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PROPERTIES OF A HYPOTHETICAL EXOTIC COMPLEX STRUCTURE ON $\mathbb{CP}^3$

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Abstract. We consider almost-complex structures on $\mathbb{CP}^3$ whose total Chern classes differ from that of the standard (integrable) almost-complex structure. E. Thomas established the existence of many such structures. We show that if there exists an "exotic" integrable almost-complex structures, then the resulting complex manifold would have specific Hodge numbers which do not vanish. We also give a necessary condition for the nondegeneration of the Frölicher spectral sequence at the second level.

Keywords: complex structure, projective space, Frölicher spectral sequence, Hodge numbers

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1. Introduction

It is well-known that the six sphere $\mathbb{S}^6$ admits almost-complex structures, for example [6, Chapter IX Ex 2.6]. Blowing up an almost-complex $\mathbb{S}^6$ at a point produces an almost-complex manifold diffeomorphic to $\mathbb{CP}^3$. We will call the resulting almost-complex structure on this manifold "exotic" because its Chern classes are topologically different from the Chern classes of the standard (integrable) almost-complex structure on $\mathbb{CP}^3$. A long standing question in differential geometry is whether or not $\mathbb{S}^6$ admits a complex structure, that is, an integrable almost-complex structure. If it does, then blowing it up at a point will give an exotic complex structure on $\mathbb{CP}^3$. This is interesting because Hirzebruch and Kodaira have shown in [3] that any Kähler manifold of odd complex dimension diffeomorphic to $\mathbb{CP}^n$ is biholomorphic to $\mathbb{CP}^n$. Yau [12], Peternell [7], and Siu [8] have subsequently proved related results for $\mathbb{CP}^2$, $\mathbb{CP}^3$, and $\mathbb{CP}^n$, respectively.

It is perhaps less well-known that $\mathbb{C}P^3$ admits other almost-complex structures. In fact Thomas gives a formula in [10] for the total Chern classes of the exotic almost-complex structures on $\mathbb{C}P^3$. Let x denote the standard generator of $H^2(\mathbb{C}P^3;\mathbb{Z})$.

Theorem 1.1 (Thomas)**.** *Consider the complex projective space* $\mathbb{C}P^3$*. The following cohomology classes, and only these, occur as the total Chern class of an almost-complex structure on* $\mathbb{C}P^3$*.*

$$c(\mathbb{C}P^3) = 1 + 2jx + 2(j^2-1)x^2 + 4x^3; \quad j \in \mathbb{Z}.$$

We denote by X_j, $j \in \mathbb{Z}$, an almost-complex manifold diffeomorphic to $\mathbb{C}P^3$ whose total Chern class is given as in the theorem. In particular, the standard almost-complex structure has $j = 2$, and the blowup of an almost-complex $\mathbb{S}^6$ has $j = -1$. It is not known whether there exist integrable almost-complex structures for $j \neq 2$. In this paper we investigate some properties of a hypothetical exotic complex structure on $\mathbb{C}P^3$. We give lower bounds on the Hodge numbers of such a hypothetical complex structure which depend on j in Theorems 3.2 and 4.5. We also present a necessary condition for the degeneration of the Frölicher spectral sequence in Corollary 4.4.

2. Dolbeault cohomology and the Frölicher spectral sequence

In this section we recall Dolbeault cohomology groups and some general facts about the Frölicher spectral sequence of a complex manifold.

Suppose X is a complex manifold of complex dimension n. A differential form of type (p,q) on X is a complex differential form φ which can be written in local complex coordinates $(z_1,\ldots,z_n)$ as

$$\varphi = \sum a_{i_1\ldots i_p j_1\ldots j_q}\, \mathrm{d}z_{i_1} \wedge \ldots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\bar{z}_{j_q}.$$

Let $\Omega^{p,q}$ denote the space of smooth (p,q) forms on X, and $\Omega^m = \bigoplus_{p+q=m} \Omega^{p,q}$.

Let $d\colon \Omega^m \to \Omega^{m+1}$ denote the exterior derivative. On a complex manifold

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1},$$
$$d = \partial + \bar{\partial},$$

where

$$\partial(\Omega^{p,q}) \subset \Omega^{p+1,q}$$

and

$$\bar{\partial}(\Omega^{p,q}) \subset \Omega^{p,q+1}.$$

Since $\overline{\partial}^2 = 0$, define the Dolbeault cohomology groups to be

$$H^{p,q}(X) = \frac{(\ker \overline{\partial}) \cap \Omega^{p,q}}{(\operatorname{im} \overline{\partial}) \cap \Omega^{p,q}}.$$

Let $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$.

