VARIABLE EXPONENT SOBOLEV SPACES WITH ZERO BOUNDARY VALUES

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Abstract. We study different definitions of the first order variable exponent Sobolev space with zero boundary values in an open subset of $\mathbb{R}^n$.

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1. Introduction

In this note we study the first order variable exponent Sobolev space with zero boundary values. Variable exponent Lebesgue and Sobolev spaces have attracted a steadily increasing interest over the last ten years although their history goes back to W. Orlicz, see for example [22], [17], [27]. These investigations were motivated by differential equations with non-standard coercivity conditions, arising for instance from modeling certain fluids (e.g, [1], [7], [24]). The properties of the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$, where $p\colon \Omega \to (1,\infty)$ is measurable and $\Omega \subset \mathbb{R}^n$ is an open set, depend crucially on the variable exponent p. For example, the class of smooth functions either can be, [6], [26], [12], or does not have to be, [29], [13], dense in $W^{1,p(\cdot)}(\Omega)$ depending on p. Hence it is easy to guess that the closure of $C_0^\infty(\Omega)$ under the Sobolev norm is not a natural definition of the Sobolev space with zero boundary values in every case.

In [11] variable exponent Sobolev spaces with zero boundary values have been defined following a method developed by Kilpeläinen, Kinnunen and Martio [15]

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for metric measure spaces. In this definition a function u belongs to a Sobolev space with zero boundary values in an open set Ω if there exists a quasicontinuous function $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ which coincides with u almost everywhere in Ω and equals zero quasieverywhere in $\mathbb{R}^n \setminus \Omega$. Here we use the Sobolev capacity studied in [10] and hence assume that $\operatorname{ess\,inf} p > 1$ and $\operatorname{ess\,sup} p < \infty$. In this definition the set of Sobolev functions with zero boundary values seems to depend on the values of p in $\mathbb{R}^n \setminus \Omega$. We show that it does not if continuous functions are dense in the Sobolev space. We also show that this class of functions can be characterized by inner traces on the boundary in the sense on [28] if the exponent p is regular enough.

Our last definition of the Sobolev space with zero boundary values is the closure of functions in $W^{1,p(\cdot)}(\Omega)$ with compact support in Ω. This is the most general condition and coincides with the two mentioned above if smooth functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. By [26], the last condition holds if p is bounded and

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|}$$

for every $x, y \in \mathbb{R}^n$ with $|x-y| \leqslant 1/2$.

2. Variable exponent spaces

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball with center x and radius r by $B(x,r)$. Let $\Omega \subset \mathbb{R}^n$ be an open set. We will now introduce the variable exponent Lebesgue and Sobolev spaces in Ω.

Let $p\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function called the *variable exponent*. Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$. We assume all the time, except in our last theorem, that p is defined in the whole $\mathbb{R}^n$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\,\mathrm{d}x < \infty$$

for some $\lambda > 0$. If $p^+ < \infty$, then we can define $u \in L^{p(\cdot)}(\Omega)$ if $\varrho_{p(\cdot)}(u) < \infty$. The function $\varrho_{p(\cdot)}\colon L^{p(\cdot)}(\Omega) \to [0,\infty)$ is called the *modular* of the space $L^{p(\cdot)}(\Omega)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon\ \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

This space is an Orlicz-Musielak space, cf. [20].

If $\|f\|_{p(\cdot)} \leqslant 1$ then $\varrho_{p(\cdot)}(f) \leqslant \|f\|_{p(\cdot)}$. Moreover, if $p^+ < \infty$, then $\varrho_{p(\cdot)}(f_i) \to 0$ if and only if $\|f_n\|_{p(\cdot)} \to 0$. Hölder's inequality, i.e. $\|fg\|_1 \leqslant C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$, holds also in the variable exponent Lebesgue spaces. For the proofs of these facts see [17].

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the space of measurable functions $u\colon \Omega \to \mathbb{R}$ such that u and the absolute value of the distributional gradient $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ are in $L^{p(\cdot)}(\Omega)$ The function $\varrho_{1,p(\cdot)}\colon W^{1,p(\cdot)}(\Omega) \to [0,\infty)$ is defined by $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. For more details on the variable exponent spaces see [17].

