$.

(iv$_\mathrm{n}$) $\widetilde{\varLambda}_n\colon [T,\infty) \to \mathbb{R}$ is a locally Lebesgue integrable function satisfying the inequalities $\beta_n'(t) \geqslant \widetilde{\varLambda}_n(t)\beta_n(t)$, $\widetilde{\varTheta}_n(t) \geqslant \widetilde{\varLambda}_n(t)$ for almost all $t \in [\tau_n,\infty)$, where $\widetilde{\varTheta}_n$ is defined by (9).

Obviously, if $A$, $B$, $\kappa$ are locally absolutely continuous on $[T,\infty)$ and $\lambda(t) \geqslant 0$, the choice $\beta(t) = -\lambda(t)$ is admissible in (iii). Similarly, if $A$, $B$, $\kappa_n$ are locally absolutely continuous on $[T,\infty)$ and $\lambda_n(t) \geqslant 0$, the choice $\beta_n(t) = -\lambda_n(t)$ is admissible in (iii$_\mathrm{n}$).

Throughout the paper we denote

$$\begin{aligned} \tilde\vartheta(t) &= \frac{\operatorname{Re}(\tilde\gamma(t)\tilde\gamma'(t) - \bar{\tilde c}(t)\tilde c'(t)) - |\tilde\gamma(t)\tilde c'(t) - \tilde\gamma'(t)\tilde c(t)|}{\tilde\gamma^2(t) - |\tilde c(t)|^2}, \\ \widetilde{\varTheta}(t) &= \operatorname{Re} a(t) + \tilde\vartheta(t) - \varkappa(t), \\ \widetilde{\varTheta}_n(t) &= \operatorname{Re} a(t) + \tilde\vartheta(t) - \varkappa_n(t) + \beta_n(t). \end{aligned} \tag{9}$$

The assumption (i) implies that

$$\begin{aligned} |\tilde\vartheta| &\leqslant \frac{|\operatorname{Re}(\tilde\gamma\tilde\gamma' - \bar{\tilde c}\tilde c')| + |\tilde\gamma\tilde c' - \tilde\gamma'\tilde c|}{\tilde\gamma^2 - |\tilde c|^2} \leqslant \frac{(|\tilde\gamma'| + |\tilde c'|)(|\tilde\gamma| + |\tilde c|)}{\tilde\gamma^2 - |\tilde c|^2} \\ &= \frac{|\tilde\gamma'| + |\tilde c'|}{|\tilde\gamma| - |\tilde c|} \leqslant \frac{1}{\mu}(|\tilde\gamma'| + |\tilde c'|), \end{aligned}$$

therefore the function $\tilde\vartheta$ is locally Lebesgue integrable on $[T,\infty)$ under this assumption. If relations $\beta_n \in AC_{\mathrm{loc}}([T,\infty), \mathbb{R}_-)$, $\varkappa_n \in L_{\mathrm{loc}}([T,\infty), \mathbb{R})$ and $\beta_n'(t)/\beta_n(t) \leqslant \widetilde{\varTheta}_n(t)$ for almost all $t \geqslant \tau_n$ together with the conditions (i), (ii$_\mathrm{n}$) are satisfied, then we can choose $\widetilde{\varLambda}_n(t) = \widetilde{\varTheta}_n(t)$ for $t \in [T,\infty)$ in (iv$_\mathrm{n}$).

In the proof of Theorem 1 below, we shall need

**Lemma 1.** *Let $a_1$, $a_2$, $b_1$, $b_2 \in \mathbb{C}$, $|a_2| > |b_2|$. Then*

$$\operatorname{Re}\frac{a_1 z + b_1\bar z}{a_2 z + b_2 \bar z} \geqslant \frac{\operatorname{Re}(a_1\bar a_2 - b_1\bar b_2) - |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

*for $z \in \mathbb{C}$, $z \neq 0$.*

The proof is similar to that of Lemma in [7], p. 131.

**Theorem 1.** *Let the assumptions* (i), (ii$_0$), (iii$_0$), (iv$_0$) *be satisfied for some* $\tau_0 \geqslant T$. *Suppose there exist* $t_1 \geqslant \tau_0$ *and* $\nu \in (-\infty, \infty)$ *such that*

$$\inf_{t \geqslant t_1} \left[ \int_{t_1}^{t} \widetilde{\Lambda}_0(s)\,\mathrm{d}s - \ln(|\tilde{\gamma}(t)| + |\tilde{c}(t)|) \right] \geqslant \nu. \tag{10}$$

*If* $z(t)$ *is any solution of* (1) *satisfying*

$$\min_{s \in [t_1 - r, t_1]} |z(s)| > R_0, \qquad \Delta(t_1) > R_0 \mathrm{e}^{-\nu}, \tag{11}$$

