STRONG SINGULARITIES IN MIXED BOUNDARY VALUE PROBLEMS

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Abstract. We study singular boundary value problems with mixed boundary conditions of the form

$$(p(t)u')' + p(t)f(t,u,p(t)u') = 0, \quad \lim_{t\to 0+} p(t)u'(t) = 0, \quad u(T) = 0,$$

where $[0,T] \subset \mathbb{R}$. We assume that $\mathscr{D} \subset \mathbb{R}^2$, f satisfies the Carathéodory conditions on $(0,T)\times\mathscr{D}$, $p \in C[0,T]$ and $1/p$ need not be integrable on $[0,T]$. Here f can have time singularities at $t=0$ and/or $t=T$ and a space singularity at $x=0$. Moreover, f can change its sign. Provided f is nonnegative it can have even a space singularity at $y=0$. We present conditions for the existence of solutions positive on $[0,T)$.

Keywords: singular mixed boundary value problem, positive solution, lower function, upper function, convergence of approximate regular problems

MSC 2000: 34B16, 34B18

1. Introduction

Assume that $[0,T] \subset \mathbb{R}$, $\mathscr{D} \subset \mathbb{R}^2$ and that f satisfies the Carathéodory conditions on $(0,T)\times\mathscr{D}$. We investigate the solvability of the singular mixed boundary value problem

$$(p(t)u')' + p(t)f(t,u,p(t)u') = 0, \tag{1.1}$$

$$\lim_{t\to 0+} p(t)u'(t) = 0, \quad u(T) = 0, \tag{1.2}$$

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where $p \in C[0,T]$ and f can have time singularities at $t = 0$ and/or $t = T$ and a space singularity at $x = 0$. In particular, f can have even a space singularity at $y = 0$ if f is nonnegative (Theorem 2.1). In [19] we have studied a special case of the above problem with $p(t) = 1$ on $[0,T]$ and in [20] we have proved solvability of (1.1), (1.2) provided $1/p \in L_1[0,T]$. Here we investigate problem (1.1), (1.2) under the assumption that $1/p$ need not be integrable on $[0,T]$. This assumption is motivated by a problem arising in the theory of shallow membrane caps (see [10], [13]), which is controlled by the equation

$$(t^3u')' + \frac{t^3}{8u^2} - a_0\frac{t^3}{u} - b_0t^{2\gamma-1} = 0, \quad a_0 \geqslant 0,\ b_0 > 0,\ \gamma > 1,$$

with $p(t) = t^3$. We see that this is the case $1/p \notin L_1[0,T]$. But in our paper, in contrast to the above example, we will investigate equations where the right-hand side f depends both on u and on u'.

Note that the importance of singular mixed problems consists also in the fact that they arise when searching for positive, radially symmetric solutions to nonlinear elliptic partial differential equations (see [9], [12]).

In this paper we prove existence of solutions of (1.1), (1.2) which are positive on $[0,T)$. For other existence results of singular mixed problems we refer to [1]–[8], [11], [14]–[22].

Here we extend results of [2], [19], [20] and offer new conditions which guarantee the existence of positive solutions of the singular problem (1.1), (1.2) provided both time and space singularities are allowed. Moreover, we also admit f to change its sign (Theorem 2.2).

First, we recall some definitions and results. Let $[a,b] \subset \mathbb{R}$, $\mathscr{M} \subset \mathbb{R}^2$. We say that a real valued function f satisfies the *Carathéodory conditions* on the set $[a,b] \times \mathscr{M}$ if

(i) $f(\cdot,x,y)\colon [a,b] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathscr{M}$,

(ii) $f(t,\cdot,\cdot)\colon \mathscr{M} \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,

(iii) for each compact set $K \subset \mathscr{M}$ there is a function $m_K \in L_1[0,T]$ such that $|f(t,x,y)| \leqslant m_K(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in K$.

We write $f \in \mathrm{Car}([a,b] \times \mathscr{M})$. By $f \in \mathrm{Car}((0,T) \times \mathscr{D})$ we mean $f \in \mathrm{Car}([a,b] \times \mathscr{D})$ for each $[a,b] \subset (0,T)$ and $f \notin \mathrm{Car}([0,T] \times \mathscr{D})$.

Definition 1.1. Let $f \in \mathrm{Car}((0,T) \times \mathscr{D})$. We say that f has a *time singularity* at $t = 0$ and/or at $t = T$ if there exists $(x,y) \in \mathscr{D}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)|\,\mathrm{d}t = \infty \quad \text{and/or} \quad \int_{T-\varepsilon}^{T} |f(t,x,y)|\,\mathrm{d}t = \infty$$

for each sufficiently small $\varepsilon > 0$. The point $t = 0$ and/or $t = T$ will be called a *singular point of* f. Let $\mathscr{D} = (0,\infty) \times I$, $I \subseteq \mathbb{R}$. We say that f has a *space singularity* at

$x = 0$ if

$$\limsup_{x\to 0+} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } y \in I.$$

Let $\mathscr{D} = (0,\infty) \times (-\infty,0)$. We say that f has a *space singularity* at $y = 0$ if

$$\limsup_{y\to 0-} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } x \in (0,\infty).$$

