A COMPARISON OF THREE RECENT SELECTION THEOREMS

Caterina Maniscalco, Palermo

(Received January 13, 2006)

Abstract. We compare a recent selection theorem given by Chistyakov using the notion of modulus of variation, with a selection theorem of Schrader based on bounded oscillation and with a selection theorem of Di Piazza-Maniscalco based on bounded $\mathscr{A}, \Lambda$-oscillation.

Keywords: variation, oscillation, modulus of variation, selection theorem

MSC 2000: 26A45

1. Introduction

In ([3]) Chistyakov proves a sufficient condition for the existence of a convergent subsequence of a given functions sequence. Such a result is based on the notion of modulus of variation introduced by Chanturiya in [2] and generalizes many selection theorems based on the notion of variation ([6]) or of generalized, in some sense, variation ([8], [10]).

In the above mentioned paper Chistyakov leaves open the problem concerning the relationship between his theorem and the selection theorems based on the notion of oscillation, contained in the works of Schrader ([9]) and Di Piazza-Maniscalco ([5]).

Here we prove that the Chistyakov theorem has no relationship both with the Schrader theorem based on bounded oscillation (see [9], Theorem 1.2) and with the Di Piazza-Maniscalco theorem based on bounded $\mathscr{A}, \Lambda$-oscillation (see [5], Theorem 2.1).

Supported by Italian MURST.

2. Notation and useful facts

A sequence $\{f_j\}$ of real functions defined on a set X is said to be *pointwise bounded* if, for each $x \in X$, the sequence $\{f_j(x)\}$ is bounded; $\{f_j\}$ is said to be *uniformly bounded* if there exists a positive constant M such that $|f_j(x)| \leqslant M$ for each $x \in X$ and for all positive integers j.

Let f be a real function defined on a bounded closed interval $[a,b]$ in $\mathbb{R}$, and let n be a positive integer. We set

$$\nu(n,f) = \sup \sum_{i=1}^{n} |f(x_{2i}) - f(x_{2i-1})|,$$

where the supremum is taken over all collections $\{x_1, x_2, \ldots, x_{2n}\}$ of $2n$ points of $[a,b]$ such that $a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{2n} \leqslant b$. The sequence $\{\nu(n,f)\}_{n=1}^{\infty}$ is called the *modulus of variation* of f in the sense of Chanturiya ([2]). The following theorem characterizes, in terms of modulus of variation, the regulated functions, i.e. the functions with finite left and right limits at each point of $[a,b]$ (see [2], Theorem 5).

Theorem 2.1. *A function $f\colon [a,b] \to \mathbb{R}$ is regulated if and only if*

$$\lim_{n\to+\infty} \frac{\nu(n,f)}{n} = 0.$$

In [3] Chistyakov proves the following selection theorem:

Theorem 2.2. *Let $\{f_j\}$ be a uniformly bounded sequence of real valued functions on $[a,b]$ such that*

$$\lim_{n\to\infty} \Big(\frac{1}{n} \limsup_{j\to\infty} \nu(n,f_j)\Big) = 0. \tag{2.1}$$

Then it contains a subsequence which converges pointwise on $[a,b]$ to a bounded function $f\colon [a,b] \to \mathbb{R}$ satisfying $\lim\limits_{n\to+\infty} \frac{1}{n}\nu(n,f) = 0$.

