GALLAI AND ANTI-GALLAI GRAPHS OF A GRAPH

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Abstract. The paper deals with graph operators—the Gallai graphs and the anti-Gallai graphs. We prove the existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free for any finite graph H. The case of complement reducible graphs—cographs is discussed in detail. Some relations between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs are also obtained.

Keywords: Gallai graphs, anti-Gallai graphs, cographs

MSC 2000: 05C99

1. Introduction

This paper mainly deals with graph operators, the Gallai graph $\Gamma(G)$ and the anti-Gallai graph $\Delta(G)$. Both the Gallai and the anti-Gallai graphs are spanning subgraphs of the well known class of line graphs. The line graph [8] $L(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are incident in G.

The Gallai graph $\Gamma(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $\Gamma(G)$ if they are incident in G, but do not span a triangle in G. In [6], it has been proved that $\Gamma(G)$ is isomorphic to G only for cycles of length greater than 3. Computing the clique number and the chromatic number of $\Gamma(G)$ are NP-complete problems. The notion of the Gallai perfect graph is discussed in [12].

The anti-Gallai graph $\Delta(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $\Delta(G)$ if they are incident in G and lie on a triangle in G. It is the complement of $\Gamma(G)$ in $L(G)$. Though $L(G)$ has a forbidden subgraph characterization, both the Gallai graphs and the anti-Gallai graphs do not have the vertex hereditary property and hence cannot be characterized using forbidden subgraphs [6]. Several other graph operators are discussed in [8].

The study of H-free graphs—graphs which do not have H as an induced subgraph—for some classes of graphs H are quite interesting. Some classes of H-free graphs are discussed in [3]. An important class of perfect graphs called the complement reducible graphs or cographs have been extensively studied. Cographs are recursively defined in [4], [11] as follows:

(1) K_1 is a cograph

(2) If G is a cograph, so is its complement $\overline{G}$ and

(3) If G and H are cographs, so is their join, $G \vee H$, where the join (sum) of two graphs G and H is defined as the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv$, where $u \in V(G)$ and $v \in V(H)\}$.

It is known [7] that a graph is a cograph if and only if it is P_4-free. Various other aspects of cographs are discussed in [4], [5], [9], [10], [11].

In this paper we prove that there exist infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai and anti-Gallai graphs. We prove the existence of a finite family of forbidden subgraphs for the Gallai graphs and anti-Gallai graphs to be H-free for any finite graph H. The list of forbidden subgraphs for $H = P_4$ is given. The connected P_4-free graphs—cographs whose Gallai and anti-Gallai graphs are also P_4-free are determined. The relationship between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs are also obtained.

All graph theoretic terminology and notation not mentioned here are from [1].

2. Gallai and anti-Gallai graphs

It is well known [1] that the only pair of non-isomorphic graphs having the same line graph is $K_{1,3}$ and K_3. But, we first observe that, in the case of both Gallai and anti-Gallai graphs, which are spanning subgraphs of $L(G)$, there are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs and anti-Gallai graphs.

Theorem 1. *There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs.*

Proof. We prove this theorem by the following two types of constructions.

T y p e 1. Let $G = P_4$ with n independent vertices joined to both its internal vertices and an end vertex attached to k of these n vertices, and $H =$ two copies of $K_{1,n+1}$ with $k+1$ distinct pairs of end vertices made adjacent.

The graph G of type 1 is as follows. Let $v_1v_2v_3v_4$ be an induced P_4. Let v_2 and v_3 be joined to n vertices $u_1, u_2, \ldots, u_n$. Introduce k end vertices $w_1, w_2, \ldots, w_k$ such that each w_i is adjacent only to u_i for $i = 1, 2, \ldots, k$. The edges $v_1v_2, v_2u_1, v_2u_2, \ldots, v_2u_n$ of G, which are vertices of $\Gamma(G)$, will induce a complete graph on $n+1$ vertices in $\Gamma(G)$. Similarly, $v_3v_4, v_3u_1, v_3u_2, \ldots, v_3u_n$ will induce another complete graph on $n+1$ vertices in $\Gamma(G)$. The vertex corresponding to the edge v_2v_3 will be adjacent to both the vertices corresponding to v_1v_2 and v_3v_4. The k vertices corresponding to the edges u_iw_i for $i = 1, 2, \ldots, k$ will be adjacent to the vertices corresponding to the edges u_iv_2 and u_iv_3 for $i = 1, 2, \ldots, k$ respectively.

