CHOVER-TYPE LAWS OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF NA SEQUENCES

Guang-hui Cai, Hangzhou

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Abstract. To derive a Baum-Katz type result, a Chover-type law of the iterated logarithm is established for weighted sums of negatively associated (NA) and identically distributed random variables with a distribution in the domain of a stable law in this paper.

Keywords: negatively associated sequence, laws of the iterated logarithm, weighted sum, stable law, Rosental type maximal inequality

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1. Introduction

Let $\{X_j, j \geqslant 1\}$ are independently identically distributed (i.i.d.) with symmetric stable distributions. And let these distributions belong to the domain of normal attraction and non-degeneration. So, their characteristic functions are of the forms:

$$\mathrm{E}\exp(\mathrm{i}tX_j) = \exp(-|t|^{\alpha}),\ \ t \in \mathbb{R},\ \ j \geqslant 1.$$

Chover (1966) has obtained that

$$\limsup_{n\to\infty} \left(n^{-1/\alpha}\left|\sum_{j=1}^{n} X_j\right|\right)^{1/\log\log n} = \mathrm{e}^{1/\alpha} \text{ a.s.} \tag{1.1}$$

We call it Chover-type LIL (Laws of the iterated logarithm). This type of LIL has been shown by Vasudeva and Divanji [11], Zinchenko [13] for delayed sums, by Chen and Huang [2] for geometric weighted sums, and by Chen [1] for weighted sums. Note that Qi and Cheng [9] extended the Chover-type law of the iterated logarithm for the partial sums to the case when the underlying distribution is in the domain of attraction of a non-symmetric stable distribution (see below for details).

Let L_α denote a stable distribution with exponent $\alpha \in (0,2)$. Recall that the distribution of X is said to be in the domain of attraction of L_α if there exist constants $A_n \in \mathbb{R}$ and $B_n > 0$ such that

$$\frac{\sum_{j=1}^n X_j - A_n}{B_n} \xrightarrow{d} L_\alpha. \tag{1.2}$$

Assuming (1.2), Qi and Cheng (1996) and Peng and Qi (2003) showed that

$$\limsup_{n\to\infty} \left(B_n^{-1} \left| \sum_{j=1}^n X_j - A_n \right| \right)^{1/\log\log n} = \mathrm{e}^{1/\alpha} \text{ a.s.}$$

It is well known that (1.2) holds if and only if

$$1 - F(x) = \frac{C_1(x)l(x)}{x^\alpha},\ F(-x) = \frac{C_2(x)l(x)}{x^\alpha},\ x > 0, \tag{1.3}$$

where $F(x)$ denotes a stable distribution with exponent $\alpha \in (0,2)$ for $x > 0$, $C_i(x) \geqslant 0$, $\lim\limits_{x\to\infty} C_i(x) = C_i$, $i = 1, 2$, $C_1 + C_2 > 0$, and $l(x) \geqslant 0$ is a slowly varying in the sense of Karamata function, i.e.,

$$\lim_{t\to\infty} \frac{l(tx)}{l(t)} = 1 \text{ for } x > 0.$$

According to Lin (1999, page 76, Exercise 21), we have $B_n = (nl(n))^{1/\alpha}$.

As for negatively associated (NA) random variables, Joag (1983) gave the following definition.

Definition (Joag, 1983). A finite family of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$, we have

$$\mathrm{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leqslant 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

To derive a Baum-Katz type result, the main purpose of this paper is to establish a Chover-type law of the iterated logarithm for weighted sums of NA and indentically distributed random variables with a distribution in the domain of a stable law.

Throughout this paper, let $h \in B[0,1]$ denote that a function h is bounded on $[0,1]$. Further, C will represent a positive constant though its value may change from one appearance to another, and $a_n = O(b_n)$ will mean $a_n \leqslant Cb_n$.

2. Main results

In order to prove our results, we need the following lemma and definition.