*where* $\Delta(t) = (|\tilde{\gamma}(t)| - |\tilde{c}(t)|)|z(t)| + \beta_0(t) \max_{s \in [t-r, t]} |z(s)| \int_{t_1 - r}^{t_1} (|\tilde{\gamma}(s)| + |\tilde{c}(s)|)\,\mathrm{d}s$, *then*

$$|z(t)| \geqslant \frac{\Delta(t_1)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} \exp\left[ \int_{t_1}^{t} \widetilde{\Lambda}_0(s)\,\mathrm{d}s \right] \tag{12}$$

*for all* $t \geqslant t_1$ *for which* $z(t)$ *is defined.*

P r o o f. Let $z(t)$ be any solution of (1) satisfying (11). Consider the function

$$V(t) = U(t) + \beta_0(t) \int_{t-r}^{t} |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)|\,\mathrm{d}s, \tag{13}$$

where

$$U(t) = |\tilde{\gamma}(t) z(t) + \tilde{c}(t) \bar{z}(t)|. \tag{14}$$

For brevity we shall denote $w(t) = z(t - r)$ and write a function of the variable $t$ simply without indicating the variable, for example, $\tilde{\gamma}$ instead of $\tilde{\gamma}(t)$.

In view of (13) we have

$$V' = U' + \beta_0' \int_{t-r}^{t} |\tilde{\gamma}(s) z(s) + \tilde{c}(s) \bar{z}(s)|\,\mathrm{d}s + \beta_0 |\tilde{\gamma} z + \tilde{c} \bar{z}| - \beta_0 |\tilde{\gamma}(t-r) w + \tilde{c}(t-r) \overline{w}| \tag{15}$$

for almost all $t \geqslant t_1$ for which $z(t)$ is defined and $U'(t)$ exists. Put $\mathcal{K} = \{t \geqslant t_1 \colon z(t)$ exists, $|z(t)| > R_0\}$. Clearly $U(t) \neq 0$ for $t \in \mathcal{K}$. The derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$. Hence we obtain

$$\begin{aligned}
UU' &= \mathrm{Re}[(\tilde{\gamma} \bar{z} + \bar{\tilde{c}} z)(\tilde{\gamma}' z + \tilde{\gamma} z' + \tilde{c} \bar{z}' + \tilde{c}' \bar{z})] \\
&= \mathrm{Re}\{(\tilde{\gamma} \bar{z} + \bar{\tilde{c}} z)[\tilde{\gamma}\,(a z + b \bar{z} + A w + B \overline{w} + g) \\
&\quad + \tilde{c}(\bar{a} \bar{z} + \bar{b} z + \bar{A} \overline{w} + \overline{B} w + \bar{g}) + \tilde{\gamma}' z + \tilde{c}' \bar{z}]\} \\
&= \mathrm{Re}\{(\tilde{\gamma} \bar{z} + \bar{\tilde{c}} z)[(\tilde{\gamma} a + \tilde{c} \bar{b}) z + (\tilde{\gamma} b + \tilde{c} \bar{a}) \bar{z} + \tilde{\gamma}(A w + B \overline{w} + g) \\
&\quad + \tilde{c}(\bar{A} \overline{w} + \overline{B} w + \bar{g}) + \tilde{\gamma}' z + \tilde{c}' \bar{z}]\}
\end{aligned}$$

for almost all $t \in \mathcal{K}$. As

$$(\tilde{\gamma} a + \tilde{c}\bar{b})\tilde{c} = (\tilde{\gamma} b + \tilde{c}\bar{a})\tilde{\gamma},$$

we get

$$\begin{aligned} UU' = \operatorname{Re}&\Big\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma} a + \tilde{c}\bar{b})\Big(z + \frac{\tilde{c}}{\tilde{\gamma}}\bar{z}\Big)\Big\} \\ &+ \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}(Aw + B\overline{w}) + \tilde{c}(\bar{A}\overline{w} + \overline{B}w))\} \\ &+ \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma} g + \tilde{c}\overline{g})\} + \operatorname{Re}\{(\tilde{\gamma}\bar{z} + \bar{\tilde{c}}z)(\tilde{\gamma}' z + \tilde{c}'\bar{z})\}. \end{aligned}$$

Consequently,

$$UU' \geqslant U^2 \operatorname{Re}\Big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\Big) - U|Aw + B\overline{w}|(|\tilde{\gamma}| + |\tilde{c}|) - U|\tilde{\gamma} g + \tilde{c}\overline{g}| + U^2 \operatorname{Re}\frac{\tilde{\gamma}' z + \tilde{c}'\bar{z}}{\tilde{\gamma} z + \tilde{c}\bar{z}} \tag{16}$$

for almost all $t \in \mathcal{K}$. Since Lemma 1 ensures

$$\operatorname{Re}\frac{\tilde{\gamma}' z + \tilde{c}'\bar{z}}{\tilde{\gamma} z + \tilde{c}\bar{z}} \geqslant \tilde{\vartheta},$$