We recall from [10, Section 3] the definition and basic properties of the Sobolev $p(\cdot)$-capacity. For $E \subset \mathbb{R}^n$ we denote

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n)\colon\ u \geqslant 1 \text{ in an open set containing } E\}.$$

The *Sobolev $p(\cdot)$-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u\in S_{p(\cdot)}(E)} \varrho_{1,p(\cdot)}(u) = \inf_{u\in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} (|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)})\,\mathrm{d}x.$$

In the case $S_{p(\cdot)}(E) = \emptyset$ we set $C_{p(\cdot)}(E) = \infty$. If $1 < p^- \leqslant p^+ < \infty$, then the set function $E \mapsto C_{p(\cdot)}(E)$ is an outer measure and a Choquet capacity, [10, Corollary 3.3 and Corollary 3.4]. If $1 < p^- \leqslant p^+ < \infty$ and $C_{p(\cdot)}(E) = 0$ then $H^s(E) = 0$ for all $s > n - p^-$, with the understanding that if $p^- > n$ then $E = \emptyset$, [10, Theorem 4.2].

A function $u\colon \mathbb{R}^n \to \mathbb{R}$ is said to be $p(\cdot)$-*quasicontinuous* (in $\mathbb{R}^n$) if for every $\varepsilon > 0$ there exists an open set O with $C_{p(\cdot)}(O) < \varepsilon$ such that u restricted to $\mathbb{R}^n \setminus O$ is continuous. For a subset E of $\mathbb{R}^n$ we say that a claim holds $p(\cdot)$-*quasieverywhere* in E if it holds everywhere except in a set $N \subset E$ with $C_{p(\cdot)}(N) = 0$.

It was proved in [10, Theorem 5.2] that if continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$, then every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ has a $p(\cdot)$-quasicontinuous representative in $\mathbb{R}^n$. Samko showed [26] that C_0^∞-functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ if $p^+ < \infty$ and if

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|} \tag{2.1}$$

for every $x, y \in \mathbb{R}^n$ with $|x-y| \leqslant 1/2$. Edmunds and Rákosník proved in [6] that a certain monotonicity condition on the exponent is also sufficient for the density of smooth functions. Hästö connected these two conditions in a single one, [12]. Zhikov showed that in the plane smooth functions are not dense if the exponent is discontinuous, [29]. Hästö gave in [13] an example in which continuous functions are not dense and the continuous exponent has growth just slightly greater than that allowed in (2.1).

3. Results

We start our study in all of $\mathbb{R}^n$. We give an example where $p^+ = \infty$ and functions with compact support are not dense in $W^{1,p(\cdot)}(\mathbb{R})$. Then we show that if $p^+ < \infty$, then every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ can be approximated by functions with compact support.

3.1. Example. Let $p(x) = \max\{|x|, 1\}$ and $u(x) = 1/2$ for every $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} |u(x)|^{p(x)}\,\mathrm{d}x \approx \int_{1}^{\infty} \Big(\frac{1}{2}\Big)^{x}\,\mathrm{d}x < \infty$$

and hence $u \in W^{1,p(\cdot)}(\mathbb{R})$. Let g be a function with compact support in $(-a, a)$. We find for every $a > 0$ that

$$\varrho_{p(\cdot)}((u-g)/\lambda) \geqslant \int_{a}^{\infty} \Big(\frac{1}{2\lambda}\Big)^{x}\,\mathrm{d}x = \infty$$

for $0 < \lambda \leqslant 1/2$ and hence $\|u - g\|_{p(\cdot)} \geqslant 1/2$ for every g with compact support.

3.2. Theorem. *If $p^+ < \infty$, then bounded Sobolev functions with compact support are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$. We write

$$u_m(x) = \max\{\min\{u(x), m\}, -m\}$$

for every $m > 0$. We obtain

$$\varrho_{1,p(\cdot)}(u - u_m) \leqslant \int_{\{x\in\mathbb{R}^n\colon\, |u(x)|\geqslant m\}} \big(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\big)\,\mathrm{d}x \to 0$$

as $m \to \infty$ since $|\{x \in \mathbb{R}^n\colon\, |u(x)| \geqslant m\}| \to 0$ as $m \to \infty$. Hence u can be approximated by bounded functions.