Lemma 2.1 (Serre Duality)**.** *Let X be a compact complex manifold of complex dimension n. Then*

$$H^{p,q}(X) = H^{n-p,n-q}(X).$$

Lemma 2.2. *Let X be a compact complex manifold of complex dimension n. There exists a natural injective map*

$$i\colon\ H^{n,0}(X) \hookrightarrow H^n_{dR}(X).$$

Proof. Since $(\operatorname{im} \overline{\partial}) \cap \Omega^{n,0} = 0$, we have $H^{n,0}(X) = (\ker \overline{\partial}) \cap \Omega^{n,0}$. In addition we have $(\ker d) \cap \Omega^{n,0} = (\ker \overline{\partial}) \cap \Omega^{n,0}$ which gives a natural map $i\colon\ H^{n,0}(X) \to H^n_{dR}(X)$. We only need to show that this map is injective.

Suppose that $\beta \in \Omega^*$ is such that $\mathrm{d}\beta \in \Omega^{n,0}$. Then

$$\int_X \mathrm{d}\beta \wedge \overline{\mathrm{d}\beta} = \int_X \mathrm{d}(\beta \wedge \overline{\mathrm{d}\beta}) = 0,$$

by Stokes' theorem. Write $\mathrm{d}\beta$ locally as $\mathrm{d}\beta = f\,\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$. Then

$$\begin{aligned}
\mathrm{d}\beta \wedge \overline{\mathrm{d}\beta} &= |f|^2\,\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge d\overline{z_1} \wedge \ldots \wedge \mathrm{d}\overline{z_n} \\
&= (-1)^{(1/2)n(n-1)}|f|^2\,\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_n} \\
&= (-1)^{(1/2)n(n-1)}|f|^2\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n,
\end{aligned}$$

where $z_j = x_j + \sqrt{-1}y_j$, $j = 1, \ldots, n$. The vanishing of the integral shows that $\mathrm{d}\beta = 0$ which gives the injectivity of i. □

Corollary 2.3. *Let X be a compact complex manifold of complex dimension n such that $b_n(X) = 0$. Any complex structure on X has the property*

$$h^{n,0} = h^{0,n} = 0.$$

Proof. The previous lemma gives that $H^{n,0}(X) \hookrightarrow H^n_{dR}(X)$, and since $b_n(X) = 0$ we have that $h^{n,0} = 0$. Then $h^{0,n} = 0$ follows by Serre duality. □

We now turn to the Frölicher spectral sequence. For a complete discussion see [5]. We form from the double complex $(\Omega^{*,*}, \partial, \overline{\partial})$ the associated de Rham complex (Ω^*, d) where

$$\Omega^m = \bigoplus_{p+q=m} \Omega^{p,q},$$
$$d = \partial + \overline{\partial}.$$

There are two filtrations on (Ω^*, d) given by

$${}'F^p\Omega^m = \bigoplus_{\substack{p'+q=m \\ p' \geqslant p}} \Omega^{p',q},$$
$${}''F^q\Omega^m = \bigoplus_{\substack{p+q''=m \\ q'' \geqslant q}} \Omega^{p,q''}.$$

Associated with each filtration is a spectral sequence $\{{}'E_r\}$ and $\{{}''E_r\}$ both of which abut to $H^*_{dR}(X)$. The first filtration ${}'F^p\Omega^m$ gives the Frölicher spectral sequence, for in this case ${}'E_1^{p,q}$ is given by

$$E_1^{p,q} = H^q_{\overline{\partial}}(X, \Omega^p) = H^{p,q}(X),$$

the Dolbeault cohomology groups of X. Henceforth we will drop this prime notation, denoting ${}'E_r^{p,q}$ by $E_r^{p,q}$.