Let $f\colon [a,b] \to \mathbb{R}$ and let $\mathscr{P}(f)$ be the family of all finite collections $P = \{x_1, x_2, \ldots, x_n\}$ with $n \geqslant 1$ and $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$ such that $(-1)^i f(x_i) > 0$ for each $i = 1,2,\ldots,n$, or $(-1)^i f(x_i) < 0$ for each $i = 1,2,\ldots,n$, or $f(x_i) = 0$ for each $i = 1,2,\ldots,n$. The *oscillation* of f in $[a,b]$ is defined in [9] by

$$T(f) = \sup_{P \in \mathscr{P}(f)} \sum_{i=1}^{n} |f(x_i)|.$$

In [9] Schrader proves the following selection theorem:

Theorem 2.3. *Let $\{f_j\}$ be a sequence of real valued functions on $[a,b]$. If there exists a positive constant M such that $T(f_j - f_l) \leqslant M$ for all j, l, then $\{f_j\}$ contains a subsequence which converges pointwise on $[a,b]$.*

Let X be a subset of $\mathbb{R}$. A family $\mathscr{A}$ of intervals in $\mathbb{R}$ is called a *complete subbase of intervals* on X ([1]) if, for almost every $x \in X$, there exists a constant $\delta(x) > 0$ such that the intervals $[x-\gamma, x]$, $[x, x+\gamma]$, $0 < \gamma < \delta(x)$, whose interior parts intersect X, are in $\mathscr{A}$. The domain of δ is denoted by $\mathscr{D}(A)$. If X is an interval we can suppose that every element of $\mathscr{A}$ is contained in X.

Let f be a real function defined on X. According to [5] we denote by $\mathscr{P}(f,\mathscr{A})$ the family of all collections $P = \{x_0, x_1, \ldots, x_n\}$ of points of $\mathscr{D}(A)$ with $n = 2k+1$, $k \in \mathbb{N}$, $x_0 < x_1 < \ldots < x_n$, $[x_{2i}, x_{2i+1}] \in \mathscr{A}$ for $i = 0, 1, \ldots, k$, and fulfilling the following condition: $f(x_{2i})f(x_{2i+1}) < 0$ for every $i = 0, 1, \ldots, k$. A sequence $\Lambda = \{\lambda_i\}$ is said to be admissible if it is a non-decreasing sequence of positive real numbers with $\lambda_1 \geqslant 1$, $\lim\limits_{i\to\infty} 1/\lambda_i = 0$ and $\sum\limits_{i=1}^{\infty} 1/\lambda_i = +\infty$. The $\mathscr{A}, \Lambda$-oscillation of f is defined by

$$T_\Lambda(f,\mathscr{A}) = \begin{cases} \sup\limits_{P \in \mathscr{P}(f,\mathscr{A})} \left(\sum\limits_{i=0}^{2k+1} \dfrac{|f(x_i)|}{\lambda_{[(i+2)/2]}} \right) & \text{if } \mathscr{P}(f,\mathscr{A}) \neq \emptyset, \\ 0 & \text{if } \mathscr{P}(f,\mathscr{A}) = \emptyset \end{cases}$$

where $[\frac{1}{2}(i+2)]$ is the integer part of $\frac{1}{2}(i+2)$.

In [5] Di Piazza and the author prove the following theorem that generalizes the above Theorem 2.3.

Theorem 2.4. *Let $\{f_j\}$ be a pointwise bounded sequence of real functions defined on a set X. If there exist a complete subbase $\mathscr{A}$ on X, an admissible sequence $\Lambda = \{\lambda_i\}$, and a positive constant M such that $T_\Lambda(f_j - f_l, \mathscr{A}) \leqslant M$ for all j, l, then $\{f_j\}$ contains a subsequence which converges pointwise on $\mathscr{D}(A)$.*

3. Comparison of selection theorems

In order to make Theorem 2.4 and Theorem 2.2 comparable we assume in this section that, in the hypotheses of Theorem 2.4, $\mathscr{D}(A) = X$.