Definition 1.2. By a *solution* of problem (1.1), (1.2) we understand a function $u \in C[0,T]$ with $pu' \in AC[0,T]$ satisfying conditions (1.2) and fulfilling

$$(p(t)u'(t))' + p(t)f(t,u(t),p(t)u'(t)) = 0 \ \text{ for a.e. } t \in [0,T]. \tag{1.3}$$

Now consider an auxiliar regular problem

$$(q(t)u')' + h(t,u,q(t)u') = 0, \quad u'(0) = 0,\ u(T) = 0, \tag{1.4}$$

where $q \in C[0,T]$ is positive on $[0,T]$ and $h \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Definition 1.3. A *solution* of the regular problem (1.4) is defined as a function $u \in C^1[0,T]$ with $qu' \in AC[0,T]$ sastisfying $u'(0) = u(T) = 0$ and fulfilling $(q(t)u'(t))' + h(t,u(t),q(t)u'(t)) = 0$ for a.e. $t \in [0,T]$.

In the proofs of our main results we will use the following lower and upper functions method for problem (1.4).

Definition 1.4. A function $\sigma \in C[0,T]$ is called a *lower function of* (1.4) if there exists a finite set $\Sigma \subset (0,T)$ such that $q\sigma' \in AC_{\mathrm{loc}}([0,T]\setminus\Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$(q(t)\sigma'(t))' + h(t,\sigma(t),q(t)\sigma'(t)) \geqslant 0 \quad \text{for a.e. } t \in [0,T] \tag{1.5}$$

and

$$\sigma'(0) \geqslant 0, \quad \sigma(T) \leqslant 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma. \tag{1.6}$$

If the inequalities in (1.5) and (1.6) are reversed, then σ is called an *upper function of* (1.4).

Lemma 1.5 ([20], Theorem 2.3). *Let σ_1 and σ_2 be a lower function and an upper function for problem* (1.4) *such that $\sigma_1 \leqslant \sigma_2$ on $[0,T]$. Assume also that there is a function $\psi \in L_1[0,T]$ such that*

$$|h(t,x,y)| \leqslant \psi(t) \quad \text{for a.e. } t \in [0,T],\ \text{all } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}. \tag{1.7}$$

Then problem (1.4) *has a solution $u \in C^1[0,T]$ satisfying $qu' \in AC[0,T]$ and*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \text{for } t \in [0,T]. \tag{1.8}$$

2. Main results

The first existence result for the singular problem (1.1), (1.2) will be proved under the assumptions

$$p \in C[0,T],\ p > 0 \text{ on } (0,T],\ 1/p \text{ need not belong to } L_1[0,T], \tag{2.1}$$

and

$$\begin{cases} \mathscr{D} = (0,\infty) \times (-\infty, 0),\ f \in \mathrm{Car}((0,T) \times \mathscr{D}), \\ f \text{ can have time singularities at } t = 0,\ t = T, \\ f \text{ can have space singularities at } x = 0,\ y = 0. \end{cases} \tag{2.2}$$

Theorem 2.1. *Let* (2.1), (2.2) *hold. Assume that there exist $\varepsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu,\infty)$ and positive functions $\varphi \in L_{1_{\mathrm{loc}}}(0,T)$, $\omega \in C(0,\infty)$, $h \in C[0,\infty)$ such that*

$$\frac{1}{p(t)} \int_0^t p(s)\varphi(s)\,\mathrm{d}s \in L_{1_{\mathrm{loc}}}[0,T), \tag{2.3}$$

$$f(t, P(t), -c) = 0 \quad \text{for a.e. } t \in (0,T), \tag{2.4}$$

$$\varepsilon \leqslant f(t,x,y) \quad \text{for a.e. } \ t \in (0,\nu],\ \text{all } x \in (0, P(t)],\ y \in [-\nu, 0), \tag{2.5}$$

and

$$0 \leqslant f(t,x,y) \leqslant \varphi(t)(\omega(x) + h(x)) \quad \text{for a.e. } t \in (0,T),\ \text{all } x \in (0,P(t)],\ y \in [-c,0), \tag{2.6}$$

where

$$P(t) = c \int_t^T \frac{\mathrm{d}s}{p(s)} \quad \text{for } t \in (0,T], \tag{2.7}$$

ω is nonincreasing, h is nondecreasing and

$$\lim_{x\to\infty} \frac{h(x)}{x} < \infty. \tag{2.8}$$

Then problem (1.1), (1.2) *has a solution* $u \in C[0,T]$ *positive and decreasing on* $[0,T)$ *with* $pu' \in AC[0,T]$.

Note. Condition $\varphi \in L_{1_{\rm loc}}(0,T)$ or $\varphi \in L_{1_{\rm loc}}[0,T)$ means that $\varphi \in L_1[a,b]$ for each $[a,b] \subset (0,T)$ or $[a,b] \subset [0,T)$, respectively. Functions satisfying (2.3) are for example $p(t) = t^{\alpha}$ and $\varphi(t) = t^{-\beta} + (T-t)^{-3}$, where $\alpha \geqslant 1, \beta \in (0,2)$.

Proof. Let $k \in \mathbb{N}$, $k \geqslant 3/T$. In the following Steps 1–5 we argue as in the proof of Theorem 3.1 in [20]. So we will show just an abridgement of these steps.