Let $\varphi_r \in C_0^\infty(\mathbb{R}^n)$ be a cut off function with $\varphi_r(x) = 1$ for $x \in B(0, r)$, $\varphi_r(x) = 0$ for $x \in \mathbb{R}^n \setminus B(0, 2r)$, $0 \leqslant \varphi_r(x) \leqslant 1$ and $|\nabla\varphi_r| \leqslant C/r$. We shall show that $(u\varphi_r)$ convergences to u in $W^{1,p(\cdot)}(\mathbb{R}^n)$ as $r \to \infty$. We have

$$\|u - u\varphi_r\|_{1,p(\cdot)} \leqslant \|u\|_{1,p(\cdot),\mathbb{R}^n\setminus B(0,2r)} + \|u - u\varphi_r\|_{1,p(\cdot),B(0,2r)\setminus B(0,r)}.$$

The first term on the right hand side tends to zero since $p^+ < \infty$ and

$$\varrho_{1,p(\cdot),\mathbb{R}^n\setminus B(0,2r)}(u) \to 0$$

as $r \to \infty$. Next we approximate the second term:

$$\begin{aligned}
&\|u - u\varphi_r\|_{1,p(\cdot),B(0,2r)\setminus B(0,r)} \\
&\qquad \leqslant \|u\|_{p(\cdot),\mathbb{R}^n\setminus B(0,r)} + \|\nabla u - (\varphi_r \nabla u + u\nabla\varphi_r)\|_{p(\cdot),B(0,2r)\setminus B(0,r)} \\
&\qquad \leqslant \|u\|_{1,p(\cdot),\mathbb{R}^n\setminus B(0,r)} + \frac{1}{r}\|u\|_{p(\cdot),\mathbb{R}^n\setminus B(0,r)},
\end{aligned}$$

and we see that it also tends to zero as $r \to \infty$. This completes the proof. $\square$

Let Ω be an open proper subset of $\mathbb{R}^n$. By $H_0^{1,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p(\cdot)}(\Omega)$. Note that $H_0^{1,p(\cdot)}(\Omega)$ is a Banach space. Clearly the values of p in $\mathbb{R}^n \setminus \Omega$ do not affect $H_0^{1,p(\cdot)}(\Omega)$.

Assume that $p\colon \mathbb{R}^n \to (1,\infty)$ with $1 < p^- \leqslant p^+ < \infty$. We denote $u \in Q_0^{1,p(\cdot)}(\Omega)$ if there exists a $p(\cdot)$-quasicontinuous function $\tilde{u} \in W^{1,p(\cdot)}(\mathbb{R}^n)$, called a *canonical representative*, such that $u = \tilde{u}$ almost everywhere in Ω and $\tilde{u} = 0$ $p(\cdot)$-quasieverywhere in $\mathbb{R}^n \setminus \Omega$. The set $Q_0^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\tilde{u}\|_{W^{1,p(\cdot)}(\mathbb{R}^n)}.$$

The space $Q_0^{1,p(\cdot)}(\Omega)$ is a Banach space, [11, Theorem 3.1].

We recall the following results, [11, Corollary 3.2 and Theorem 3.3].

3.3. Theorem. *Let* $p\colon \mathbb{R}^n \to (1,\infty)$.

(i) *If* $1 < p^- \leqslant p^+ < \infty$ *then* $H_0^{1,p(\cdot)}(\Omega) \subset Q_0^{1,p(\cdot)}(\Omega)$.

(ii) *If* $1 < p^- \leqslant p^+ < \infty$ *and if* C^∞ *functions are dense in* $W^{1,p(\cdot)}(\mathbb{R}^n)$ *, then* $H_0^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega)$.

The definition of $Q_0^{1,p(\cdot)}(\Omega)$ raises the following question: Do the values of the exponent p in $\mathbb{R}^n \setminus \Omega$ affect the function space? We will show that the answer is negative. Before that we prove that the functions with compact support are dense in $Q_0^{1,p(\cdot)}(\Omega)$; the proof is essentially the same as in [11, Theorem 3.3], and it is based on the arguments of [2, Section 9.2].

3.4. Lemma. *If $1 < p^- \leqslant p^+ < \infty$ and if continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, then the set of functions with compact support is dense in $Q_0^{1,p(\cdot)}(\Omega)$.*

P r o o f. Let $u \in Q_0^{1,p(\cdot)}(\Omega)$ and let $\tilde{u}$ be its canonical representative. We need to show that there exist functions $\varphi_i \in Q_0^{1,p(\cdot)}(\Omega)$ with compact support in Ω that tend to $\tilde{u}$ in Ω.