The graph H of type 1 is as follows. Let u adjacent to $u_1, u_2, \ldots, u_{n+1}$ and v adjacent to $v_1, v_2, \ldots, v_{n+1}$ be the two $K_{1,n+1}$-s in H. Let $u_1v_1, u_2v_2, \ldots, u_{k+1}v_{k+1}$ be the $k+1$ distinct pairs of adjacent vertices in H. The vertices corresponding to the edges $uu_1, uu_2, \ldots, uu_{n+1}$ will induce a complete graph on $n+1$ vertices in $\Gamma(H)$. Similarly, the vertices corresponding to $vv_1, vv_2, \ldots, vv_{n+1}$ will also induce another complete graph on $n+1$ vertices in $\Gamma(H)$. Again, the vertices corresponding to the edges u_iv_i for $i = 1, 2, \ldots, k+1$ will be adjacent to the vertices corresponding to the edges uu_i and vv_i for $i = 1, 2, \ldots, k+1$ respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of complete graphs on $n+1$ vertices together with $k+1$ new vertices made adjacent to $k+1$ distinct vertices of both the complete graphs.

T y p e 2. Let $G = P_4$ with n independent vertices joined to both its internal vertices and an end vertex attached to k of them with $k \geqslant 1$ together with one end vertex attached to each of the end vertices of P_4, and $H =$ two copies of $K_{1,n+1}$ with $k+1$ distinct pairs of end vertices (one from each star) made adjacent and a single pair made adjacent to another vertex.

The graph G of type 2 can be obtained from the graph G of type 1 by attaching two end vertices x and y to v_1 and v_2 respectively. In $\Gamma(G)$ the vertices corresponding to the edges v_1x and v_4y will be adjacent to the vertices corresponding to the edges v_1v_2 and v_3v_4 respectively.

The graph H of type 2 can be obtained from the graph H of type 1 by adding a new vertex w and making it adjacent to both u_1 and v_1. In $\Gamma(H)$ the vertices corresponding to the edges wu_1 and wv_1 will be adjacent to the vertices corresponding to the edges uu_1 and vv_1 respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of complete graphs on $n+1$ vertices together with $k+1$ vertices made adjacent to $k+1$ distinct vertices of both the complete graphs and two end vertices adjacent to one vertex from each of the complete graphs.

The constructions mentioned in type 1 and type 2 are illustrated in Table 1. In both the cases, the graphs G and H have the same Gallai graph. If $n = k$ and $n = k - 1$ in type 1 and type 2 respectively, then the order of G and H is the same.

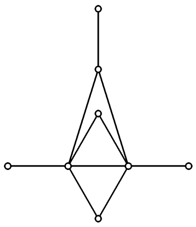
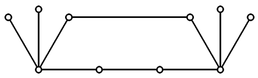
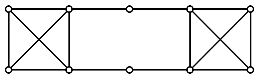
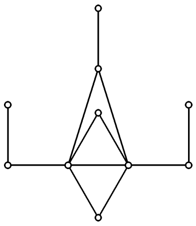
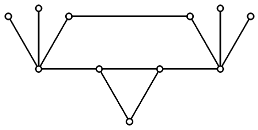
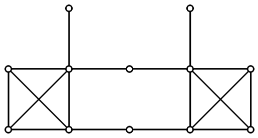
	G	H	$\Gamma(G) = \Gamma(H)$
Type 1 $n = 3$ $k = 1$			
Type 2 $n = 3$ $k = 1$			

Table 1.