hence using (8) with $n = 0$, the relation $\operatorname{Re}\big(a + \frac{\tilde{c}}{\tilde{\gamma}}\bar{b}\big) = \operatorname{Re} a$ and the assumption (ii$_0$), we obtain

$$\begin{aligned} UU' &\geqslant U^2(\operatorname{Re} a + \tilde{\vartheta} - \varkappa_0) - U(|A| + |B|)|w|(|\tilde{\gamma}| + |\tilde{c}|) \\ &\quad - U\kappa_0|\tilde{\gamma}(t-r)w + \tilde{c}(t-r)\overline{w}| \\ &\geqslant U^2(\operatorname{Re} a + \tilde{\vartheta} - \varkappa_0) - U\lambda_0|\tilde{\gamma}(t-r)w + \tilde{c}(t-r)\overline{w}|. \end{aligned}$$

Therefore

$$U' \geqslant U(\operatorname{Re} a + \tilde{\vartheta} - \varkappa_0) - \lambda_0|\tilde{\gamma}(t-r)w + \tilde{c}(t-r)\overline{w}| \tag{17}$$

for almost all $t \in \mathcal{K}$. Combining relations (15) (17), we get

$$\begin{aligned} V' \geqslant U(\operatorname{Re} a + \tilde{\vartheta} - \varkappa_0 + \beta_0) - |\tilde{\gamma}(t-r)w + \tilde{c}(t-r)\overline{w}|(\lambda_0 + \beta_0) \\ + \beta_0' \int_{t-r}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)|\,\mathrm{d}s. \end{aligned}$$

Using (7) and (9) for $n = 0$, we obtain

$$V'(t) \geqslant U(t)\Theta_0(t) + \beta_0'(t)\int_{t-r}^{t} |\tilde{\gamma}(s)z(s) + \tilde{c}(s)\bar{z}(s)|\,\mathrm{d}s.$$

Hence, in view of ($\mathrm{iv}_0$), we have

$$V'(t) - \widetilde{\Lambda}_0(t)V(t) \geqslant 0 \tag{18}$$

for almost all $t \in \mathcal{K}$. Multiplying (18) by $\exp\bigl[-\int_{t_1}^t \widetilde{\Lambda}_0(s)\,\mathrm{d}s\bigr]$ and integrating over $[t_1, t]$, we get

$$V(t)\exp\left[-\int_{t_1}^t \widetilde{\Lambda}_0(s)\,\mathrm{d}s\right] - V(t_1) \geqslant 0$$

on any interval $[t_1, \omega)$ where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. Now, with respect to (13), (14) and $\beta_0 \leqslant 0$, we have

$$(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)|z(t)| \geqslant V(t) \geqslant V(t_1)\exp\left[\int_{t_1}^t \widetilde{\Lambda}_0(s)\,\mathrm{d}s\right] \geqslant \Delta(t_1)\exp\left[\int_{t_1}^t \widetilde{\Lambda}_0(s)\,\mathrm{d}s\right].$$

If (11) is fulfilled, there is an $R > R_0$ such that $\Delta(t_1) > R\mathrm{e}^{-\nu}$. By virtue of (10) and (11) we can easily see that

$$|z(t)| \geqslant \frac{\Delta(t_1)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|}\exp\left[\int_{t_1}^t \widetilde{\Lambda}_0(s)\,\mathrm{d}s\right] \geqslant R\mathrm{e}^{-\nu}\mathrm{e}^{\nu} = R$$

for all $t \geqslant t_1$ for which $z(t)$ is defined. $\square$

In the next example we give an equation of the form (1) to which Theorem 2 of [1] is not applicable, nonetheless Theorem 1 of the present paper can be applied.

Example 1. Consider the equation (1) where $a(t) \equiv 8 + 6\mathrm{i}$, $b(t) \equiv 5$, $A(t) \equiv 0$, $B(t) \equiv 0$, $r > 0$, $g(t, z, w) = 6z + 2\mathrm{e}^{-t}w$. Suppose $t_0 = 1$ and $T \geqslant 1 + r$. Then $\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2} \equiv 10 + 5\sqrt{3}$, $c(t) = \bar{a}(t)b(t)/|a(t)| \equiv 4 - 3\mathrm{i}$, $\tilde{\gamma} \equiv 6 + \sqrt{11}$, $\tilde{c}(t) \equiv -5\mathrm{i}$. Further,

$$\begin{aligned}
|\gamma(t)g(t, z, w) + c(t)\bar{g}(t, z, w)| &\leqslant 6|\gamma(t)z + c(t)\bar{z}| + 2\mathrm{e}^{-t}|\gamma(t - r)w + c(t - r)\overline{w}|,\\
|\tilde{\gamma}(t)g(t, z, w) + \tilde{c}(t)\bar{g}(t, z, w)| &\leqslant 6|\tilde{\gamma}(t)z + \tilde{c}(t)\bar{z}| + 2\mathrm{e}^{-t}|\tilde{\gamma}(t - r)w + \tilde{c}(t - r)\overline{w}|.
\end{aligned}$$