If we can construct such a sequence for $\tilde{u}_+(x) = \max\{\tilde{u}(x), 0\}$, then we can do it for $\tilde{u}_-$ as well, and combining these two gives the result for $\tilde{u} = \tilde{u}_+ + \tilde{u}_-$. We therefore assume that $\tilde{u}$ is positive. By Theorem 3.2 we may assume that $\tilde{u}$ is bounded and has compact support in $\mathbb{R}^n$. If we look at the proof of Theorem 3.2 we note that $\tilde{u}$ after all is still quasicontinuous.

For $0 < \varepsilon < 1$ define $\tilde{u}_\varepsilon(x) = \max\{\tilde{u}(x) - \varepsilon, 0\}$. Let G be an open set such that $\tilde{u}$ restricted to $\Omega \setminus G$ is continuous. Let $\delta > 0$ and let $\omega \in W^{1,p(\cdot)}(\mathbb{R}^n)$ be quasicontinuous, $0 \leqslant \omega \leqslant 1$, $\omega|_G = 1$ and $\|\omega\|_{1,p(\cdot)} < \delta$. For the existence of this function see [11, Theorem 2.2]. The function $(1-\omega)\tilde{u}_\varepsilon$ is quasicontinuous and zero at a point x if either $\tilde{u}(x) \leqslant \varepsilon$ or $x \in G$. Hence it vanishes in a neighborhood of $\mathbb{R}^n \setminus \Omega$. We obtain that

$$\|\tilde{u} - (1-\omega)\tilde{u}_\varepsilon\|_{1,p(\cdot)} \leqslant \|\tilde{u} - \tilde{u}_\varepsilon\|_{1,p(\cdot)} + \|\omega\tilde{u}_\varepsilon\|_{1,p(\cdot)}.$$

We have

$$\varrho_{1,p(\cdot)}\left(\tilde{u} - \tilde{u}_\varepsilon\right) \leqslant \varepsilon^{p^-} \varrho_{p(\cdot)}\left(\chi_{\operatorname{spt}\tilde{u}}\right) + \varrho_{p(\cdot)}\left(\chi_{\{0<\tilde{u}(x)\leqslant\varepsilon\}}\nabla\tilde{u}\right).$$

Since $\chi_{\{0<\tilde{u}(x)\leqslant\varepsilon\}}(x)\nabla\tilde{u}(x)$ goes to zero with ε we see by the dominated convergence theorem that $\|\tilde{u} - \tilde{u}_\varepsilon\|_{1,p(\cdot)}$ goes to zero with ε. Since $\tilde{u}$ is bounded we also find that

$$\begin{aligned}
\varrho_{1,p(\cdot)}(\omega\tilde{u}) &\leqslant \int_{\mathbb{R}^n} |\omega(x)\tilde{u}(x)|^{p(x)}\,\mathrm{d}x + 2^{p^+}\int_{\mathbb{R}^n} \omega(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,\mathrm{d}x \\
&\quad + 2^{p^+}\int_{\mathbb{R}^n} |\nabla\omega(x)|^{p(x)}|\tilde{u}(x)|^{p(x)}\,\mathrm{d}x \\
&\leqslant (2^{p^+}+1)\delta(\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\tilde{u}(x)+1)^{p^+} + 2^{p^+}\int_{\mathbb{R}^n} \omega(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,\mathrm{d}x.
\end{aligned}$$

Since $\omega \to 0$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as $\delta \to 0$, we can choose a subsequence ω_i which tends to 0 pointwise almost everywhere. Then $\int_{\mathbb{R}^n}\omega_i(x)^{p(x)}|\nabla\tilde{u}(x)|^{p(x)}\,\mathrm{d}x \to 0$ by the dominated convergence theorem. Therefore $\varrho_{1,p(\cdot)}(\omega_i\tilde{u}) \to 0$ and so also $\|\omega_i\tilde{u}\|_{1,p(\cdot)} \to 0$. Thus we see that $(1-\omega_i)\tilde{u}_\varepsilon \to \tilde{u}$ as $\varepsilon \to 0$ and $i \to \infty$. $\square$