Theorem 3.1. *Selection Theorem 2.2 has no relation both with the selection Theorem 2.3 based on bounded oscillation and with the selection Theorem 2.4 based on bounded $\mathscr{A},\Lambda$-oscillation.*

P r o o f. The proof is constructive and is divided into three steps:

S t e p I. We construct on $[0,1]$ a functions sequence $\{f_j\}$ fulfilling the hypotheses of Theorem 2.2 and the ones of Theorem 2.4 but not fulfilling the hypotheses of Theorem 2.3. Define

$$f_j(x)=\begin{cases}(-1)^{m+j}\dfrac{1}{mj}+\displaystyle\sum_{i=1}^{m}(-1)^i\frac{1}{i} & \text{if } x\in I_m=\Big[\dfrac{m-1}{m},\dfrac{m}{m+1}\Big[,\\ -\log 2 & \text{if} \quad x=1.\end{cases}$$

For calculation of $\nu(n,f_j)$, let us consider $2n$ points $0\leqslant x_1\leqslant x_2\leqslant\ldots\leqslant x_{2n}\leqslant 1$. For each $i\in\mathbb{N}$ let m_i be the natural index such that the point $x_i\in I_{m_i}$, except possibly the case $x_{2n}=1$. In case $x_{2n}=1$, we put $m_{2n}=+\infty$. Obviously $m_i\leqslant m_{i+1}$. Without loss of generality, we may suppose that $m_{2h-1}\neq m_{2h}$, $h=1,2,\ldots,n$. Indeed, if x_{2h} and x_{2h-1} are in the same interval, then $|f_j(x_{2h})-f_j(x_{2h-1})|=0$.

By the Leibniz Theorem we obtain

$$\begin{aligned}
&|f_j(x_{2h})-f_j(x_{2h-1})|\\
&\quad=\Big|\sum_{i=1}^{m_{2h}}(-1)^i\frac{1}{i}+(-1)^{m_{2h}+j}\frac{1}{m_{2h}j}-\sum_{i=1}^{m_{2h-1}}(-1)^i\frac{1}{i}-(-1)^{m_{2h-1}+j}\frac{1}{m_{2h-1}j}\Big|\\
&\quad\leqslant\Big|\sum_{i=1}^{m_{2h}}(-1)^i\frac{1}{i}-\sum_{i=1}^{m_{2h-1}}(-1)^i\frac{1}{i}\Big|+\frac{1}{m_{2h}j}+\frac{1}{m_{2h-1}j}\\
&\quad<\frac{2}{m_{2h-1}+1}+\frac{2}{m_{2h-1}j}\leqslant\frac{2}{m_{2h-1}}\Big(1+\frac{1}{j}\Big)\leqslant\frac{4}{m_{2h-1}}.
\end{aligned}$$

Hence we infer

$$\sum_{h=1}^{n}|f_j(x_{2h})-f_j(x_{2h-1})|<\sum_{h=1}^{n}\frac{4}{m_{2h-1}}\leqslant 4\sum_{h=1}^{n}\frac{1}{h}.$$

Therefore, for each positive integer j we have $\nu(n,f_j)\leqslant 4\sum\limits_{h=1}^{n}1/h=4\log n+4\gamma+o(1)$ when $n\to+\infty$, where γ is the Euler-Mascheroni constant. So the sequence $\{f_j\}$ fulfils (2.1). Moreover, $|f_j(x)|\leqslant 2$, hence the hypotheses of Theorem 2.2 are verified.

We are going to prove now that $\{f_j\}$ doesn't fulfil the hypotheses of Theorem 2.3. Let j and l, with $j>l$, be positive integers and let $x\in[\frac{m-1}{m},\frac{m}{m+1}[$. It is easy to see that, if j and l are both even or both odd, we have

$$|f_j(x) - f_l(x)| = \frac{j-l}{mjl},$$

while if j is even and l odd, or vice-versa, we have

$$|f_j(x) - f_l(x)| = \frac{j+l}{mjl}.$$

Let us fix now j and l. For each positive integer n there exists a collection $P = \{x_1, x_2, \dots, x_n\} \in \mathscr{P}(f_j - f_l)$ with $x_i \in I_i$, $i = 1, 2, \dots, n$. Then, if j and l are both odd or both even, we obtain

$$\sum_{i=1}^{n} |f_j(x_i) - f_l(x_i)| = \frac{j-l}{jl} \sum_{i=1}^{n} \frac{1}{i},$$

while if j is even and l is odd, or vice-versa, we have

$$\sum_{i=1}^{n} |f_j(x_i) - f_l(x_i)| = \frac{j+l}{jl} \sum_{i=1}^{n} \frac{1}{i}.$$

In any case $T(f_j - f_l) = +\infty$. Then $\{f_j\}$ doesn't fulfil the hypotheses of Theorem 2.3.