Following Theorem 2 of [1] we obtain $\varkappa(t) \equiv 6$, $\kappa(t) = 2\mathrm{e}^{-t}$, $\vartheta(t) \equiv 0$, $\alpha(t) \equiv 1/2$, $\Lambda_0(t) \leqslant \Theta_0(t) = -2 + \beta_0(t) \leqslant -2 < 0$ and we see that Theorem 2 of [1] is not applicable, because the relation (10) in [1] cannot be fulfilled. On the other hand, taking $\varkappa_0(t) \equiv 6$, $\kappa_0(t) \equiv 2\mathrm{e}^{-t}$, $\tau_0 = T$, $R_0 = 0$, $\tilde{\vartheta}(t) \equiv 0$, $\lambda_0(t) = 2\mathrm{e}^{-t}$, $\beta_0(t) = -2\mathrm{e}^{-t}$, $\widetilde{\Lambda}_0(t) = \widetilde{\Theta}_0(t) = 2 - 2\mathrm{e}^{-t}$ ($> 0$) in Theorem 1 of the present paper, we have $\beta_0(t) \leqslant -\lambda_0(t)$, $\beta_0'(t) \geqslant \widetilde{\Theta}_0(t)\beta_0(t)$ for $t \in [T, \infty)$ and Theorem 1 is applicable to the equation considered.

**Corollary 1.** *Let the assumptions of Theorem* 1 *be fulfilled with* $R_0 > 0$. *If*

$$\liminf_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)\,\mathrm{d}s-\ln(|\tilde\gamma(t)|+|\tilde c(t)|)\right]=\varsigma>\nu, \tag{19}$$

*then for any* $\varepsilon$, $0<\varepsilon<R_0\mathrm{e}^{\varsigma-\nu}$, *there is a* $t_2\geqslant t_1$ *such that*

$$|z(t)|>\varepsilon \tag{20}$$

*for all* $t\geqslant t_2$ *for which* $z(t)$ *is defined.*

Proof. Without loss of generality we can assume $\varepsilon>R_0$. Choose $\chi$, $0<\chi<1$ such that $R_0<\varepsilon<\chi R_0\mathrm{e}^{\varsigma-\nu}$. In view of (19) there is a $t_2\geqslant t_1$ such that

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)\,\mathrm{d}s-\ln(|\tilde\gamma(t)|+|\tilde c(t)|)>\varsigma+\ln\chi$$

for $t\geqslant t_2$. Hence

$$\int_{t_1}^{t}\widetilde{\Lambda}_0(s)\,\mathrm{d}s-\ln(|\tilde\gamma(t)|+|\tilde c(t)|)>\nu+\ln\frac{\varepsilon}{R_0}$$

for $t\geqslant t_2$. The estimate (12) together with (11) now yields

$$|z(t)|>R_0\mathrm{e}^{-\nu}\mathrm{e}^{\nu}\frac{\varepsilon}{R_0}=\varepsilon$$

for all $t\geqslant t_2$ for which $z(t)$ is defined. □

**Corollary 2.** *Let the assumptions of Theorem* 1 *be fulfilled with* $R_0>0$. *If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t}\widetilde{\Lambda}_0(s)\,\mathrm{d}s-\ln(|\tilde\gamma(t)|+|\tilde c(t)|)\right]=\infty,$$

*then for any* $\varepsilon>0$ *there exists a* $t_2\geqslant t_1$ *such that* (20) *holds for all* $t\geqslant t_2$ *for which* $z(t)$ *is defined.*

In the proof of the next theorem we shall use results of K. P. Rybakowski [9] on a Ważewski topological principle for retarded functional differential equations of Carathéodory type.

**Theorem 2.** *Let the conditions* (i), (ii), (iii) *be fulfilled and let* $\widetilde{\Lambda}$ *be a continuous function satisfying the inequality* $\widetilde{\Lambda}(t) \leqslant \widetilde{\Theta}(t)$ *a.e. on* $[T,\infty)$*, where* $\widetilde{\Theta}$ *is defined by* (9). *If* $\xi\colon [T-r,\infty) \to \mathbb{R}$ *is a continuous function such that*

$$\widetilde{\Lambda}(t) + \beta(t) \exp\left[-\int_{t-r}^{t} \xi(s)\,\mathrm{d}s\right] - \xi(t) > \varrho(t) C^{-1} \exp\left(-\int_{T}^{t} \xi(s)\,\mathrm{d}s\right) \tag{21}$$

*for* $t \in [T,\infty]$ *and some constant* $C > 0$*, then there exists a* $t_2 > T$ *and a solution* $z_0(t)$ *of* (1) *satisfying*

$$|z_0(t)| \leqslant \frac{C}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \exp\left[\int_{T}^{t} \xi(s)\,\mathrm{d}s\right] \tag{22}$$

*for* $t \geqslant t_2$.