Here we note that if X is a Kähler manifold, then the Frölicher spectral sequence degenerates at the E_1 level and we have the Hodge decomposition

$$H^m(X) = \bigoplus_{p+q=m} H^{p,q}(X)$$

as well as

$$H^{p,q}(X) = \overline{H^{q,p}}(X).$$

As above we let $h^{p,q} = \dim H^{p,q}(X) = \dim E_1^{p,q}$, and we also define $h_r^{p,q} = \dim E_r^{p,q}$ where

$$d_r\colon\ E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$E_{r+1}^{p,q} = \frac{(\ker d_r) \cap E_r^{p,q}}{(\operatorname{im} d_r) \cap E_r^{p,q}}.$$

For each p, let

$$\chi_p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q}.$$

Observe that $h_{r+1}^{p,q} \leqslant h_r^{p,q}$, and that if $p = 0$, then following Hirzebruch [2], $\chi_0(X)$ is the familiar arithmetic genus. In [11] Ugarte gives the following useful proposition.

Proposition 2.4 (Ugarte). *Let X be a compact complex manifold of complex dimension n. If there are no holomorphic n-forms on X, then $E_n \cong E_\infty$.*

This proposition follows from noting that the holomorphic n-forms are by definition $\Omega^{n,0} \cap (\ker \overline{\partial})$ which by the proof of lemma (2.2) is $H^{n,0}(X)$. If there are no holomorphic n-forms, then $d_r \colon E_r^{p,q} \to E_r^{p+n,q-n+1}$ is identically zero for any $r \geqslant n$.

3. Cohomology relations for exotic complex structures and the Atiyah-Singer index theorem

In this section we consider the relations among the Hodge numbers for an exotic complex structure on $\mathbb{C}\mathrm{P}^3$. We employ the Hirzebruch-Riemann-Roch theorem as it appears in [1] and [2]. Suppose X is a compact complex manifold of complex dimension n.

Consider the Dolbeault complex

$$\Omega^{0,*}\colon\ 0 \to \Omega^{0,0} \to \ldots \to \Omega^{0,q} \xrightarrow{\overline{\partial}} \Omega^{p,q+1} \to \ldots \to \Omega^{0,n} \to 0.$$

We apply the Atiyah-Singer Index theorem

$$\operatorname{index} \overline{\partial} = \left\{\operatorname{ch} \sigma(\overline{\partial}) \operatorname{Td}(X)\right\} [TX], \tag{1}$$

where $\operatorname{ch}\sigma(\overline{\partial})$ is the Chern character of the symbol of the operator $\overline{\partial}$, $\operatorname{Td}(X)$ is the Todd class of X and $[TX]$ is the fundamental class of the tangent bundle. The left hand side of equation (1) is the arithmetic genus given by

$$\operatorname{index} \overline{\partial} = \sum_{q=0}^{3} (-1)^q H^q(X, \mathcal{O}) = \sum_{q=0}^{3} (-1)^q h^{0,q} = \chi_0(X).$$

The expression on the right hand side of equation (1) can be rewritten in terms of a universal expression in Chern classes $c_k \in H^{2k}(X)$ evaluated on the fundamental class $[X] \in H_{2n}(X)$. In particular, for a complex manifold of complex dimension three, the formula simplifies to

$$\left\{\operatorname{ch} \sigma(\overline{\partial}) \operatorname{Td}(X)\right\} [TX] = \operatorname{Td}(X)[X] = \frac{1}{24} c_1 c_2 [X].$$

In the special case of $X = \mathbb{S}^6$ we have a theorem of Gray [4] for a hypothetical complex structure on X.

Theorem 3.1 (Gray)**.** *Any complex structure on* $\mathbb{S}^6$ *has the property that*

$$h^{0,1}(\mathbb{S}^6) \geqslant 1.$$

P r o o f. Any complex structure on $\mathbb{S}^6$ satisfies

$$\chi_0(\mathbb{S}^6) = \frac{1}{24} c_1 c_2 [X].$$

Since the cohomology $H^k(X)$ vanishes for all $k \neq 0, 6$ we have $h^{0,3} = 0$ and $1/24 c_1 c_2 [X] = 0$ so that

$$1 - h^{0,1} + h^{0,2} = 0,$$

which gives

$$h^{0,1} = 1 + h^{0,2} \geqslant 1.$$

$\square$

We can extend this result to the exotic manifolds X_j from the introduction.