Step 1. Approximate solutions. For $t \in [0,T]$, $x, y \in \mathbb{R}$ put

$$\alpha_k(t,x) = \begin{cases} P(t) & \text{if } x > P(t), \\ x & \text{if } 1/k \leqslant x \leqslant P(t), \\ 1/k & \text{if } x < 1/k, \end{cases} \tag{2.9}$$

and

$$\beta_k(y) = \begin{cases} -1/k & \text{if } y > -1/k, \\ y & \text{if } -c \leqslant y \leqslant -1/k, \\ -c & \text{if } y < -c, \end{cases}$$

and

$$\gamma(y) = \begin{cases} \varepsilon & \text{if } y \geqslant -\nu, \\ \varepsilon(c+y)(c-\nu)^{-1} & \text{if } -c < y < -\nu, \\ 0 & \text{if } y \leqslant -c. \end{cases} \tag{2.10}$$

For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define

$$f_k(t,x,y) = \begin{cases} \gamma(y) & \text{if } t \in [0,1/k), \\ f(t,\alpha_k(t,x),\beta_k(y)) & \text{if } t \in [1/k, T-1/k], \\ 0 & \text{if } t \in (T-1/k, T] \end{cases}$$

and

$$p_k(t) = \begin{cases} \max\{p(t), p(1/k)\} & \text{if } t \in [0,1/k), \\ p(t) & \text{if } t \in [1/k, T]. \end{cases} \tag{2.11}$$

Then $p_k \in C[0,T]$, $p_k > 0$ on $[0,T]$, and there is $\psi_k \in L_1[0,T]$ such that

$$|p_k(t) f_k(t,x,y)| \leqslant \psi_k(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}. \tag{2.12}$$

We have got a sequence of auxiliary regular problems

$$(p_k(t)u')' + p_k(t)f_k(t,u,p_k(t)u') = 0, \quad u'(0) = 0, \ u(T) = 0, \tag{2.13}$$

$k \in \mathbb{N}$, $k \geqslant 3/T$. If we put

$$\sigma_1(t) = 0, \ \sigma_{2k}(t) = c\int_t^T \frac{\mathrm{d}s}{p_k(s)} \quad \text{for } t \in [0,T],$$

then σ_1 and σ_{2k} are lower and upper functions of (2.13) and, by Lemma 1.5, problem (2.13) has a solution $u_k \in C^1[0,T]$ satisfying

$$0 \leqslant u_k(t) \leqslant \sigma_{2k}(t) \quad \text{for } t \in [0,T]. \tag{2.14}$$

Step 2. A priori estimates of approximate solutions u_k. Conditions (2.14) and $u_k(T) = \sigma_{2k}(T) = 0$, $p_k(0)u_k'(0) = 0$ and the monotonicity of $p_k u_k'$ give

$$-c \leqslant p_k(t)u_k'(t) \leqslant 0 \quad \text{on } [0,T]. \tag{2.15}$$

Choose an arbitrary compact interval $J \subset (0,T)$. By virtue of (2.5) and (2.15) there is $k_J \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geqslant k_J$

$$\begin{cases} 1/k_J \leqslant u_k(t) \leqslant k_J, \quad -k_J \leqslant u_k'(t) \leqslant -1/k_J, \\ -c \leqslant p_k(t)u_k'(t) \leqslant -1/k_J \quad \text{for } t \in J, \end{cases} \tag{2.16}$$

and hence there is $\psi \in L_1(J)$ such that

$$|p_k(t)f_k(t,u_k(t),p_k(t)u_k'(t))| \leqslant \psi(t) \quad \text{a.e. on } J. \tag{2.17}$$

Step 3. Convergence of a sequence of approximate solutions. Using conditions (2.16), (2.17) we see that the sequences $\{u_k\}$ and $\{p_k u_k'\}$ are equibounded and equicontinuous on J. Therefore by the Arzelà-Ascoli theorem and the diagonalization principle we can choose $u \in C(0,T)$ and subsequences of $\{u_k\}$ and of $\{p_k u_k'\}$ which we denote for simplicity in the same way such that

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} p_k u_k' = pu' \quad \text{locally uniformly on } (0,T), \tag{2.18}$$

$$0 < u(t) \leqslant P(t), \quad -c \leqslant p(t)u'(t) < 0 \quad \text{for } t \in (0,T). \tag{2.19}$$

Step 4. Convergence of a sequence of approximate problems.

Choose an arbitrary $\xi \in (0,T)$ such that

$$f(\xi,\cdot,\cdot) \quad \text{is continuous on } (0,\infty)\times(-\infty,0).$$

There exists a compact interval $J_\xi \subset (0,T)$ with $\xi \in J_\xi$ and, by (2.16), we can find $k_\xi \in \mathbb{N}$ such that for each $k \geqslant k_\xi$

$$u_k(\xi) \geqslant \frac{1}{k_\xi}, \quad p_k(\xi)u_k'(\xi) \leqslant -\frac{1}{k_\xi}, \quad J_\xi \subset \Big[\frac{1}{k}, T-\frac{1}{k}\Big].$$

Therefore

$$\lim_{k\to\infty} p_k(t)f_k(t,u_k(t),p_k(t)u_k'(t)) = p(t)f(t,u(t),p(t)u'(t)) \quad \text{for a.e. } t\in(0,T). \tag{2.20}$$

Integrating (2.13), letting $k\to\infty$ and using the Lebesgue convergence theorem we get for an arbitrary $t\in(0,T)$

$$p\Big(\frac{T}{2}\Big)u'\Big(\frac{T}{2}\Big) - p(t)u'(t) = \int_{\frac12 T}^{t} p(\tau)f(\tau,u(\tau),p(\tau)u'(\tau))\,\mathrm{d}\tau, \tag{2.21}$$

i.e. (1.3) is valid.