P r o o f. Consider the equation (1) written in the form (1′). Let $\tau > T$. Put

$$\begin{aligned}
\widetilde{U}(t,z,\overline{z}) &= |\tilde{\gamma}(t) z + \tilde{c}(t)\overline{z}| - \varphi(t),\\
\varphi(t) &= C \exp\left[\int_{T}^{t} \xi(s)\,\mathrm{d}s\right],\\
\Omega^0 &= \{(t,z) \in (\tau,\infty) \times \mathbb{C}\colon\ \widetilde{U}(t,z,\overline{z}) < 0\},\\
\Omega_{\widetilde{U}} &= \{(t,z) \in (\tau,\infty) \times \mathbb{C}\colon\ \widetilde{U}(t,z,\overline{z}) = 0\}.
\end{aligned}$$

Clearly $\Omega^0$ is a polyfacial set generated by functions $\widehat{U}(t) = \tau - t$, $\widetilde{U}(t,z,\overline{z})$ (see Rybakowski [9, p. 134]) and $\Omega_{\widetilde{U}} \subset \partial\Omega^0$. Since $(|\tilde{\gamma}(t)| + |\tilde{c}(t)|)|z(t)| \geqslant |\tilde{\gamma}(t) z + \tilde{c}(t)\overline{z}|$, we have

$$|z| \geqslant \frac{\varphi(t)}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} = \frac{C}{|\tilde{\gamma}(t)| + |\tilde{c}(t)|} \exp\left[\int_{T}^{t} \xi(s)\,\mathrm{d}s\right] > 0$$

for $(t,z) \in \Omega_{\widetilde{U}}$. Further,

$$D^+\widehat{U}(t) = \frac{\partial}{\partial t}(\tau - t) = -1 < 0.$$

Let $(t^*,\zeta) \in \Omega_{\widetilde{U}}$ and let $\phi \in \mathcal{C}$ be such that $\phi(0) = \zeta$ and $(t^* + \theta, \phi(\theta)) \in \Omega^0$ for all $\theta \in [-r,0)$. If $(t,\psi) \in (\tau,\infty) \times \mathcal{C}$, then

$$\begin{aligned}
D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) &:= \limsup_{h\to 0+} (1/h)[\widetilde{U}(t+h,\psi(0)\\
&\qquad + hF(t,\psi), \overline{\psi}(0) + h\overline{F}(t,\psi)) - \widetilde{U}(t,\psi(0),\overline{\psi}(0))]\\
&= \frac{\partial \widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial t} + \frac{\partial \widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial z} F(t,\psi)\\
&\qquad + \frac{\partial \widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial \overline{z}} \overline{F}(t,\psi).
\end{aligned}$$

Hence

$$
\begin{aligned}
D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) = {}& |\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)| \operatorname{Re}\frac{\tilde{\gamma}'(t)\psi(0)+\tilde{c}'(t)\overline{\psi}(0)}{\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)} - \varphi'(t)\\
&+ \tfrac{1}{2}|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)|^{-1} \times \operatorname{Re}\{[\bar{\tilde{\gamma}}(t)(\tilde{\gamma}(t)\overline{\psi}(0)+\bar{\tilde{c}}(t)\psi(0))\\
&+ \bar{\tilde{c}}(t)(\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0))]F(t,\psi)\\
&+ [\tilde{c}(t)(\bar{\tilde{\gamma}}(t)\overline{\psi}(0)+\bar{\tilde{c}}(t)\psi(0)) + \tilde{\gamma}(t)(\tilde{\gamma}(t)\psi(0)\\
&+ \tilde{c}(t)\overline{\psi}(0))]\overline{F}(t,\psi)\}
\end{aligned}
$$

provided the derivatives $\tilde{\gamma}'(t)$, $\tilde{c}'(t)$ exist and $\psi(0)\neq 0$.

Similarly to the proof of Theorem 5 of [1] we obtain

$$
\begin{aligned}
&D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) \geqslant (\operatorname{Re} a(t)+\tilde{\vartheta}(t)-\varkappa(t))|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)|\\
&\quad - \lambda(t)|\tilde{\gamma}(t-r)\psi(-r)+\tilde{c}(t-r)\overline{\psi}(-r)| - \varrho(t) - \varphi'(t)\\
&\geqslant \widetilde{\Theta}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)| + \beta(t)|\gamma(t-r)\psi(-r)+c(t-r)\overline{\psi}(-r)| - \varrho(t) - \varphi'(t)\\
&\geqslant \widetilde{\Lambda}(t)|\tilde{\gamma}(t)\psi(0)+\tilde{c}(t)\overline{\psi}(0)| + \beta(t)|\tilde{\gamma}(t-r)\psi(-r)+\tilde{c}(t-r)\overline{\psi}(-r)| - \varrho(t) - \varphi'(t)
\end{aligned}
$$