Theorem 3.2. *Let* X_j *be a complex manifold diffeomorphic to* $\mathbb{C}\mathrm{P}^3$ *whose total Chern class is given by* $c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3$*, where* x *generates* $H^2(X_j, \mathbb{Z})$*.*

(a) *If* $j < 2$*, then*

$$h^{0,1}(X_j) \geqslant 1, \quad \textit{and} \quad h^{1,1} + h^{2,0} \geqslant 2.$$

(b) *If* $j > 2$*, then*

$$h^{0,2}(X_j) \geqslant 3, \quad \textit{and} \quad h^{1,0} + h^{1,2} \geqslant 2.$$

R e m a r k 1. If $j \neq 2$, then X_j is not Kähler because this is inconsistent with Hodge decomposition. The results of [3] imply this as well. We can also see that if $j \neq 2$, then X_j is not Kähler since the Frölicher spectral sequence lives to E_2. We will explore this further in section 4.

P r o o f. From Thomas' theorem (1.1) for each $j \in \mathbb{Z}$, the total Chern class of X_j is given by

$$c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3.$$

As above

$$\chi_0(X_j) = 1 - h^{0,1}(X_j) + h^{0,2}(X_j)$$

since $h^{3,0}(X_j) = 0$. Combining this with the index theorem gives

$$\begin{aligned}
1 - h^{0,1}(X_j) + h^{0,2}(X_j) &= \frac{j(j^2-1)}{6}, \\
h^{0,1}(X_j) &\geqslant 1 - \frac{j(j^2-1)}{6} \geqslant 1, \quad \text{for } j < 2, \\
h^{0,2}(X_j) &\geqslant \frac{j(j^2-1)}{6} - 1 \geqslant 3, \quad \text{for } j > 2.
\end{aligned}$$

Additionally, the topological Euler characteristic may be expressed

$$\begin{aligned}
\chi_{\mathrm{Top}}(X_j) &= \sum_{p=0}^{3}\sum_{q=0}^{3}(-1)^{p+q}h^{p,q} \\
&= 2\left(\sum_{q=0}^{3}(-1)^q h^{0,q} - \sum_{q=0}^{3}(-1)^q h^{1,q}\right) \\
&= 2(\chi_0 - \chi_1).
\end{aligned}$$

In particular, $\chi_1 = \chi_0 - 2$. This expression for χ_1 along with Serre duality give

$$\chi_1 = h^{1,0} - h^{1,1} + h^{1,2} - h^{2,0} = \frac{j(j^2-1)}{6} - 2,$$

so that

$$\begin{aligned}
h^{1,1} + h^{2,0} &\geqslant 2 - \frac{j(j^2-1)}{6} \geqslant 2 \quad \text{for } j < 2, \\
h^{1,0} + h^{1,2} &\geqslant \frac{j(j^2-1)}{6} - 2 \geqslant 2 \quad \text{for } j > 2.
\end{aligned}$$

□

In section 4 we prove a sharper inequality for $h^{1,2}$ using the Frölicher spectral sequence.

4. Frölicher spectral sequence computations

Since $b_1(X_j) = 0$ and $b_2(X_j) = 1$, it is clear from the preceding proposition that if $j \neq 2$, the Frölicher spectral sequence lives at least to $E_2(X_j)$. We also have that $E_3(X_j) \cong E_\infty(X_j)$, so we would like to know under what conditions does the spectral sequence live to $E_3(X_j)$. For a compact complex manifold X of complex dimension three, consider the dimension grids below.

E_1

0	$h^{1,3}$	$h^{2,3}$	1
$h^{0,2}$	$h^{1,2}$	$h^{2,2}$	$h^{3,2}$
$h^{0,1}$	$h^{1,1}$	$h^{2,1}$	$h^{3,1}$
1	$h^{1,0}$	$h^{2,0}$	0

E_2

0	$h_2^{1,3}$	$h_2^{2,3}$	1
$h_2^{0,2}$	$h_2^{1,2}$	$h_2^{2,2}$	$h_2^{3,2}$
$h_2^{0,1}$	$h_2^{1,1}$	$h_2^{2,1}$	$h_2^{3,1}$
1	$h_2^{1,0}$	$h_2^{2,0}$	0

E_3

0	$h_3^{1,3}$	$h_3^{2,3}$	1
$h_3^{0,2}$	$h_3^{1,2}$	$h_3^{2,2}$	$h_3^{3,2}$
$h_3^{0,1}$	$h_3^{1,1}$	$h_3^{2,1}$	$h_3^{3,1}$
1	$h_3^{1,0}$	$h_3^{2,0}$	0

R e m a r k 2. We recall two facts about the dimension grids above: First, each entry $h_r^{p,q}$ is a non-negative integer, and second, $\dim H^n_{dR}(X) = \sum\limits_{p+q=n} h_\infty^{p,q} = \sum\limits_{p+q=n} h_3^{p,q}$. The computations in the subsections that follow use the basic homological algebra fact that the Euler characteristic of a complex of vector spaces equals the Euler characteristic of the cohomology of the complex.