S t e p 5. P r o p e r t i e s o f pu'. According to (2.13) and (2.15) we have for each $k\geqslant 3/T$

$$\int_0^T p_k(s)f_k(s,u_k(s),p_k(s)u_k'(s))\,\mathrm{d}s = -p_k(T)u_k'(T) \in (0,c],$$

which together with (2.6), (2.19) and (2.20) yields, by the Fatou lemma, that $p(t)f(t,u(t),p(t)u'(t)) \in L_1[0,T]$. Therefore, by (2.21), $pu' \in AC[0,T]$.

S t e p 6. P r o p e r t i e s o f u. Since pu' is continuous on $[0,T]$ and $1/p$ is continuous on $(0,T]$, we get $u\in C(0,T]$. It remains to prove that $u\in C[0,T]$. By (2.19) u is decreasing on $(0,T)$, which yields

$$0 < A = \lim_{t\to 0+} u(t).$$

Therefore it is sufficient to prove that $A<\infty$.

By (1.3), (2.6) and (2.19) we deduce that

$$-(p(t)u'(t))' \leqslant p(t)\varphi(t)(\omega(u(t)) + h(u(t)) \text{ for a.e. } t\in(0,T). \tag{2.22}$$

Let $B_0 \in (0,\infty)$ and $x_0 \in (0,A)$ be such that

$$\omega(x_0) = h(x_0) + B_0 \in (0,\infty).$$

Then there is $t_0 \in (0,T)$ such that

$$u(t_0) = x_0, \quad x_0 < u(t) < A \text{ for } t \in (0,t_0),$$

and having in mind monotonicity of ω and h we obtain

$$-(p(t)u'(t))' \leqslant p(t)\varphi(t)(2h(A)+B_0) \quad \text{for a.e. } t \in (0,t_0], \tag{2.23}$$

where $h(A) = \lim_{x\to A} h(x)$. By virtue of (2.8) we can find $a \in (0,\infty)$ such that

$$\lim_{x\to\infty} \frac{h(x)}{x} \leqslant a$$

and due to (2.3) there is $t_a \in (0,t_0)$ satisfying

$$\int_0^{t_a} \frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \leqslant \frac{1}{3a}.$$

Integrating (2.23) we get

$$-u'(s) \leqslant (2h(A)+B_0)\frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau, \quad s \in (0,t_0],$$

and integrating the last inequality we obtain

$$u(t) - u(t_a) \leqslant (2h(A)+B_0) \int_t^{t_a} \frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in (0,t_a).$$

Hence, for $t \to 0+$ we get

$$A \leqslant u(t_a) + (2h(A)+B_0) \int_0^{t_a} \frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \leqslant u(t_a) + \frac{2h(A)+B_0}{3a}$$

and

$$1 \leqslant \frac{u(t_a)}{A} + \frac{2h(A)+B_0}{3aA} = F(A).$$

Since $\lim_{x\to\infty} F(x) \leqslant 2/3$, there exists $A^* \in (0,\infty)$ such that $F(x) < 1$ for each $x \geqslant A^*$. Since $F(A) \geqslant 1$, we have $A \leqslant A^*$. $\square$

The second existence result is applicable to sign-changing nonlinearities. Now we will assume (2.1) and

$$\begin{cases} \mathscr{D} = (0,\infty) \times \mathbb{R}, \ f \in \mathrm{Car}((0,T) \times \mathscr{D}), \\ f \text{ can have time singularities at } t = 0, \ t = T, \\ f \text{ can have a space singularity at } x = 0. \end{cases} \tag{2.24}$$

Theorem 2.2. *Let* (2.1) *and* (2.24) *hold. Assume that there exist* $r, \varepsilon, \mu, \nu \in (0,\infty)$, $c \in (\nu,\infty)$ *and positive functions* $\varphi \in L_{1_{\mathrm{loc}}}(0,T)$, $\psi \in L_1[0,T]$, $\omega \in C(0,\infty)$, $h \in C[0,\infty)$ *such that*

$$\frac{1}{p(t)}\int_0^t p(s)\psi(s)\,\mathrm{d}s \in L_1[0,T], \tag{2.25}$$

$$f(t,P(t),-c) \leqslant 0 \quad \text{for a.e. } t \in (0,T), \tag{2.26}$$

$$\varepsilon \leqslant f(t,x,y) \quad \text{for a.e. } t \in (0,T),\ \text{all } x \in (0,\nu], y \in [-\nu,\nu], \tag{2.27}$$

and

$$\begin{cases} -\psi(t) \leqslant f(t,x,y) \leqslant \varphi(t)(\omega(x)+h(x))(|y|+1)+ry^2, \\ \text{for a.e. } t \in (0,T),\ \text{all } x \in (0,P(t)],\ y \in \mathbb{R}, \end{cases} \tag{2.28}$$

hold, where ω *is nonincreasing,* h *is nondecreasing,* φ *and* h *satisfy* (2.3) *and* (2.8), *respectively, and* P *is given by* (2.7). *Then problem* (1.1), (1.2) *has a positive solution* $u \in C[0,T]$ *with* $pu' \in AC[0,T]$.