for almost all $t\in(\tau,\infty)$ and for $\psi\in\mathcal{C}$ sufficiently close to $\phi$. Replacing $t$ and $\psi$ in the last expression by $t^*$ and $\phi$, respectively, we get

$$
\begin{aligned}
&\widetilde{\Lambda}(t^*)|\tilde{\gamma}(t^*)\phi(0)+\tilde{c}(t^*)\bar{\phi}(0)| + \beta(t^*)|\tilde{\gamma}(t^*-r)\phi(-r)\\
&\qquad + \tilde{c}(t^*-r)\bar{\phi}(-r)| - \varrho(t^*) - \varphi'(t^*)\\
&\geqslant \widetilde{\Lambda}(t^*)|\tilde{\gamma}(t^*)\zeta+\tilde{c}(t^*)\bar{\zeta}| + \beta(t^*)\varphi(t^*-r) - \varrho(t^*) - \varphi'(t^*)\\
&\geqslant \widetilde{\Lambda}(t^*)\varphi(t^*) + \beta(t^*)\varphi(t^*-r) - \varrho(t^*) - \varphi'(t^*)\\
&= \widetilde{\Lambda}(t^*)C\exp\left[\int_T^{t^*}\xi(s)\,\mathrm{d}s\right] + \beta(t^*)C\exp\left[\int_T^{t^*-r}\xi(s)\,\mathrm{d}s\right]\\
&\qquad - \varrho(t^*) - C\xi(t^*)\exp\left[\int_T^{t^*}\xi(s)\,\mathrm{d}s\right]\\
&= \left\{\widetilde{\Lambda}(t^*) + \beta(t^*)\exp\left[-\int_{t^*-r}^{t^*}\xi(s)\,\mathrm{d}s\right] - \xi(t^*)\right\}C\exp\left[\int_T^{t^*}\xi(s)\,\mathrm{d}s\right] - \varrho(t^*) > 0,
\end{aligned}
$$

where the last inequality follows from (21). Therefore, in view of the continuity, $D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0))>0$ holds for $\psi$ sufficiently close to $\phi$ and almost all $t$ sufficiently close to $t^*$. Hence $\Omega^0$ is a regular polyfacial set with respect to (1′).

Choose $Z=\{(t_2,z)\in\Omega^0\cup\Omega_{\widetilde{U}}\}$, where $t_2>\tau+r$ is fixed. It can be easily verified that $Z\cap\Omega_{\widetilde{U}}$ is a retract of $\Omega_{\widetilde{U}}$, but $Z\cap\Omega_{\widetilde{U}}$ is not a retract of $Z$. Let $\eta\in\mathcal{C}$ be such that $\eta(0)=1$ and $0\leqslant\eta(\theta)<1$ for $\theta\in[-r,0)$. Define a mapping $p\colon Z\to\mathcal{C}$ for

$(t_2, z) \in Z$ by the relation

$$p(t_2, z)(\theta) = \frac{\varphi(t_2+\theta)\eta(\theta)}{(\tilde{\gamma}^2(t_2+\theta) - |\tilde{c}(t_2+\theta)|^2)\varphi(t_2)}[(\tilde{\gamma}(t_2)\tilde{\gamma}(t_2+\theta) - \bar{\tilde{c}}(t_2)\tilde{c}(t_2+\theta))z + (\tilde{\gamma}(t_2+\theta)\tilde{c}(t_2) - \tilde{\gamma}(t_2)\tilde{c}(t_2+\theta))\bar{z}].$$

The mapping $p$ is continuous and

$$p(t_2, z)(0) = z \quad \text{for } (t_2, z) \in Z, \qquad p(t_2, 0)(\theta) = 0 \quad \text{for } \theta \in [-r, 0].$$

Since

$$\tilde{\gamma}(t_2+\theta)p(t_2, z)(\theta) + \tilde{c}(t_2+\theta)\overline{p(t_2, z)(\theta)} = \frac{\varphi(t_2+\theta)\eta(\theta)}{\varphi(t_2)}(\tilde{\gamma}(t_2)z + \tilde{c}(t_2)\bar{z}),$$

we have

$$|\tilde{\gamma}(t_2)z + \tilde{c}(t_2)\bar{z}| < \varphi(t_2)$$

and

$$\left|\tilde{\gamma}(t_2+\theta)p(t_2, z)(\theta) + \tilde{c}(t_2+\theta)\overline{p(t_2, z)(\theta)}\right| < \varphi(t_2+\theta) \tag{23}$$

for $(t_2, z) \in Z \cap \Omega^0$ and $\theta \in [-r, 0]$. Clearly, the inequality (23) holds also for $(t_2, z) \in Z \cap \Omega_{\widetilde{U}}$ and $\theta \in [-r, 0)$.