3.5. Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $p, q\colon \mathbb{R}^n \to (1, \infty)$ be such that $p(x) = q(x)$ for every $x \in \Omega$, $1 < p^- \leqslant p^+ < \infty$, $1 < q^- \leqslant q^+ < \infty$ and let continuous functions be dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ and in $W^{1,q(\cdot)}(\mathbb{R}^n)$. Then $Q_0^{1,p(\cdot)}(\Omega) = Q_0^{1,q(\cdot)}(\Omega)$.*

Proof. By symmetry it is enough to show that $Q_0^{1,p(\cdot)}(\Omega) \subset Q_0^{1,q(\cdot)}(\Omega)$. Fix $u \in Q_0^{1,p(\cdot)}(\Omega)$ and let $\tilde{u}$ be its canonical representative in $W^{1,p(\cdot)}(\mathbb{R}^n)$. By Lemma 3.4 we may approximate u by functions u_i with compact support in Ω. Since $Q_0^{1,q(\cdot)}(\Omega)$ is a Banach space it is enough to show that each u_i belongs to $Q_0^{1,q(\cdot)}(\Omega)$. Hence we assume that u has compact support. We denote this support by E. We have to show that u has a $q(\cdot)$-quasicontinuous canonical representative.

We choose a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in E. Now $\varphi\tilde{u}$ is $p(\cdot)$-quasicontinuous, $\varphi\tilde{u} = u$ almost everywhere in Ω and $\varphi\tilde{u}$ vanishes in the complement of $\operatorname{spt}(\varphi)$, i.e. in some open neighborhood of $\mathbb{R}^n \setminus \Omega$.

Since $\varphi\tilde{u}$ is $p(\cdot)$-quasicontinuous there exists a sequence $O_i \subset \mathbb{R}^n$ of open sets such that $\varphi\tilde{u}$ restricted to $\mathbb{R}^n \setminus O_i$ is continuous and $C_{p(\cdot)}(O_i) \leqslant 2^{-i}$. Let U be an open set with $\operatorname{spt}\varphi\tilde{u} \subset U$ and $\overline{U} \subset \Omega$. Evidently $\varphi\tilde{u}$ restricted to $\mathbb{R}^n \setminus (O_i \cap U)$ is continuous and $C_{p(\cdot)}(O_i \cap U) \leqslant 2^{-i}$. Again we choose a cut-off function $\Phi \in C_0^\infty(\Omega)$ such that $\Phi = 1$ in U. Let $g_i \in S^{p(\cdot)}(O_i \cap U)$ with $\varrho_{1,p(\cdot)}(g_i) \leqslant 2 \cdot 2^{-i}$. Now $\Phi g_i \in S^{q(\cdot)}(O_i \cap U)$ and hence

$$C_{q(\cdot)}(O_i \cap U) \leqslant \varrho_{1,q(\cdot)}(\Phi g_i) = \varrho_{1,p(\cdot)}(\Phi g_i) \leqslant C(\Phi)2^{-i}.$$

We have shown that $\varphi\tilde{u}$ is $q(\cdot)$-quasicontinuous and hence $u \in Q^{1,q(\cdot)}(\Omega)$. This completes the proof. $\square$

If p is regular enough then we get a characterization of $Q_0^{1,p(\cdot)}(\Omega)$ by inner traces. We follow the proof of [28, Theorem 2.2], see also [2, Theorem 9.1.3].

3.6. Theorem. *Assume that $p\colon \mathbb{R}^n \to (1, \infty)$ satisfies $1 < p^- \leqslant p^+ < \infty$,*

$$|p(x) - p(y)| \leqslant \frac{C_1}{-\log|x - y|}$$

holds for every $|x - y| \leqslant 1/2$ and

$$|p(x) - p(y)| \leqslant \frac{C_2}{\log(\mathrm{e} + |x|)}$$

holds for every $|y| \geqslant |x|$. Then $u \in Q_0^{1,p(\cdot)}(\Omega)$ if and only if $u \in W^{1,p(\cdot)}(\Omega)$ and

$$\lim_{r \to \infty} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} u(y)\,\mathrm{d}y = 0 \tag{3.7}$$

for $p(\cdot)$-quasievery $x \in \partial\Omega$.

Note that the assumptions in Theorem 3.6 guarantee that the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself, see [3] and also [4], [21], [23], [18].