Proof. Let $k \in \mathbb{N}$, $k \geqslant 3/T$.

Step 1. Approximate solutions. For $t \in [0,T]$, $x, y \in \mathbb{R}$ define α_k, γ and p_k by (2.9), (2.10) and (2.11), respectively. Consider a sequence $\{\varrho_k\} \subset (1,\infty)$ satisfying $\lim\limits_{k\to\infty} \varrho_k = \infty$, and put for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$

$$\beta_k(y) = \begin{cases} y \text{ if } |y| \leqslant \varrho_k, \\ \varrho_k \operatorname{sign} y \text{ if } |y| > \varrho_k, \end{cases}$$

$$f_k(t,x,y) = \begin{cases} \gamma(y) \text{ if } t \in [0,1/k) \cup (T-1/k,T], \\ f(t,\alpha_k(t,x),\beta_k(y)) \text{ if } t \in [1/k,T-1/k]. \end{cases}$$

In such a way we have got a sequence of regular problems (2.13) fulfilling (2.12) and consequently a sequence of their solutions $\{u_k\}$ satisfying (2.14).

Step 2. A priori estimates of approximate solutions u_k. Without loss of generality we can assume that $\varepsilon > 0$ is so small that

$$\varepsilon \int_0^T p(s)\,\mathrm{d}s < \nu. \tag{2.29}$$

(I) Assume that $u_k(0) \geqslant \nu$. Since $u_k(T) = 0$ there exist $s_0 \in [0,T)$, $\tau_0 \in (s_0,T]$ such that

$$u_k(t) \geqslant \nu \quad \text{for } t \in [0,s_0] \tag{2.30}$$

and

$$u_k(s_0) = \nu, \quad u_k(t) < \nu \text{ for } t \in (s_0, \tau_0].$$

Then $u_k'(s_0) \leqslant 0$ and we will consider two cases: $-\nu < p_k(s_0)u_k'(s_0) \leqslant 0$ and $p_k(s_0)u_k'(s_0) \leqslant -\nu$.

C a s e A. Let $-\nu < p_k(s_0)u_k'(s_0) \leqslant 0$. Then there exists $t_0 \in (s_0, T]$ such that for $t \in [s_0, t_0]$

$$0 \leqslant u_k(t) \leqslant \nu, \quad |p_k(t)u_k'(t)| \leqslant \nu.$$

By (2.27) we get

$$p_k(t)u_k'(t) \leqslant -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s + p_k(s_0)u_k'(s_0) \leqslant -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s, \ t \in (s_0, t_0],$$

i.e. for $t \in [s_0, t_0]$

$$p_k(t)u_k'(t) \leqslant -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s. \tag{2.31}$$

Therefore $u_k(t) < \nu$, $u_k'(t) < 0$ and $p_k(t)u_k'(t) \geqslant -\nu$ on $(s_0, t_0]$. Assume that $t_0 < T$. Then there exists $t_1 \in (t_0, T]$ such that $p_k(t)u_k'(t) < -\nu$ for $t \in (t_0, t_1]$, which yields $u_k(t) < \nu$ and (2.31) on $[t_0, t_1]$. Assume that $t_1 < T$. Then there exists $t_2 \in (t_1, T]$ such that

$$-\nu < -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s < p_k(t)u_k'(t) \leqslant 0 \text{ for } t \in (t_1, t_2].$$

This implies that $u_k < \nu$ on $(t_1, t_2]$ and, by (2.27),

$$p_k(t)u_k'(t) \leqslant -\varepsilon \int_{t_1}^t p(s)\,\mathrm{d}s + p_k(t_1)u_k'(t_1) \leqslant -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s \text{ for } t \in (t_1, t_2],$$

a contradiction. So, we have proved $t_1 = T$ and hence, by (2.29),

$$(2.31) \quad \text{and} \quad u_k(t) < \nu \quad \text{hold on } (s_0, T]. \tag{2.32}$$

C a s e B. Let $p_k(s_0)u_k'(s_0) \leqslant -\nu$. Then there exists $s_1 \in (s_0, T]$ such that $0 \leqslant u_k(t) < \nu$ for $t \in (s_0, s_1]$ and, by (2.29),

$$p_k(t)u_k'(t) \leqslant -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s, \ t \in (s_0, s_1].$$

Assume that $s_1 < T$. Then there exists $s_2 \in (s_1, T]$ such that

$$-\nu < -\varepsilon \int_{s_0}^t p(s)\,\mathrm{d}s < p_k(t)u_k'(t) \leqslant 0 \text{ for } t \in (s_1, s_2].$$