Using the topological principle for retarded functional differential equations (see Rybakowski [9, Theorem 2.1]), we infer that there is a solution $z_0(t)$ of (1) such that $(t, z_0(t)) \in \Omega^0$ for all $t \geqslant t_2$ for which the solution $z_0(t)$ exists. Obviously $z_0(t)$ exists for all $t \geqslant t_2$ and

$$(|\tilde{\gamma}(t)| - |\tilde{c}(t)|)|z_0(t)| \leqslant |\tilde{\gamma}(t)z_0(t) + \tilde{c}(t)\bar{z}_0(t)| \leqslant \varphi(t) \qquad \text{for } t \geqslant t_2.$$

Hence

$$|z_0(t)| \leqslant \frac{\varphi(t)}{|\tilde{\gamma}(t)| - |\tilde{c}(t)|} \qquad \text{for } t \geqslant t_2.$$

□

Remark 1. If $\eta_1(t)\widetilde{\Lambda}(t) > |\beta(t)| + C^{-1}\varrho(t) > 0$, where $0 < \eta_1(t) \leqslant 1$, the functions $\eta_1$, $\widetilde{\Lambda}$ are continuous on $[T, \infty)$ and $\widetilde{\Lambda}(t) \leqslant \widetilde{\Theta}(t)$ a.e. on $[T, \infty)$, then the choice $\xi(t) = \eta_1(t)\widetilde{\Lambda}(t) + \beta(t) - C^{-1}\varrho(t)$ is possible in (21). Moreover, the condition $|\beta(t)| + C^{-1}\varrho(t) > 0$ can be omitted if Theorem 2 is used. Indeed, the identity $|\beta(t)| + C^{-1}\varrho(t) \equiv 0$ implies $\beta(t) \equiv 0$, $\varrho(t) \equiv 0$ and consequently, in view of (5), (6), (ii), we have $\lambda(t) \equiv 0$, $\kappa(t) \equiv 0$, $A(t) \equiv 0$, $B(t) \equiv 0$, $g(t, 0, 0) \equiv 0$. Thus the equation (1) has the trivial solution $z_0(t) \equiv 0$ in this case.

**Corollary 3.** *Let the assumptions of Theorem 2 be satisfied. If*

$$\limsup_{t\to\infty}\left[\frac{1}{|\tilde\gamma(t)|-|\tilde c(t)|}\exp\Big(\int_T^t \xi(s)\,\mathrm{d}s\Big)\right]<\infty,$$

*then there is a bounded solution $z_0(t)$ of* (1). *If*

$$\lim_{t\to\infty}\left[\frac{1}{|\tilde\gamma(t)|-|\tilde c(t)|}\exp\Big(\int_T^t \xi(s)\,\mathrm{d}s\Big)\right]=0,$$

*then there is a solution $z_0(t)$ of* (1) *such that*

$$\lim_{t\to\infty} z_0(t)=0.$$

**Theorem 3.** *Suppose that the hypotheses* (i), (ii), (ii$_\mathrm{n}$), (iii), (iii$_\mathrm{n}$), (iv$_\mathrm{n}$) *are fulfilled for $\tau_n\geqslant T$ and $n\in\mathbb{N}$, where $R_n>0$, $\inf\limits_{n\in\mathbb{N}} R_n=0$. Let $\widetilde\Lambda$ be a continuous function satisfying the inequality $\widetilde\Lambda(t)\leqslant\widetilde\Theta(t)$ a.e. on $[T,\infty)$, where $\widetilde\Theta$ is defined by* (9). *Assume that $\xi\colon[T-r,\infty)\to\mathbb{R}$ is a continuous function such that*

$$\widetilde\Lambda(t)+\beta(t)\exp\left[-\int_{t-r}^t \xi(s)\,\mathrm{d}s\right]-\xi(t)>\varrho(t)C^{-1}\exp\Big(-\int_T^t\xi(s)\,\mathrm{d}s\Big)\tag{24}$$

*for $t\in[T,\infty)$ and some constant $C>0$. Suppose*

$$\limsup_{t\to\infty}\left[\int_T^t(\widetilde\Lambda_n(s)-\xi(s))\,\mathrm{d}s+\ln\frac{|\tilde\gamma(t)|-|\tilde c(t)|}{|\tilde\gamma(t)|+|\tilde c(t)|}\right]=\infty,\tag{25}$$

$$\lim_{t\to\infty}\left[\beta_n(t)\max_{s\in[t-r,t]}\frac{\exp\big[\int_T^s\xi(\sigma)\,\mathrm{d}\sigma\big]}{|\tilde\gamma(s)|-|\tilde c(s)|}\int_{t-r}^t(|\tilde\gamma(s)|+|\tilde c(s)|)\,\mathrm{d}s\right]=0,\tag{26}$$

$$\inf_{\tau_n\leqslant s\leqslant t<\infty}\left[\int_s^t\widetilde\Lambda_n(\sigma)\,\mathrm{d}\sigma-\ln(|\tilde\gamma(t)|+|\tilde c(t)|)\right]\geqslant\nu\tag{27}$$