Lemma 2.1 (Shao, 2000). *Let $\{X_i, i \geqslant 1\}$ be a sequence of NA random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geqslant 2$ and for every $i \geqslant 1$. Then there exists $C = C(p)$, such that*

$$\mathrm{E}\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big|^p \leqslant C\Big\{\sum_{i=1}^{n}\mathrm{E}\,|X_i|^p + \Big(\sum_{i=1}^{n}\mathrm{E}\,X_i^2\Big)^{p/2}\Big\}.$$

Definition (Lin and Lu, 1997). A function $f(x) > 0$ $(x > 0)$ is said to be quasi-monotone non-decreasing, if

$$\limsup_{x\to\infty}\sup_{0\leqslant t\leqslant x}\frac{f(t)}{f(x)} < \infty.$$

Now we state the main results and their proofs.

Theorem 1. *Let $\{X, X_i, i \geqslant 1\}$ be an NA sequence of identically distributed random variables with distribution $F(x)$, where $F(x)$ denotes a stable distribution with exponent $\alpha \in (0,2)$. Let h be a bounded function on $[0,1]$, $S_n = \sum\limits_{i=1}^{n} h(i/n)X_i$. We have $\mathrm{E}\,X = 0$, $\alpha > 1$. Let $f(x) > 0$ be quasi-monotone non-decreasing and $\int_1^\infty 1/(xf(x))\,\mathrm{d}x < \infty$. $l(x) \geqslant 0$ is a slowly varying in the sense of Karamata function, $\sup\limits_{n\geqslant 1} l(a_n)/l(n) < \infty$, where $a_n = (nf(n)l(n))^{1/\alpha}$. Then under condition (1.2), for any $\varepsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\leqslant j\leqslant n}|S_j| > \varepsilon(nf(n)l(n))^{1/\alpha}\Big) < \infty. \tag{2.1}$$

Proof of Theorem 1. For any $i \geqslant 1$, define $X_i^{(n)} = X_iI(|X_i| \leqslant a_n)$, $S_j^{(n)} = \sum\limits_{i=1}^{j}(h(i/n)X_i^{(n)} - \mathrm{E}\,h(i/n)X_i^{(n)})$, where $a_n = (nf(n)l(n))^{1/\alpha}$. Then for any $\varepsilon > 0$, we have

(2.2)
$$P\Big(\max_{1\leqslant j\leqslant n}|S_j| > \varepsilon a_n\Big) \leqslant P\Big(\max_{1\leqslant j\leqslant n}|X_j| > a_n\Big) + P\Big(\max_{1\leqslant j\leqslant n}|S_j^{(n)}| > \varepsilon a_n - \max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}\mathrm{E}\,h(i/n)X_i^{(n)}\Big|\Big).$$

First we show that

$$\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathrm{E}\,h(i/n)X_i^{(n)}\right|\to 0,\ \text{ as } n\to\infty. \tag{2.3}$$

Let us consider two cases, (i) when $0<\alpha\leqslant 1$, notice that $h\in B[0,1]$. Then for any positive integers n, N,

$$\begin{aligned}
&\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathrm{E}\,h(i/n)X_i^{(n)}\right|\leqslant\frac{1}{a_n}\sum_{i=1}^{n}\mathrm{E}\,|h(i/n)X_i^{(n)}|\\
&\quad\leqslant\frac{Cn}{a_n}\int_{|x|\leqslant a_n}|x|\,\mathrm{d}F(x)\leqslant\frac{Cn}{a_n}a_N+\frac{Cn}{a_n}\int_{a_N<|x|\leqslant a_n}|x|\,\mathrm{d}F(x)\\
&\quad=:C(A+B).
\end{aligned}$$