4.1. The Frölicher spectral sequence for $\mathbb{S}^6$. We recall some of L. Ugarte's main results in [11], since we know that $\dim H^n_{dR}(\mathbb{S}^6) = 0$ for all $n \neq 0, 6$ we have $h_3^{p,q} = 0$ for all pairs (p,q) except $(0,0)$ and $(3,3)$, so that the E_3 term becomes:

E_3

0	0	0	1
0	0	0	0
0	0	0	0
1	0	0	0

Since the E_3 term comes from the following sequences

$$0 \to E_2^{p,q} \xrightarrow{d_2} E_2^{p+2,q-1} \to 0, \tag{2}$$

and $E_2^{p,q} = 0$ for all $p, q < 0$, $p, q > 3$, and $(p,q) = (0,3), (3,0)$ we know that

$$h_2^{1,0} = h_2^{2,3} = h_2^{1,1} = h_2^{2,2} = h_2^{3,0} = h_2^{0,3} = 0.$$

We also know that for the cohomology of the complex (2) to vanish we need $E_2^{p,q} \cong E_2^{p+2,q-1}$ hence we have

$$h_2^{0,1} = h_2^{2,0},$$
$$h_2^{0,2} = h_2^{2,1},$$
$$h_2^{1,2} = h_2^{3,1},$$
$$h_2^{1,3} = h_2^{3,2}.$$

On the other hand the entries of the E_2 term arise from the following sequences

(3) $$0 \to E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q} \xrightarrow{d_1} E_1^{p+2,q} \xrightarrow{d_1} E_1^{p+3,q} \to 0,$$

so that

$$h_2^{0,q} - h_2^{1,q} + h_2^{2,q} - h_2^{3,q} = h^{0,q} - h^{1,q} + h^{2,q} - h^{3,q}.$$

By Serre duality we know that $h^{p,q} = h^{3-p,3-q}$. Then we have

$$1 + h_2^{2,0} = 1 - h^{1,0} + h^{2,0} = 1 - h^{2,3} + h^{1,3} = 1 + h_2^{1,3},$$

which gives

$$h_2^{0,1} = h_2^{2,0} = h_2^{1,3} = h_2^{3,2}.$$

We also have

$$\begin{aligned} h_2^{0,1} + h_2^{2,1} - h_2^{3,1} &= h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} \\ &= h^{3,2} - h^{2,2} + h^{1,2} - h^{0,2} \\ &= h_2^{3,2} + h_2^{1,2} - h_2^{0,2} \\ &= h_2^{0,1} + h_2^{3,1} - h_2^{2,1}, \end{aligned}$$

which gives

$$h_2^{0,2} = h_2^{1,2} = h_2^{2,1} = h_2^{3,1}.$$

Let $a = h_2^{0,1} = \dim((\ker d_1) \cap H^{0,1}(\mathbb{S}^6))$ and $b = h_2^{0,2} = \dim((\ker d_1) \cap H^{0,2}(\mathbb{S}^6))$. Then the E_2 term is

E_2

0	a	0	1
b	b	0	a
a	0	b	b
1	0	a	0

Proposition 4.1 (Ugarte)**.** *If $X = \mathbb{S}^6$, then either*

(a) $H^{1,1}(X) \neq 0$, *or*

(b) $H_2^{2,0}(X) \neq 0$ *and* $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$.

4.2. The Frölicher spectral sequence for X_j. Consider now the case $X = X_j$. Since $b_0 = b_2 = b_4 = b_6 = 1$ and $b_1 = b_3 = b_5 = 0$ we have

$$\begin{aligned} &h_3^{0,0} = h_3^{3,3} = 1, \\ &h_3^{0,1} = h_3^{1,0} = h_3^{0,3} = h_3^{1,2} = h_3^{2,1} = h_3^{3,0} = h_3^{2,3} = h_3^{3,2} = 0, \\ &h_3^{0,2} + h_3^{1,1} + h_3^{2,0} = 1, \\ &h_3^{1,3} + h_3^{2,2} + h_3^{3,1} = 1, \end{aligned}$$

so the E_3 term becomes

$$E_3 \qquad \begin{array}{|c|c|c|c|} \hline 0 & h_3^{1,3} & 0 & 1 \\ \hline h_3^{0,2} & 0 & h_3^{2,2} & 0 \\ \hline 0 & h_3^{1,1} & 0 & h_3^{3,1} \\ \hline 1 & 0 & h_3^{2,0} & 0 \\ \hline \end{array}$$