P r o o f. Assume first that $u \in Q_0^{1,p(\cdot)}(\Omega)$ and $\tilde{u}$ is its canonical representative. By [9, Theorem 4.7] the $p(\cdot)$-quasicontinuous representative of $\tilde{u}$ is

$$\lim_{r\to\infty} \fint_{B(x,r)} \tilde{u}(y)\,\mathrm{d}y$$

and hence by [14]

$$\tilde{u} = \lim_{r\to\infty} \fint_{B(x,r)} \tilde{u}(y)\,\mathrm{d}y$$

except on a set of zero $p(\cdot)$-capacity. This implies that for $p(\cdot)$-quasievery $x \in \partial\Omega$

$$0 = \lim_{r\to\infty} \fint_{B(x,r)} \tilde{u}(y)\,\mathrm{d}y = \lim_{r\to\infty} \frac{C}{r^n} \int_{B(x,r)\cap\Omega} u(y)\,\mathrm{d}y.$$

Assume now that $u \in W^{1,p(\cdot)}(\Omega)$ and the condition (3.7) holds. We follow the proof of [28, Theorem 2.2] and sketch the rest of the proof.

Let $U_i \subset \Omega$ be an increasing sequence of bounded open sets with $\overline{U}_i \subset \Omega$ and $\lim\limits_{i\to\infty} U_i = \Omega$. Let $\varphi_i \in C_0^\infty(\Omega)$ be a cut-off function which is 1 in U_i. Hence $u\varphi_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and by [9, Theorem 4.7],

$$\lim_{r\to\infty} \fint_{B(x,r)} u(y)\varphi_i(y)\,\mathrm{d}y$$

is a $p(\cdot)$-quasicontinuous representative of $u\varphi_i$. Letting $i \to \infty$ the subadditivity of the capacity implies that

$$\tilde{u}(x) = \lim_{r\to\infty} \fint_{B(x,r)} u(y)\,\mathrm{d}y$$

exists $p(\cdot)$-quasieverywhere in Ω and that it is the $p(\cdot)$-quasicontinuous representative of u in Ω.

We define $u^*(x) = \tilde{u}(x)$ for $x \in \Omega$ and $u^*(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. By the assumption we have

$$\lim_{r\to 0} \fint_{B(x,r)} u^*(y)\,\mathrm{d}y = \lim_{r\to 0} \frac{C}{r^n} \int_{B(x,r)\cap\Omega} u(y)\,\mathrm{d}y = 0$$

for $p(\cdot)$-quasievery $x \in \partial\Omega$. Since u^* is identically zero in the complement of Ω and since the capacity can be approximated by the Hausdorff measure, [10, Theorem 4.2], we obtain that u^* is approximately continuous at H^{n-1}-almost every $x \in \mathbb{R}^n$. Since

$u \in W^{1,p^-}_{\mathrm{loc}}(\Omega)$, we find as in the proof of [28, Theorem 2.2] that u^* is absolutely continuous on almost every line segment parallel to coordinate axis and hence $u^* \in W^{1,p(\cdot)}(\mathbb{R}^n)$. Now by [9, Theorem 4.7],

$$\lim_{r\to 0} \fint_{B(x,r)} u^*(y)\,\mathrm{d}y$$

is the $p(\cdot)$-quasicontinuous representative of u^* and hence $u \in Q_0^{1,p(\cdot)}(\Omega)$. □

We write $K_0^{1,p(\cdot)}(\Omega)$ to denote the closure of

$$\{u \in W^{1,p(\cdot)}(\Omega) \text{ and } u \text{ has compact support in } \Omega\}$$

in the space $W^{1,p(\cdot)}(\Omega)$. It is easy to see that this is a Banach space and the set of functions with compact support are dense there. Clearly the values of p in $\mathbb{R}^n \setminus \Omega$ do not affect $K_0^{1,p(\cdot)}(\Omega)$.