Moreover, the sequence $\{f_j\}$ fulfils the hypotheses of Theorem 2.4 with $\Lambda = \{i\}$ and the complete subbase $\mathscr{A}$ of $[0,1]$ such that $\mathscr{D}(A) = [0,1]$ and, for $x \in]\frac{m-1}{m}, \frac{m}{m+1}[$, we have $\delta(x) < \min(d(x, \frac{m-1}{m}), d(x, \frac{m}{m+1}))$, where d is the Euclidean distance in $\mathbb{R}$. Let us fix j, l and let $P = \{x_0, x_1, \dots, x_n\}$ be a collection of $\mathscr{P}(f_j - f_l, \mathscr{A})$ with $n = 2k+1$. Then, for each $h = 0, 1, 2, \dots, k$, at least one of the extremes of $[x_{2h}, x_{2h+1}]$ is a point of the form $\frac{m}{m+1}$. Therefore,

$$\sum_{i=0}^{2k+1} \frac{1}{\lambda_{[(i+2)/2]}} |f_j(x_i) - f_l(x_i| \leqslant 2 \sum_{i=1}^{k+1} \frac{l+j}{i^2 lj} \leqslant 2 \frac{l+j}{lj} \sum_{i=1}^{\infty} \frac{1}{i^2} < 4 \sum_{i=1}^{\infty} \frac{1}{i^2}.$$

So $T_\Lambda(f_j - f_l, \mathscr{A}) \leqslant 4 \sum\limits_{i=1}^{\infty} 1/i^2$ whenever $1 \leqslant l < j < +\infty$.

Step II. Now we construct on $[0,1]$ a functions sequence $\{g_j\}$ fulfilling the hypotheses of Theorem 2.2 but not the ones of Theorem 2.4. Define

$$g_j(x) = \begin{cases} (-1)^{m+j} j^{-1} + \sum\limits_{i=1}^{m} (-1)^i i^{-1} & \text{if } x \in I_m = [\frac{m-1}{m}, \frac{m}{m+1}[, \\ -\log 2 & \text{if } \quad x = 1. \end{cases}$$

Using the same formalism and technique as in Step I, for $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{2n} \leqslant 1$ we obtain

$$|g_j(x_{2h}) - g_j(x_{2h-1})| \leqslant \frac{2}{m_{2h-1}} + \frac{2}{j} \qquad h = 1, 2, \ldots, n. \tag{3.1}$$

Then

$$\sum_{h=1}^{n} |g_j(x_{2h}) - g_j(x_{2h-1})| \leqslant \sum_{h=1}^{n} \frac{2}{m_{2h-1}} + \frac{2n}{j} \leqslant 2\sum_{h=1}^{n} \frac{1}{h} + \frac{2n}{j}.$$

So, for each positive integer j we have $\nu(n, g_j) \leqslant 2 \sum\limits_{h=1}^{n} 1/h + 2n/j$. Therefore

$$\limsup_{j\to\infty} \nu(n, g_j) \leqslant 2\sum_{h=1}^{n} \frac{1}{h} = 2\log n + 2\gamma + o(1) \quad \text{when } n \to +\infty. \tag{3.2}$$

Hence the sequence $\{g_j\}$ fulfils (2.1). Moreover, $|g_j(x)| \leqslant 2$. Hence the hypotheses of Theorem 2.2 are verified.