This implies that $u_k < \nu$ on $(s_1, s_2]$ and, by (2.27),

$$p_k(t)u_k'(t) < -\varepsilon \int_{s_1}^{t} p(s)\,\mathrm{d}s + p_k(s_1)u_k'(s_1) \leqslant -\varepsilon \int_{s_0}^{t} p(s)\,\mathrm{d}s \text{ for } t \in (s_1, s_2],$$

a contradiction. So, we have proved $s_1 = T$, which yields (2.32). Denote

$$M = \max\{p(t)\colon\ t \in [0, T]\}. \tag{2.33}$$

Then, using (2.30) and integrating (2.31), we obtain

$$u_k(t) \geqslant \begin{cases} \nu \text{ for } t \in [0, s_0], \\ \varepsilon M^{-1} \int_t^T \int_{s_0}^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \text{ for } t \in [s_0, T]. \end{cases} \tag{2.34}$$

(II) Assume that $u_k(0) \in [0, \nu)$. Since $p_k(0)u_k'(0) = 0$, we can argue as in (I) Case A with $s_0 = 0$ and derive

$$p_k(t)u_k'(t) \leqslant -\varepsilon \int_0^t p(s)\,\mathrm{d}s \quad \text{for } t \in [0, T]. \tag{2.35}$$

Integrating this inequality and using (2.33), we have

$$u_k(t) \geqslant \varepsilon M^{-1} \int_t^T \int_0^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \quad \text{for } t \in [0, T]. \tag{2.36}$$

Choose an arbitrary interval

$$J = [a, b] \subset (0, T).$$

According to (2.7), (2.14), (2.34) and (2.36) there exists $k_0 \in \mathbb{N}$ such that for each $k \geqslant k_0$

$$J \subset [1/k, T - 1/k] \quad \text{and} \quad c_b \leqslant u_k(t) \leqslant P(a) \quad \text{for} \ \ t \in J, \tag{2.37}$$

where

$$c_b = \min\left\{\nu, \varepsilon M^{-1} \int_b^T \int_b^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\}.$$

Step 3. A priori estimates of $|p_k u_k'|$ on J. By virtue of (2.37) there exists $\xi_k \in (a, b)$ such that

$$p_k(\xi_k)u_k'(\xi_k) = \frac{u_k(b) - u_k(a)}{b - a} p_k(\xi_k)$$

and, using (2.33) and (2.37), we have

$$|p_k(\xi_k)u_k'(\xi_k)| \leqslant \frac{MP(a)}{T} = m_J. \tag{2.38}$$

Let $\max\{|p_k(t)u_k'(t)|\colon\ t \in [a,b]\} = |p_k(\eta_k)u_k'(\eta_k)| = R_k > m_J$. Then we can find $\zeta_k \in [a,b]$ such that

$$|p_k(\zeta_k)u_k'(\zeta_k)| = m_J \quad \text{and} \quad |p_k(t)u_k'(t)| \geqslant m_J \quad \text{for } t \in [\min\{\zeta_k,\eta_k\},\max\{\zeta_k,\eta_k\}].$$

Assume that $p_k(\eta_k)u_k'(\eta_k) = R_k$ and $\zeta_k > \eta_k$. By (2.9), (2.11), (2.28), (2.33), (2.37),

$$\int_{\zeta_k}^{\eta_k} \frac{(p_k(t)u_k'(t))'\,\mathrm{d}t}{p_k(t)u_k'(t)+1} \leqslant M\left[\bigl(\omega(c_b)+h(P(a))\bigr)\int_a^b \varphi(t)\,\mathrm{d}t + rMP(a)\right] = M_J,$$

and consequently

$$\int_{m_J}^{R_k} \frac{\mathrm{d}s}{s+1} \leqslant M_J. \tag{2.39}$$

Assume that $p_k(\eta_k)u_k'(\eta_k) = -R_k$ and $\zeta_k < \eta_k$. Similarly as above we get

$$\int_{\zeta_k}^{\eta_k} \frac{-(p_k(t)u_k'(t))'\,\mathrm{d}t}{-p_k(t)u_k'(t)+1} \leqslant M_J,$$

which gives (2.39). Since there exists $\varrho_J > 0$ such that $\int_{m_J}^{\varrho_J}(s+1)^{-1}\,\mathrm{d}s > M_J$, we get $R_k < \varrho_J$. If $p_k(\eta_k)u_k'(\eta_k) = R_k$ and $\zeta_k < \eta_k$ or $p_k(\eta_k)u_k'(\eta_k) = -R_k$ and $\zeta_k > \eta_k$, we get by (2.28)

$$R_k \leqslant m_J + \int_a^b p(t)\psi(t)\,\mathrm{d}t.$$

We can choose

$$\varrho_J \geqslant m_J + \int_a^b p(t)\psi(t)\,\mathrm{d}t$$

and then we have

$$|p_k u_k'(t)| \leqslant \varrho_J, \quad |u_k'(t)| \leqslant \frac{\varrho_J}{c_J} \quad \text{for } t \in J, \tag{2.40}$$

where $c_J = \min\{p(t)\colon\ t \in J\}$.

Step 4. Convergence of sequences of approximate solutions and problems. Having in mind (2.37) and (2.40) we get (2.17) and hence condition (2.18) and the inequality

$$0 < u(t) \leqslant P(t) \quad \text{for } t \in (0,T) \tag{2.41}$$

are valid. Further we can follow Step 4 of the proof of Theorem 2.1 to obtain (2.20) and (2.21).