Theorem 2. *There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic anti-Gallai graphs.*

Proof. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and an edge v_iv_j such that G is not isomorphic to a graph obtained under permutations of the index set of the vertices which interchange i and j and $\Delta(G)$ is connected. Introduce a vertex u adjacent to v_i and v_j. Let H_1 be the graph obtained by introducing one more vertex u_1 adjacent to u and v_i. Let H_2 be the graph obtained by introducing another vertex u_2 (u_1 is absent here) adjacent to u and v_j. Then by construction H_1 and H_2 are non-isomorphic. $\Delta(H_1)$ is $\Delta(G)$ together with four more vertices corresponding to uv_i, uv_j, uu_1, v_iu_1 in which uv_i and uv_j are adjacent to each other and to v_iv_j, uu_1 and v_iu_1 are adjacent to each other and to uv_i. $\Delta(H_2)$ is $\Delta(G)$ together with four more vertices corresponding to uv_i, uv_j, uu_2, v_ju_2 in which uv_i and uv_j are adjacent to each other and to v_iv_j, uu_2 and v_ju_2 are adjacent to each other and to uv_j. Therefore, $\Delta(H_1)$ is isomorphic to $\Delta(H_2)$.

However, the following problem is open.

Problem. Characterize all pairs of non-isomorphic graphs of the same order having isomorphic Gallai graph and anti-Gallai graph.

3. Forbidden subgraph characterizations

A property P of a graph G is vertex hereditary if every induced subgraph of G has the property P.

Notation 3. For a connected graph H, let $G(H) = \{G\colon \Gamma(G) \text{ is } H\text{-free}\}$ and $G^*(H) = \{G\colon \Delta(G) \text{ is } H\text{-free}\}$.

Theorem 4. *The properties of being an element of $G(H)$ and $G^*(H)$ are vertex hereditary.*

Proof. Let $G \in G(H)$ and $v \in V(G)$. Consider $G' = G - \{v\}$. We desire to prove that $G' \in G(H)$. On the contrary assume that $\Gamma(G')$ has H as an induced subgraph. Let $v_1, v_2, \ldots, v_t$ be neighbors of v. Therefore $\Gamma(G)$ has the vertex set $V(\Gamma(G') \cup \{vv_1, vv_2, \ldots, vv_t\}$. In $\Gamma(G)$, vv_i is adjacent to vv_j if v_i is not adjacent to v_j, and vv_i will be adjacent to all edges which have v_i as one end vertex and other end vertex is not v_j for $j = 1, 2, \ldots, t$. Hence if H is an induced subgraph of $\Gamma(G')$ then H is an induced subgraph of $\Gamma(G)$ also, which is a contradiction.

The case of $G^*(H)$ follows similarly.

Corollary 5. *$G(H)$ and $G^*(H)$ have vertex minimal forbidden subgraph characterization.*

Though many well known classes of graphs admit forbidden subgraph characterizations, the number of such forbidden subgraphs need not be finite (as in the case of planar graphs). However, for $G(H)$ and $G^*(H)$ we have

Theorem 6. *For every vertex minimal forbidden subgraph of $G(H)$ and $G^*(H)$, the number of vertices is bounded above by $n(H)+1$, where $n(H)$ denotes the number of vertices in H.*

Proof. Let $F(H)$ be the collection of all vertex minimal forbidden subgraphs of $G(H)$. Let $L \in F(H)$. Therefore, $\Gamma(L)$ has H as an induced subgraph. The $n(H)$ vertices of H, which correspond to $n(H)$ edges of L, say $e_1, e_2, \ldots, e_{n(H)}$, can cover a maximum of $n(H) + 1$ vertices of L, since H is connected.

We should prove that $n(L) \leqslant n(H) + 1$. To the contrary assume that $n(L) > n(H) + 1$. Then there exists at least one vertex $v \in V(L)$ which is not an end vertex of any of $e_1, e_2, \ldots, e_{n(H)}$. Therefore, $\Gamma(L - v)$ still has H as an induced subgraph, which contradicts that L is a vertex minimal forbidden subgraph of $G(H)$. Hence, $n(L) \leqslant n(H) + 1$.

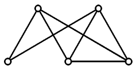
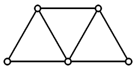
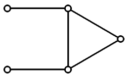
A similar argument holds for $G^*(H)$ also.

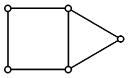
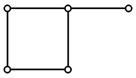
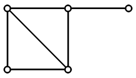
Corollary 7. *The number of vertex minimal forbidden subgraphs for $G(H)$ and $G^*(H)$ is finite.*

In the next theorem, we obtain a forbidden subgraph characterization of G for $\Gamma(G)$ to be a cograph.