Notice that $f(x)>0$ is quasi-monotone non-decreasing and (1.3) holds. We have for $n\geqslant N$, N large enough,

$$\begin{aligned}
B&=\frac{n}{a_n}\sum_{k=N+1}^{n}\int_{a_{k-1}<|x|\leqslant a_k}|x|\,\mathrm{d}F(x)\leqslant\frac{n}{a_n}\sum_{k=N+1}^{n}a_kP(a_{k-1}<|X|\leqslant a_k)\\
&\leqslant C\sum_{k=N+1}^{n}kP(a_{k-1}<|X|\leqslant a_k)\leqslant CNP(|X|\geqslant a_N)+C\sum_{k=N}^{\infty}P(|X|\geqslant a_k)\\
&\leqslant C\frac{1}{f(N)}+C\sum_{k=N}^{\infty}\frac{1}{kf(k)}\leqslant C\frac{1}{f(N)}+C\int_N^{\infty}\frac{\mathrm{d}x}{kf(k)}<\frac{\varepsilon}{4}.
\end{aligned}$$

It is obvious that for each given N,

$$A\leqslant C\frac{a_N}{(f(n))^{1/\alpha}}\to 0,\ n\to\infty.$$

So, for $0<\alpha\leqslant 1$, we have (2.3).

(ii) When $1<\alpha<2$, using $EX_i=0$, $h\in B[0,1]$ and (1.3), when $n\to\infty$, then

$$\begin{aligned}
&\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathrm{E}\,h(i/n)X_i^{(n)}\right|=\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathrm{E}\,h(i/n)X_iI(|X_i|>a_n)\right|\\
&\quad\leqslant\frac{1}{a_n}\sum_{i=1}^{n}\mathrm{E}\,|h(i/n)X_i|I(|X_i|>a_n)\leqslant\frac{Cn}{a_n}\mathrm{E}\,|X|I(|X|>a_n)\\
&\quad=\frac{Cn}{a_n}\int_{a_n}^{\infty}P(|X|\geqslant x)\,\mathrm{d}x=\frac{Cn}{a_n}\int_{a_n}^{\infty}\frac{Cl(x)}{x^{\alpha}}\,\mathrm{d}x\\
&\quad=\frac{n}{a_n}Ca_n^{1-\alpha}l(a_n)\leqslant\frac{C}{f(n)}<\frac{\varepsilon}{2}.
\end{aligned}$$

So, for $1 < \alpha < 2$, we also have (2.3). Further, (i) and (ii) imply (2.3).

By (2.2) and (2.3), we have that

$$P\Big(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon a_n\Big) \leqslant \sum_{j=1}^{n} P(|X_j| > a_n) + P\Big(\max_{1\leqslant j\leqslant n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n\Big),$$

for n large enough. Hence we need only to prove

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > a_n) < \infty, \tag{2.4}$$

$$II =: \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leqslant j\leqslant n} |S_j^{(n)}| > \frac{\varepsilon}{2} a_n\Big) < \infty. \tag{2.5}$$

From (1.3), it is easily seen that

$$I = \sum_{n=1}^{\infty} P(|X| > a_n) \leqslant \sum_{n=1}^{\infty} \frac{C}{nf(n)} \leqslant C \int_1^{\infty} \frac{\mathrm{d}x}{xf(x)} < \infty. \tag{2.6}$$

Lemma 2.1 and the fact that $h \in B[0,1]$ imply that

(2.7)

$$\begin{aligned}
II &\leqslant C \sum_{n=1}^{\infty} n^{-1} \,\mathrm{E} \max_{1\leqslant j\leqslant n} |S_j^{(n)}|^2 \frac{1}{a_n^2} \leqslant C \sum_{n=1}^{\infty} n^{-1} \frac{1}{a_n^2} \Big(\sum_{i=1}^{n} \mathrm{E}\, |h(i/n) X_i^{(n)}|^2 \Big) \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \,\mathrm{E}\, |X|^2 I(|X| \leqslant a_n) = C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \int_{|x|\leqslant a_n} x^2 \,\mathrm{d}F(x) \\
&= C \sum_{n=1}^{\infty} \frac{1}{a_n^2} \sum_{k=1}^{n} \int_{a_{k-1}<|x|\leqslant a_k} x^2 \,\mathrm{d}F(x) \leqslant C \sum_{k=1}^{\infty} a_k^2 P(a_{k-1} < |X| \leqslant a_k) \sum_{n=k}^{\infty} \frac{1}{a_n^2} \\
&\leqslant C \sum_{k=1}^{\infty} k P(a_{k-1} < |X| \leqslant a_k) \leqslant C \int_1^{\infty} \frac{\mathrm{d}x}{xf(x)} < \infty.
\end{aligned}$$

Now we complete the proof of Theorem 1.