Unlike the case of $\mathbb{S}^6$ we cannot determine all of the entries of the E_3 term exactly, but we do know that either $h_3^{0,2}, h_3^{1,1}$, or $h_3^{2,0}$ is 1, and $h_3^{1,3}, h_3^{2,2}$, or $h_3^{3,1}$ is 1. This observation allows us to regard the nine cases of E_3 individually. Before we do this we can make some general observations.

Since

$$h_3^{0,1} = h_3^{1,0} = h_3^{0,3} = h_3^{1,2} = h_3^{2,1} = h_3^{3,0} = h_3^{2,3} = h_3^{3,2} = 0,$$

we can conclude that

$$h_2^{0,3} = h_2^{1,0} = h_2^{2,3} = h_2^{3,0} = 0.$$

By Serre Duality at the E_1 level we have

$$h_2^{1,3} = h_2^{2,0}.$$

We can also conclude

$$\begin{aligned} h_2^{1,1} &= h_3^{1,1}, \\ h_2^{2,2} &= h_3^{2,2}, \\ h_3^{0,2} &= h_2^{0,2} - h_2^{2,1}, \\ h_3^{2,0} &= h_2^{2,0} - h_2^{0,1}, \\ h_3^{1,3} &= h_2^{1,3} - h_2^{3,2}, \\ h_3^{3,1} &= h_2^{3,1} - h_2^{1,2}, \\ h_2^{0,1} - h_2^{1,1} + h_2^{2,1} - h_2^{3,1} &= h_2^{3,2} - h_2^{2,2} + h_2^{1,2} - h_2^{0,2}. \end{aligned}$$

In all of the cases that follow let $a = h_2^{0,1} = \dim((\ker d_1) \cap H^{0,1}(X_j))$ and $b = h_2^{0,2} = \dim((\ker d_1) \cap H^{0,2}(X_j))$.

Case 1: $h_3^{0,2} = 1$ and $h_3^{1,3} = 1$.

E_3

0	1	0	1
1	0	0	0
0	0	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	0	$a-1$
a	0	$b-1$	b
1	0	a	0

from which we conclude that $a, b > 0$ so that

(i) $H^{0,1}(X_j) \neq 0$, $H^{0,2}(X_j) \neq 0$ and

(ii) this spectral sequence lives to E_3.

Case 2: $h_3^{0,2} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
1	0	1	0
0	0	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	1	a
a	0	$b-1$	b
1	0	a	0

from which we conclude that $b > 0$ so that

(i) $H^{0,2}(X_j) \neq 0$ and

(ii) this spectral sequence lives to E_3.

Case 3: $h_3^{0,2} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
1	0	0	0
0	0	0	1
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	$b-1$	0	a
a	0	$b-1$	b
1	0	a	0

from which we conclude that $b > 0$ so that
(i) $H^{0,2}(X_j) \neq 0$ and
(ii) $E_2 \cong E_\infty$ if and only if $a = 0$ and $b = 1$.

Case 4: $h_3^{1,1} = 1$ and $h_3^{1,3} = 1$

E_3

0	1	0	1
0	0	0	0
0	1	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	0	$a-1$
a	1	b	b
1	0	a	0

from which we conclude that $a > 0$ so that
(i) $H^{0,1}(X_j) \neq 0$ and
(ii) this spectral sequence lives to E_3.

Case 5: $h_3^{1,1} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
0	0	1	0
0	1	0	0
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	b	1	a
a	1	b	b
1	0	a	0

from which we conclude

(i) $E_2 \cong E_\infty$ if and only if $a = b = 0$.

Case 6: $h_3^{1,1} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
0	0	0	0
0	1	0	1
1	0	0	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	a	0	1
b	$b-1$	0	a
a	1	b	b
1	0	a	0

from which we conclude that $b > 0$ so that

(i) $H^{0,2}(X_j) \neq 0$ and

(ii) this spectral sequence lives to E_3.