Theorem 8. *Let G be a graph. Then, $\Gamma(G)$ is a cograph if and only if G does not have the following graphs as induced subgraphs.*

(i) P_5 (ii) C_5 (iii) $K_{2,3}$

(iv)  (v)  (vi) 

(vii)  (viii)  (ix) 

P r o o f. If $\Gamma(G)$ is not a cograph then there exists an induced P_4 in $\Gamma(G)$, say $e_1e_2e_3e_4$. In G, let $e_1 = u_{11}u_{12}$, $e_2 = u_{21}u_{22}$, $e_3 = u_{31}u_{32}$ and $e_4 = u_{41}u_{42}$.

Since e_1 is adjacent to e_2, let $u_{12} = u_{21}$ and let u_{11} be not adjacent to u_{22}. Since e_2 is adjacent to e_3, either $u_{21} = u_{31}$ or $u_{22} = u_{31}$.

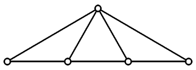
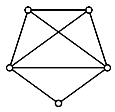
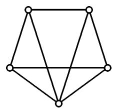
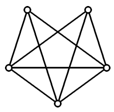
If $u_{21} = u_{31}$, then since e_1 is not adjacent to e_3, u_{11} is adjacent to u_{32}. Since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then since e_1 and e_2 are not adjacent to e_4, both u_{11} and u_{21} are adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is not adjacent to u_{42}.

If $u_{22} = u_{31}$, then u_{21} is not adjacent to u_{32}. Again, since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then since e_2 and e_4 are not adjacent, u_{21} is adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is not adjacent to u_{42}. The above four resulting graphs are respectively (iv), (vi), (vi) and (i).

In (iv), if we add even a single edge the property of $\Gamma(G)$ not being a cograph will be lost. In (vi), u_{22} adjacent to u_{42} gives (vii), u_{11} adjacent to u_{42} gives (ix) and the combination of both gives (iv). The addition of these edges will not change the required property either. In (i), u_{11} adjacent to u_{42} gives (ii), u_{11} adjacent to u_{41} gives (viii) and a combination of both gives (iii). Again, the addition of these edges will not change the required property. However, if we add any other edge then the property will be lost.

The converse can be easily proved.

Theorem 9. *Let G be a graph. Then $\Delta(G)$ is a cograph if and only if G does not have the following graphs as induced subgraphs.*

(i) K_5 (ii)  (iii)  (iv)  (v) 

P r o o f. If $\Delta(G)$ is not a cograph then there exists an induced P_4 in $\Delta(G)$, say $e_1e_2e_3e_4$. In G, let $e_1 = u_{11}u_{12}$, $e_2 = u_{21}u_{22}$, $e_3 = u_{31}u_{32}$ and $e_4 = u_{41}u_{42}$.

Since e_1 is adjacent to e_2, let $u_{12} = u_{21}$ and let u_{11} be adjacent to u_{22}. Since e_2 is adjacent to e_3, either $u_{21} = u_{31}$ or $u_{22} = u_{31}$.

If $u_{21} = u_{31}$ then u_{22} is adjacent to u_{32} and u_{11} is not adjacent to u_{31}. Since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then u_{32} is adjacent to u_{42} and u_{11} and u_{22} are not adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is adjacent to u_{42}.

If $u_{22} = u_{31}$ then u_{12} is adjacent to u_{32}. Again, since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then u_{32} is adjacent to u_{42} and u_{21} is not adjacent to u_{42}. If $u_{32} = u_{42}$ then u_{31} is adjacent to u_{42}.

All the four resulting graphs are isomorphic to (ii) itself. Also, addition of any of the possible edges will leave an induced P_4 in $\Delta(G)$ and hence any graph with 5 vertices which contains (ii) as a (not induced) subgraph are also forbidden. Hence all the above graphs are forbidden.

Conversely, it can be verified that the anti-Gallai graph will not be a cograph if any of the nine graphs listed above is an induced subgraph of G.