3.8. Theorem. *Let $p\colon \mathbb{R}^n \to (1,\infty)$.*

(i) *If $1 < p^- \leqslant p^+ < \infty$, then $Q_0^{1,p(\cdot)}(\Omega) \subset K_0^{1,p(\cdot)}(\Omega)$.*

(ii) *If $1 < p^- \leqslant p^+ < \infty$ and continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, then $Q_0^{1,p(\cdot)}(\Omega) = K_0^{1,p(\cdot)}(\Omega)$.*

Proof. Let $u \in Q_0^{1,p(\cdot)}(\Omega)$ and let $\tilde u$ be its canonical representative. We need to show that there exist functions $\varphi_i \in W^{1,p(\cdot)}(\Omega)$ with compact support in Ω that tend to $\tilde u$ in Ω. As in the proof of Lemma 3.4 we may assume that $\tilde u$ is bounded and has compact support in $\mathbb{R}^n$.

For $0 < \varepsilon < 1$ define $\tilde u_\varepsilon(x) = \max\{\tilde u(x) - \varepsilon, 0\}$. Let G be an open set such that $\tilde u$ restricted to $\Omega \setminus G$ is continuous. Let $\delta > 0$ and let $\omega \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $0 \leqslant \omega \leqslant 1$, ω is one in an open set containing G and $\|\omega\|_{1,p(\cdot)} < \delta$. The function $(1-\omega)\tilde u_\varepsilon$ is zero at a point x if either $\tilde u(x) \leqslant \varepsilon$ or $x \in G$. Hence it vanishes in a neighborhood of $\mathbb{R}^n \setminus \Omega$. Using the same arguments as in the proof of Lemma 3.4, we obtain the desired sequence (φ_i).

To prove the claim (ii), assume that $u \in K_0^{1,p(\cdot)}(\Omega)$. By definition there exists a sequence (u_i) of functions with compact support converging to u. Since $Q_0^{1,p(\cdot)}(\Omega)$ is a Banach space, it is enough to show that each u_i belongs to $Q_0^{1,p(\cdot)}(\Omega)$. Since continuous functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, every u_i has a quasicontinuous representative [10, Theorem 5.2]. Since u_i is zero everywhere in $\mathbb{R}^n \setminus \operatorname{spt}(u_i)$ its quasicontinuous representative is zero quasieverywhere in $\mathbb{R}^n \setminus \operatorname{spt}(u_i)$, [14] (or [11, Lemma 2.1]) and $\mathbb{R}^n \setminus \Omega \subset \mathbb{R}^n \setminus \operatorname{spt}(u_i)$. This completes the proof. □

3.9. E x a m p l e. P. Hästö constructed a function $u \in W^{1,p(\cdot)}(B(0,\frac{1}{4}))$ which is not continuous at the origin (even if we redefine it in a set of measure zero) and $C_{p(\cdot)}(\{0\}) > 0$, i.e. the function u does not have a quasicontinuous representative, [13]. In this example the exponent is continuous and satisfies

$$|p(x) - p(0)| \approx \frac{\log_2 \log_2(1/x)}{-\log x}.$$

Multiplying u with a suitable Lipschitz continuous cut-off function φ which is 1 in $B(0,\frac{1}{8})$ and zero outside $B(0,\frac{3}{8})$, we get a function which belongs to $K_0^{1,p(\cdot)}(B(0,\frac{1}{4}))$ but not to $Q_0^{1,p(\cdot)}(B(0,\frac{1}{4}))$.

In the conclusion we collect our results. Assume that $p\colon \Omega \to (1,\infty)$ with $1 < p^- \leqslant p^+ < \infty$ and that

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|} \tag{3.1}$$

holds for every $|x-y| \leqslant 1/2$. Now p can be extended by the McShane extension, see [19], to $\mathbb{R}^n$ so that p^- and p^+ do not change and (3.10) holds, possibly with a different constant, for every $x, y \in \mathbb{R}^n$ with $|x-y| \leqslant 1/2$. Hence by [26] $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Keeping Theorem 3.5 in mind we obtain by Theorems 3.3 and 3.8 the following theorem.

3.11. Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p\colon \Omega \to (1,\infty)$.*

(i) *If $1 < p^- \leqslant p^+ < \infty$, then*

$$H_0^{1,p(\cdot)}(\Omega) \subset Q_0^{1,p(\cdot)}(\Omega) \subset K_0^{1,p(\cdot)}(\Omega).$$

(ii) *If $1 < p^- \leqslant p^+ < \infty$ and*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|}$$

for every $|x-y| \leqslant 1/2$, then

$$H_0^{1,p(\cdot)}(\Omega) = Q_0^{1,p(\cdot)}(\Omega) = K_0^{1,p(\cdot)}(\Omega).$$

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