Step 5. Properties of pu'. By (2.32) and (2.35) we have $p_k(T)u_k'(T) < 0$. The conditions (2.14) and $u_k(T) = \sigma_{2k}(T) = 0$ give

$$p_k(t)\frac{u_k(T) - u_k(t)}{T-t} \geqslant p_k(t)\frac{\sigma_{2k}(T) - \sigma_{2k}(t)}{T-t} \quad \text{for } t \in (0,T),$$

which yields

$$-c \leqslant p_k(T)u_k'(T) < 0. \tag{2.42}$$

According to (2.13) and (2.42) we have for each $k \geqslant 3/T$

$$\int_0^T p_k(s)f_k(s, u_k(s), p_k(s)u_k'(s))\,\mathrm{d}s = -p_k(T)u_k'(T) \in (0,c].$$

This together with (2.28), (2.41), (2.20) yields, by the Fatou lemma, that

$$p(t)f(t,u(t),p(t)u'(t)) \in L_1[0,T].$$

Therefore, by (2.21), $pu' \in AC[0,T]$.

Step 6. Properties of u. We will prove that $u \in C[0,T]$. Since pu' is continuous on $[0,T]$ and $1/p$ is continuous on $(0,T]$, we get $u \in C(0,T]$. It remains to prove that u is right continuous at $t=0$. Denote

$$\limsup_{t\to 0+} u(t) = A. \tag{2.43}$$

(i) Assume $A < \nu$. By (2.41) and (1.2) there is a $\delta_0 > 0$ such that

$$u(t) \in (0,\nu), \quad |p(t)u'(t)| \leqslant \nu \quad \text{for } t \in (0,\delta_0),$$

and so, due to (2.27), u is strictly decreasing on $(0,\delta_0)$. Hence

$$\lim_{t\to 0+} u(t) = A \in (0,\nu),$$

which yields $u \in C[0,T]$.

(ii) Assume $A \geqslant \nu$. Then there exist $t_0 \in [0,T)$ and $t_1 \in (t_0,T]$ such that $u(t_0+) = \nu$ and $u(t) < \nu$ for $t \in (t_0,t_1]$. If $t_0 = 0$, we get $u \in C[0,T]$ as in (i). Now, assume that $t_0 > 0$. Then we argue as in Step 2 and deduce $t_1 = T$. Hence, according to (1.2), we can find $t^* \in (0,T)$ such that $\nu \leqslant u(t)$ for $t \in (0,t^*)$. By (2.8) we can find $a \in (0,\infty)$ such that

$$\lim_{x\to\infty} \frac{h(x)}{x} \leqslant a.$$

Further, by (2.3), (2.43) and (1.2), there is $\delta^* \in (0,t^*)$ such that

$$\begin{gathered}\int_0^{\delta^*} \frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \leqslant \frac{1}{2(\nu+1)a},\\ \nu \leqslant u(t) \leqslant A+1, \quad |p(t)u'(t)| \leqslant \nu \quad \text{for } t \in (0,\delta^*).\end{gathered} \tag{2.44}$$

Moreover, (2.27) and (2.28) yield $\varepsilon \leqslant \varphi(t)[\omega(\nu)+h(\nu)]$ for a.e. $t \in (0,T)$. Thus for $t \in [0,T]$

$$0 \leqslant \frac{\varepsilon}{\omega(\nu)+h(\nu)} \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \leqslant \int_0^t \frac{1}{p(s)} \int_0^s p(\tau)\varphi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

and so, due to (2.3),

$$\int_0^{\delta^*} \frac{1}{p(s)} \int_0^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = c^* \in (0,\infty). \tag{2.45}$$

Integrating (2.28) and using (2.44) we get for $t \in (0,\delta^*)$

$$-p(t)u'(t) \leqslant (\omega(\nu)+h(A+1))(\nu+1)\int_0^t p(\tau)\varphi(\tau)\,\mathrm{d}\tau + r\nu^2 \int_0^t p(\tau)\,\mathrm{d}\tau$$

and integrating this inequality once more and using (2.44) and (2.45) we have for $t \in (0,\delta^*)$

$$u(t) \leqslant u(\delta^*) + (\omega(\nu)+h(A+1))\,\frac{1}{2a} + r\nu^2 c^*.$$

According to (2.43) we can choose a sequence $\{t_n\} \subset (0,\delta^*)$, $t_n \to 0$, and $u(t_n) \to A$. Therefore

$$A \leqslant u(\delta^*) + (\omega(\nu)+h(A+1))\,\frac{1}{2a} + r\nu^2 c^*$$

and

$$1 \leqslant \frac{1}{A}\Big[u(\delta^*) + \frac{\omega(\nu)}{2a} + r\nu^2 c^*\Big] + \frac{(A+1)h(A+1)}{2aA(A+1)} = F(A).$$

Since $\lim\limits_{x\to\infty} F(x) \leqslant 1/2$, there exists $A^* \in (0,\infty)$ such that $F(x) < 1$ for each $x \geqslant A^*$. Since $F(A) \geqslant 1$, we get $A \leqslant A^*$, which means that u is bounded on $[0,T]$. Due to (2.44) and (2.28)

$$-p(t)\psi(t) \leqslant -(p(t)u'(t))' \leqslant p(t)[\varphi(t)(\omega(\nu)+h(A+1))(\nu+1)+r\nu^2]$$

holds for a.e. $t \in (0,\delta^*)$. If we put $K_1 = (\omega(\nu) + h(A+1))(\nu+1)$, $K_2 = r\nu^2$ and integrate the above inequalities, we get on $(0,\delta^*)$

$$-\frac{1}{p(t)}\int_0^t p(\tau)\psi(\tau)\,\mathrm{d}\tau \leqslant -u'(t) \leqslant K_1\frac{1}{p(t)}\int_0^t p(\tau)\varphi(\tau)\,\mathrm{d}\tau + K_2\frac{1}{p(t)}\int_0^t p(\tau)\,\mathrm{d}\tau.$$