4. Cographs

Theorem 10. *If G is a connected cograph without a vertex of full degree then $\Gamma(G)$ is a cograph if and only if $G = ({}^pK_2)^c$, the complement of p copies of K_2.*

P r o o f. Let $G = ({}^pK_2)^c$. Then the number of vertices of G is $2p$ and the number of edges of G is $2p(p-1)$. Let the vertices of G be $\{v_{11}, v_{12}, \ldots, v_{1p}, v_{21}, v_{22}, \ldots, v_{2p}\}$ with v_{1j} and v_{2j} as the only pair of non-adjacent vertices. Therefore, vertices of the Gallai graph are of the form $v_{ij}v_{i'j'}$ where $j \neq j'$. By definition of the Gallai graph, $v_{ij}v_{i'j'}$ will be adjacent only to $v_{ij}v_{1j'}$ or $v_{ij}v_{2j'}$ and $v_{1j}v_{i'j'}$ or $v_{2j}v_{i'j'}$ according to the value of i and i'. Therefore, $\Gamma(G) = ({}^pC_2)C_4$, which is a cograph.

Conversely, assume that G is a cograph without a vertex of full degree and $\Gamma(G)$ is also a cograph. For every $u \in V(G)$, there exist at least one $u' \in V(G)$ which is not adjacent to u.

Claim: u' is the only vertex which is not adjacent to u.

To the contrary assume that there exists another vertex u'' which is not adjacent to u. Since G is a connected cograph, $G = G_1 \vee G_2$. Let $u \in V(G_1)$. Since u is not adjacent to both u' and u'', both of them belong to $V(G_1)$. Since G has no vertex of full degree, G_2 must contain at least two non-adjacent vertices v_1 and v_2. Then the edges $u''v_1$, v_1u, uv_2, v_2u' will induce a P_4 in $\Gamma(G)$, which is a contradiction.

Therefore $G = ({}^pK_2)^c$, where $2p = n$.

Notation 11. Consider the class of graphs which are recursively defined as follows:

$\mathcal{H}_1 = \{G\colon\ G = (pK_2)^c \vee (K_q)$, where $p, q \geqslant 0\}$.
$\mathcal{H}_i = \{G\colon\ G = (\bigcup H_{i-1}) \vee K_r$, where $H_{i-1} \in \mathcal{H}_{i-1}$ and $r \geqslant 0\}$ for $i > 1$.
$\mathcal{H} = \bigcup \mathcal{H}_i$

Theorem 12. *For a connected cograph G, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$.*

Proof. Let G be a cograph other than K_q with a vertex of full degree. Let V_1 be the collection of all full degree vertices in G. Define $G_1 = \langle V - V_1\rangle$. $\Gamma(G_1)$ is and induced subgraph of $\Gamma(G)$. More precisely, $\Gamma(G) = \Gamma(G_1)$ together with some isolated vertices. Therefore, $\Gamma(G)$ is a cograph if and only if $\Gamma(G_1)$ is a cograph. If G_1 is a connected cograph then G_1 has no vertex of full degree and hence $\Gamma(G_1)$ is a cograph if and only if $G_1 = ({}^pK_2)^c$. Therefore, $\Gamma(G)$ is a cograph if and only if $G = ({}^pK_2)^c \vee (K_q) \in \mathcal{H}_1$.

If G_1 is disconnected, then consider each of the connected components of G_1. If the removal of all full degree vertices from each of the components of G_1 preserves connectedness then as above each of these components must be of the form $({}^pK_2)^c \vee (K_q)$. Therefore, $G = (F_1 \cup F_2 \cup \ldots \cup F_p) \vee K_q$ where each $F_i \in \mathcal{H}_1$ and $q \geqslant 0$. Consequently, $G \in \mathcal{H}_2$.

If any of the components of G_1, say G_2, is disconnected then repeat the above process to get $G_1 \in \mathcal{H}_2$ and hence $G = (H_1 \cup H_2 \cup \ldots \cup H_r) \vee K_s$ where each $H_i \in \mathcal{H}_2$ and $r \geqslant 0$. Consequently, $G \in \mathcal{H}_3$.

This process must terminate since the number of vertices of G is finite. Therefore for a connected cograph G, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$.