Case 7: $h_3^{2,0} = 1$ and $h_3^{1,3} = 1$

E_3

0	1	0	1
0	0	0	0
0	0	0	0
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	b	0	a
a	0	b	b
1	0	$a+1$	0

from which we conclude:

(i) $E_2 \cong E_\infty$ if and only if $a = b = 0$.

Case 8: $h_3^{2,0} = 1$ and $h_3^{2,2} = 1$

E_3

0	0	0	1
0	0	1	0
0	0	0	0
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	b	1	$a+1$
a	0	b	b
1	0	$a+1$	0

from which we conclude:

(i) this spectral sequence lives to E_3.

Case 9: $h_3^{2,0} = 1$ and $h_3^{3,1} = 1$

E_3

0	0	0	1
0	0	0	0
0	0	0	1
1	0	1	0

Then the E_2 term becomes for all $j \in \mathbb{Z}$:

E_2

0	$a+1$	0	1
b	$b-1$	0	$a+1$
a	0	b	b
1	0	$a+1$	0

from which we conclude that $b > 0$ so that

(i) $H^{0,2}(X_j) \neq 0$ and

(ii) this spectral sequence lives to E_3.

4.3. General descriptions of the terms of the Frölicher spectral sequence. We combine the preceding nine cases to make some general case-independent observations about when the spectral sequence lives to E_3, and when it degenerates at the E_2 level. For the remaining statements we make no assumptions on the vanishing of specific terms at the various levels of the spectral sequence.

Proposition 4.2. *If $E_2 \cong E_\infty$, then $h_2^{p,q} = h_2^{3-p,3-q}$.*

Proposition 4.3. *$h_2^{p,q} = h_2^{3-p,3-q}$ if and only if $h_3^{p,q} = h_3^{3-p,3-q}$.*

Combining these together gives a necessary condition for the degeneration of the Frölicher spectral sequence at the second level.

Corollary 4.4. *If $h_3^{p,q} \neq h_3^{3-p,3-q}$, then $E_1 \ncong E_2 \ncong E_3$.*

We complement Theorem 3.2 with the following.

Theorem 4.5. *Let X_j be a complex manifold diffeomorphic to $\mathbb{CP}^3$ whose total Chern class is given by $c(X_j) = 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3$, where x generates $H^2(X_j, \mathbb{Z})$. If $j > 2$, then $h^{1,2} = h^{2,1} \geqslant 2$. Moreover, if $h_3^{0,2} \neq 1$ or $h_3^{3,1} \neq 1$, then $h^{1,2} \geqslant h^{0,2} \geqslant 3$.*

P r o o f. Observe that in all nine cases above either $h_2^{1,2} = h_2^{0,2}$ or $h_2^{1,2} = h_2^{0,2} - 1$. Let us suppose $h_2^{1,2} = h_2^{0,2}$. To simplify the notation we consider the complex

$$0 \to E_1^{0,2} \xrightarrow{\alpha} E_1^{1,2} \xrightarrow{\beta} \dots$$

where α and β are the maps d_1. We know $h_2^{0,2} = \dim(\ker\alpha)$ and $h_2^{1,2} = \dim(\ker\beta) - \operatorname{rank}(\alpha)$, thus giving

$$\begin{aligned} h^{0,2} &= \dim(\ker\alpha) + \operatorname{rank}(\alpha) \\ &= h_2^{1,2} + \operatorname{rank}(\alpha) \\ &= \dim(\ker\beta) - \operatorname{rank}(\alpha) + \operatorname{rank}(\alpha) \\ &\leqslant h^{1,2}. \end{aligned}$$

We assumed that $h_2^{1,2} = h_2^{0,2}$, but suppose instead that $h_2^{1,2} = h_2^{0,2} - 1$. If this occurs, then unless $h_3^{0,2} = h_3^{3,1} = 1$, we have $h^{3,1} = h^{2,1}$. We can repeat the above argument for $h^{3,1}$ and $h^{2,1}$. Serre duality again gives

$$h^{0,2} = h^{3,1} \leqslant h^{2,1} = h^{1,2}.$$

In case $h_3^{0,2} = h_3^{3,1} = 1$ we have $h^{2,1} = h^{1,2} = h^{0,2} - 1$. The same arguments go through except that now we have

$$h^{0,2} \leqslant h^{1,2} + 1.$$

$\square$

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