Due to (2.3), (2.25) and (2.45) there exists $h_0 \in L_1[0,\delta^*]$ such that $|u'(t)| \leqslant h_0(t)$ for a.e. $t \in (0,\delta^*)$. Therefore $u \in C[0,\delta^*]$, which completes the proof. $\square$

3. Examples

In Theorems 2.1 and 2.2 we assume that $\omega \in C(0,\infty)$ is positive and nonincreasing but no additional assumption about the behaviour of ω near the singularity $x=0$ is required. Therefore $\omega(x)$ can go to $+\infty$ for $x \to 0+$ very quickly, which means that $f(t,x,y)$ can have at $x=0$ a strong singularity.

Example 3.1. Let $\alpha,\gamma,\theta \in (0,\infty)$, $c_1,c_2 \in [0,\infty)$, $\beta \in [0,1]$, $0<\delta<\min\{2,\theta+1\}$. By Theorem 2.1 the problem

$$(t^\theta u')' + t^{\theta-\delta}(c_1u^{-\alpha} + c_2u^\beta + 1)(1-(t^\theta|u'|)^\gamma) = 0, \tag{3.1}$$

$$\lim_{t\to 0+} t^\theta u'(t) = 0, \quad u(1) = 0 \tag{3.2}$$

has a positive decreasing solution.

To see this we put $p(t) = t^\theta$, $\varphi(t) = t^{-\delta}$, $\nu = 1/2$, $\varepsilon = 1-(1/2)^\gamma$, $c=1$, $\omega(x) = c_1x^{-\alpha}+1$, $h(x) = c_2x^\beta + 1$ and $f(t,x,y) = t^{-\delta}(c_1x^{-\alpha} + c_2x^\beta + 1)(1-|y|^\gamma)$.

Remark 3.2. Note that:

1. Since α can be chosen in $(0,\infty)$, equation (3.1) can have both a weak singularity at $x=0$ (if we choose $\alpha \in (0,1)$) and a strong singularity at $x=0$ (if we choose $\alpha \geqslant 1$). Hence we generalize the results of [2] where only weak singularities are admitted. See Examples 2.2 and 2.3 in [2].
2. $\theta \in (0,\infty)$ implies that we can choose $\theta \geqslant 1$ and get $1/p \notin L_1[0,1]$.
3. Similarly, $0<\delta<\min\{2,\theta+1\}$ implies that if $\theta \geqslant 1$ we can choose $\delta \in [1,2)$ and get $\varphi \notin L_1[0,1]$.
4. Since $\beta \in [0,1]$, the function f can have for $x \to \infty$ either a sublinear growth (if $\beta \in (0,1)$) or a linear growth (if $\beta = 1$) or f can be bounded for large x (if $\beta = 0$).
5. $\gamma \in (0,\infty)$ yields that f can have a similar behaviour for large y as for large x but, moreover, f can have also a superlinear growth for $|y| \to \infty$ (if we choose $\gamma > 1$).

Example 3.3. Let $\alpha \in [0,\infty)$, $\beta \in [0,1]$, $\gamma, \theta \in [1,\infty)$, $\delta \in [1,2)$. Denote $q(t) = t^{-\delta} + (1-t)^{-\gamma}$, $q_1(t) = 1/\sqrt{t} + 1/\sqrt{1-t}$ and consider the equation

$$(t^\theta u')' + t^\theta q(t)[(u^{-\alpha} + u^\beta + 1)|1 + t^\theta u'| + 4(1 + t^\theta u')^2] \\ - t^\theta q_1(t)(\sin^2(u+1) + 1) = 0.$$

By Theorem 2.2 the problem (3.3), (3.2) has a positive solution.

To see this we put $p(t) = t^\theta$, $\varphi(t) = q(t) + 2q_1(t)$, $\psi(t) = 2q_1(t)$, $r = 4$, $\varepsilon = 1$, $\nu = 1/3$, $c = 1$, $\omega(x) = x^{-\alpha} + 1$, $h(x) = x^\beta + 1$ and $f(t,x,y) = q(t)[(x^{-\alpha} + x^\beta + 1)|1 + y| + 4(1+y)^2] - q_1(t)(\sin^2(x+1) + 1)$.

Remark 3.4. In Example 3.1 the function f is nonnegative on the set where we have found solutions, i.e. for $t \in (0,1]$, $x \in (0,\infty)$, $y \in [-1,0)$. Let us show that in Example 3.3 the function f changes its sign. We can see that $f(t,x,-1) < 0$ for $t \in (0,1)$, $x \in (0,\infty)$. On the other hand, for $t \in (0,1)$, $x \in (0,1/3]$, $y \in [-1/3,1/3]$ we have $f(t,x,y) > 1$.

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