Corollary 1. *Under the conditions of Theorem* 1, *we have*

$$\limsup_{n\to\infty} \Big(\frac{|S_n|}{B_n}\Big)^{1/\log\log n} \leqslant \mathrm{e}^{1/\alpha} \ \text{a.s.} \tag{2.8}$$

P r o o f o f C o r o l l a r y 1. Notice that for any positive integer n there exists a non-negative integer k, such that $2^k \leqslant n < 2^{k+1}$. And there exists a $t \in [0,1)$, such that $n = 2^{k+t}$. Using (2.1), we obtain

$$\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (2^{k+1}-1)^{-1} P\Big(\max_{1\leqslant j\leqslant 2^{k+t}} |S_j| > \varepsilon(2^{k+1} f(2^{k+t}) l(2^{k+t}))^{1/\alpha}\Big) < \infty.$$

Then

$$\sum_{k=0}^{\infty} P\Big(\max_{1\leqslant j\leqslant 2^{k+t}} |S_j| > \varepsilon(2^{k+1} f(2^{k+t}) l(2^{k+t}))^{1/\alpha}\Big) < \infty,$$

and consequently

$$\frac{\max\limits_{1\leqslant j\leqslant 2^{k+t}} |S_j|}{(2^{k+1} f(2^{k+t}) l(2^{k+t}))^{1/\alpha}} \to 0 \text{ a.s.}$$

So

$$\begin{aligned} \frac{|S_n|}{(nf(n)l(n))^{1/\alpha}} &\leqslant \frac{\max\limits_{1\leqslant j\leqslant 2^{k+t}} |S_j|}{(2^{k+1} f(2^{k+t}) l(2^{k+t}))^{1/\alpha}} \frac{(2^{k+1} f(2^{k+t}) l(2^{k+t}))^{1/\alpha}}{(nf(n))^{1/\alpha}} \\ &\leqslant 2^{1/\alpha} \frac{\max\limits_{1\leqslant j\leqslant 2^{k+t}} |S_j|}{(2^{k+1} f(2^{k+t}))^{1/\alpha}} \to 0 \text{ a.s.} \end{aligned}$$

Then

$$\limsup_{n\to\infty} \frac{|S_n|}{(nf(n)l(n))^{1/\alpha}} = 0 \text{ a.s.} \tag{2.9}$$

Given $\varepsilon > 0$, let $f(x) = \log^{1+\varepsilon} x$. It is obvious that $\int_1^\infty 1/(xf(x))\,\mathrm{d}x < \infty$. By (2.9), we have

$$\limsup_{n\to\infty} \frac{|S_n|}{(nl(n)\log^{1+\varepsilon} n)^{1/\alpha}} = 0 \text{ a.s.}$$

Then

$$\limsup_{n\to\infty} \Big(\frac{|S_n|}{B(n)}\Big)^{1/\log\log n} \leqslant \mathrm{e}^{(1+\varepsilon)/\alpha} \text{ a.s.}$$

Therefore

$$\limsup_{n\to\infty} \Big(\frac{|S_n|}{B(n)}\Big)^{1/\log\log n} \leqslant \mathrm{e}^{1/\alpha} \text{ a.s.}$$

Now we complete the proof of (2.8). □

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Author's address: *Guang-hui Cai*, Department of Mathematics and Statistics, Zhejiang Gongshang University, Hangzhou 310035, P. R. China, e-mail: cghzju@163.com.