Theorem 13. *For a connected cograph G, $\Delta(G)$ is a cograph if and only if*

(i) *$G = G_1 \vee G_2$, where G_1 is edgeless and G_2 does not contain P_4 as a subgraph (which need not be induced) or*

(ii) *G is C_4.*

Proof. Let G be a connected cograph whose $\Delta(G)$ is also a cograph. Since G is a connected cograph, $G = G_1 \vee G_2$. Let G_1 be an edgeless graph and $u \in V(G_1)$.

If G_2 contains a P_4, say $v_1v_2v_3v_4$, then the edges v_1v_2, v_2u, uv_3, v_3v_4 of G induce a P_4 in $\Delta(G)$, which is a contradiction. Therefore, if G_1 is edgeless then G_2 does not contain P_4 as a subgraph.

Let $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. If G_1 contains one more vertex, say v, not adjacent to u_1 and v_1, then the edges u_1v_1, v_1u_2, u_2v_2, u_2u of G induce a P_4 in $\Delta(G)$, which is a contradiction. If v is adjacent to at least one of the vertices, say v_1, then the edges u_1u_2, u_2v_1, v_1v_2, v_2v of G induce a P_4 in $\Delta(G)$, which is a contradiction. A similar argument holds also for the vertex set of G_2. Therefore both G_1 and G_2 are K_2's and $G = C_4$.

Conversely, assume that G is a cograph of type (i) or (ii). Then G does not contain any of the graphs from Theorem 9 as an induced subgraph and hence $\Delta(G)$ is a cograph.

5. Chromatic number

Theorem 14. *Given two positive integers a, b, where $a > 1$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Gamma(G)) = b$.*

Proof. If $a = 1$ then G must be a graph without edges, which makes $\Gamma(G)$ empty. So we can assume that $a > 1$.

Let G be the graph K_a together with $b-1$ end vertices attached to any one of the vertices. Then $\Gamma(G)$ is $a-1$ copies of K_b sharing $b-1$ vertices in common together with some isolated vertices. Clearly, $\chi(G) = a$ and $\chi(\Gamma(G)) = b$.

Lemma 15. *The anti-Gallai graph of any graph G cannot be bipartite except for the triangle free graphs.*

Proof. If u_1 is adjacent to u_2 in $\Delta(G)$ then the corresponding edges, say e_1 and e_2, lie in a triangle, say $e_1e_2e_3$. Then the vertex u_3 in $\Delta(G)$ which corresponds to e_3 will be adjacent to both u_1 and u_3. Therefore, $u_1u_2u_3$ induces a cycle of odd length in $\Delta(G)$ and hence $\Delta(G)$ cannot be bipartite.

Theorem 16. *Given two positive integers a, b, where $b < a$, $b \neq 2$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Delta(G)) = b$. Further, for any odd number a, there exists a graph G such that $\chi(G) = \chi(\Delta(G)) = a$.*

Proof. First note that the anti-Gallai graph of a graph G cannot be bipartite except for the triangle free graphs by the above lemma. Hence, $b = \chi(\Delta(G)) \neq 2$ for any G.

Using Myceilski's construction [1] there exists a triangle-free graph H with chromatic number a. For H, $\Delta(G)$ is a trivial graph and hence $b = 1$. For $2 < b < a$, there exists an induced subgraph H' of H whose chromatic number is b. Let $v_1, v_2, \ldots, v_n$ be the vertices of H'. Let G be the graph obtained from H by joining all vertices of H' to a new vertex u. Since $b < a$, $\chi(G) = a$ itself. If v_i and v_j are adjacent (or non-adjacent) in H' then the vertices corresponding to uv_i and uv_j are adjacent (or non-adjacent) in $\Delta(G)$. Therefore, the vertices corresponding to the edges $uv_1, uv_2, \ldots, uv_n$ induce an H' in $\Delta(G)$. Again for any pair of adjacent vertices, say v_i and v_j in H', the vertices corresponding to the edges uv_i and uv_j are adjacent to the vertex corresponding to v_1v_2. Therefore $\Delta(G)$ is H' together with one vertex each adjacent to both the end vertices of each edge in H'. For $b > 2$, $\chi(\Delta(G)) = \chi(H') = b$.

If a is an odd integer then $\chi(K_a) = a$ and $\chi(\Delta(G)) = \chi(L(G)) = \chi'(K_a) = a$, where χ' is the edge chromatic number.