*for $n\in\mathbb{N}$, where $\nu\in(-\infty,\infty)$. Then there exists a solution $z_0(t)$ of* (1) *such that*

$$\lim_{t\to\infty}\min_{s\in[t-r,t]}|z_0(s)|=0.\tag{28}$$

P r o o f. Using Theorem 2 we observe that there is a $t_2\geqslant T$ and a solution $z_0(t)$ of (1) with the property

$$|z_0(t)|\leqslant\frac{C}{|\tilde\gamma(t)|-|\tilde c(t)|}\exp\left[\int_T^t\xi(s)\,\mathrm{d}s\right]\tag{29}$$

for $t \geqslant t_2$. Suppose that (28) is not satisfied. Then there is $\varepsilon_0 > 0$ such that

$$\limsup_{t\to\infty} \min_{s\in[t-r,t]} |z_0(s)| > \varepsilon_0.$$

Choose $N \in \mathbb{N}$ such that

$$\max\Big\{R_N, \frac{2}{\mu} R_N \mathrm{e}^{-\nu}\Big\} < \varepsilon_0.$$

Then

$$\min_{s\in[\tau-r,\tau]} |z_0(s)| > \max\Big\{R_N, \frac{2}{\mu} R_N \mathrm{e}^{-\nu}\Big\} \tag{30}$$

for some $\tau > \max\{T, \tau_N, t_2\}$. In view of (26) we can suppose that

$$|\beta_N(\tau)| C \max_{s\in[\tau-r,\tau]} \frac{\exp\big[\int_T^s \xi(\sigma)\,\mathrm{d}\sigma\big]}{|\tilde\gamma(s)| - |\tilde c(s)|} \int_{\tau-r}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\,\mathrm{d}s < \frac{1}{2} R_N \mathrm{e}^{-\nu}. \tag{31}$$

Hence, taking into account (4), (29), (30), (31) and the nonpositiveness of $\beta_N$, we have

$$\begin{aligned}
&(|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| + \beta_N(\tau) \max_{s\in[\tau-r,\tau]} |z_0(s)| \int_{\tau-r}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\,\mathrm{d}s \\
&\geqslant (|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| + \beta_N(\tau) C \max_{s\in[\tau-r,\tau]} \frac{\exp\big[\int_T^s \xi(\sigma)\,\mathrm{d}\sigma\big]}{|\tilde\gamma(s)| - |\tilde c(s)|} \int_{\tau-r}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\,\mathrm{d}s \\
&\geqslant \mu \frac{2}{\mu} R_N \mathrm{e}^{-\nu} - \frac{1}{2} R_N \mathrm{e}^{-\nu} > R_N \mathrm{e}^{-\nu}.
\end{aligned}$$

Moreover, (27) implies

$$\inf_{\tau\leqslant t<\infty} \Big[\int_\tau^t \widetilde\Lambda_N(s)\,\mathrm{d}s - \ln(|\tilde\gamma(t)| + |\tilde c(t)|)\Big] \geqslant \nu > -\infty.$$

By Theorem 1 we obtain the estimate

$$|z_0(t)| \geqslant \frac{\Psi(\tau)}{|\tilde\gamma(t)| + |\tilde c(t)|} \exp\Big[\int_\tau^t \widetilde\Lambda_N(s)\,\mathrm{d}s\Big]$$

for all $t \geqslant \tau$, $\Psi$ being defined by

$$\Psi(\tau) = (|\tilde\gamma(\tau)| - |\tilde c(\tau)|)|z_0(\tau)| + \beta_N(\tau) \max_{s\in[\tau-r,\tau]} |z_0(s)| \int_{\tau-r}^{\tau} (|\tilde\gamma(s)| + |\tilde c(s)|)\,\mathrm{d}s.$$

(29) together with (32) yield

$$\frac{\Psi(\tau)}{|\tilde{\gamma}(t)|+|\tilde{c}(t)|}\exp\left[\int_\tau^t \widetilde{\Lambda}_N(s)\,\mathrm{d}s\right] \leqslant \frac{C}{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}\exp\left[\int_T^t \xi(s)\,\mathrm{d}s\right],$$

i.e.

$$\int_T^t [\widetilde{\Lambda}_N(s)-\xi(s)]\,\mathrm{d}s + \ln\frac{|\tilde{\gamma}(t)|-|\tilde{c}(t)|}{|\tilde{\gamma}(t)|+|\tilde{c}(t)|} \leqslant \int_T^\tau \widetilde{\Lambda}_N(s)\,\mathrm{d}s - \ln[C^{-1}\Psi(\tau)]$$

for $t \geqslant \tau$. However, the last inequality contradicts (25) and Theorem 3 is proved. □

Acknowledgement. The research was supported by grant IAA1163401/04 of the Grant Agency of the Academy of Sciences of the Czech Republic. This support is gratefully acknowledged.

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