6. Radius and diameter

In this section $r(G)$ and $d(G)$ denote the radius and the diameter of a graph G respectively.

Theorem 17. *Let G be a graph such that $\Gamma(G)$ is connected. Then $r(\Gamma(G)) \geqslant r(G) - 1$ and $d(\Gamma(G)) \geqslant d(G) - 1$.*

Proof. Let $r(\Gamma(G)) = r$. Then there exists an edge, say uv, in G which is at a distance less than or equal to r from every other edge in G. Hence, any vertex of G is at a distance less than or equal to $r+1$ from both u and v. We have $r(G) \leqslant r+1$, which implies $r(\Gamma(G)) \geqslant r(G) - 1$.

Let $d(G) = d$. There exist two vertices u and v such that the distance between u and v is $d(u, v) = d$. Let $uu_1u_2u_{a-1}v$ be a shortest path connecting u and v in G.

Claim: $d_{\Gamma(G)}(uu_1, u_{a-1}v) = a - 1$. $uu_1, u_1u_2, u_{a-1}v$ is a path of length $a-1$ connecting uu_1 and $u_{a-1}v$ in $\Gamma(G)$. Therefore, $d_{\Gamma(G)}(uu_1, u_{a-1}v) \leqslant a - 1$.

It is required to prove that $d_{\Gamma(G)}(uu_1, u_{a-1}v) = a - 1$. On the contrary assume that there exists an induced path $uu_1, v_1v_1', v_2v_2', v_{k-1}v_{k-1}', u_{a-1}v$ of length k in $\Gamma(G)$ connecting uu_1 and $u_{a-1}v$, where $k < a - 1$. Then there exists a path of length less than or equal to $a-1$ connecting u and v in G, which contradicts $d(u, v) = a$. Hence, $d_{\Gamma(G)}(uu_1, u_{a-1}v) = a - 1$.

Since there exist two vertices of $\Gamma(G)$ which are at a distance $a-1$, $d(\Gamma(G))$ must be greater than or equal to $a - 1$.

Note 18. If a and b are two positive integers such that $a > 1$ and $b \geqslant a - 1$ then there exist graphs G and H whose Gallai graphs are connected and $r(G) = a$, $r(\Gamma(G)) = b$, $d(H) = a$ and $d(\Gamma(H)) = b$.

Theorem 19. *If G is a graph such that $\Delta(G)$ is connected and $r(G) > 1$, $r(\Delta(G)) \geqslant 2(r(G)-1)$ and $d(\Delta(G)) \geqslant 2(d(G)-1)$.*

P r o o f. Let $r(\Delta(G)) = r > 1$. There exists an edge, say uv, in G such that any edge is at a distance less than or equal to r from uv in $\Delta(G)$. Let $w \in V(G)$. Since G is connected there exists at least one edge with w as an end vertex, say ww'. There exists a path of length less than or equal to r from ww' to uv in $\Delta(G)$. Any two adjacent edges in $\Delta(G)$ belong to a triangle and hence this path induces a path of length less than or equal to $\frac{1}{2}r$ from either u or v to w. Therefore, any vertex is at a distance less than or equal to $\frac{1}{2}r+1$ from both u and v. Hence $r(G) \leqslant \frac{1}{2}r+1$, which implies that $r(\Delta(G)) \geqslant 2(r(G)-1)$.

Let $d(G) = d$. There exist two vertices u and v such that $d(u,v) = d$. Let $uu_1u_2\ldots u_{d-1}v$ be a shortest path connecting u and v. Consider $d(uu_1, u_{d-1}v)$ in $\Delta(G)$. If it is k, then there exists a path of length less than or equal to $\frac{1}{2}k+1$ in G connecting u and v. Therefore, $\frac{1}{2}k+1 \geqslant d$, which implies $k = 2(d-1)$. However, $d(\Delta(G)) \geqslant k$. Hence, $d(\Delta(G)) \geqslant 2(d(G)-1)$.

N o t e 20. If a and b are two positive integers such that $a > 1$ and $b \geqslant 2(a-1)$ then there exist graphs G and H whose anti-Gallai graphs are connected with $r(G) = a$, $r(\Delta(G)) = b$, $d(H) = a$ and $d(\Delta(H)) = b$.

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