In order to prove that $\{g_j\}$ doesn't fulfil the hypotheses of Theorem 2.4, let us fix positive integers j and l, with $j > l$ and with j even and l odd, or vice-versa. For each $x \in [0, 1[$ we have

$$|g_j(x) - g_l(x)| = \frac{j+l}{jl}.$$

Whichever complete subbase $\mathscr{A}$ with $\mathscr{D}(A) = [0, 1]$ we consider, for each positive integer $n = 2k + 1$ there exists a collection $P = \{x_0, x_1, \ldots, x_n\} \in \mathscr{P}(g_j - g_l, \mathscr{A})$ with $x_{2i} \in I_{i+1}, i = 0, 1, \ldots, k$. Then, if $\Lambda = \{\lambda_i\}$ is any admissible sequence, we get

$$\sum_{i=0}^{n} \frac{|g_j(x_i) - g_l(x_i)|}{\lambda_{[(i+2)/2]}} = 2\frac{j+l}{jl} \sum_{i=1}^{k+1} \frac{1}{\lambda_i}.$$

So $T_\Lambda(g_j - g_l, \mathscr{A}) = +\infty$. Hence $\{g_j\}$ doesn't fulfil the hypotheses of Theorem 2.4.

S t e p III. Finally, in this step we construct a functions sequence $\{h_j\}$ fulfilling the hypotheses of Theorem 2.3 (and consequently also the ones of Theorem 2.4) but not fulfilling the hypotheses of Theorem 2.2.

Let h_0 be a non regulated function on $[a, b]$ (i.e. h_0 has at least one non simple discontinuity), and define

$$h_j(x) = h_0(x) + \frac{1}{j}.$$

If j, l are positive integers with $j > l$, then for each x we have

$$h_j(x) - h_l(x) = \frac{1}{j} - \frac{1}{l} < 0.$$

Then every collection $P \in \mathscr{P}(h_j - h_l)$ contains only a single point. So $T(h_j - h_l) = 1/l - 1/j < 1$. Therefore, $\{h_j\}$ fulfils the hypotheses of Theorem 2.3 and, consequently, the ones of Theorem 2.4. On the other hand, since the sequence $\{h_j\}$ converges to h_0, in view of Theorem 2.1, $\{h_j\}$ cannot fulfil the hypotheses of Theorem 2.2.

References

[1] *Bongiorno B., Vetro P.*: Su un teorema di F. Riesz. Atti Acc. Sc. Lett. Arti Palermo, Ser. IV *37* (1977–78), 3–13.

[2] *Chanturiya Z. A.*: The modulus of variation of a function and its application in the theory of Fourier series. Soviet. Math. Dokl. *15* (1974), 67–71.

[3] *Chistyakov V. V.*: A selection principle for functions of a real variable. Atti Sem. Mat. Fis. Univ. Modena e Reggio Emilia *53* (2005), 25–43.

[4] *Chistyakov V. V.*: The optimal form of selection principle for functions of a real variable. J. Math. Anal. Appl. *310* (2005), 609–625.

[5] *Di Piazza L., Maniscalco C.*: Selection theorems, based on generalized variation and oscillation. Rend. Circ. Mat. Palermo, Ser. II *35* (1986), 386–396.

[6] *Helly E.*: Über linear Funktionaloperationen. Sitzungsber. Naturwiss. Kl. Kaiserlichen Akad. Wiss. Wien *121* (1912), 265–297.

[7] *Henstock H.*: The General Theory of Integration. Clarendon Press, Oxford, U.K., 1991.

[8] *Musielak J., Orlicz W.*: On generalized variations (I). Studia Math. *18* (1959), 11–41.

[9] *Schrader K.*: A generalization of the Helly selection theorem. Bull. Amer. Math. Soc. *78* (1972), 415–419.

[10] *Waterman D.*: On Λ-bounded variation. Studia Math. *57* (1976), 33–45.

Author's address: *Caterina Maniscalco*, Dipartimento di Matematica e Applicazioni, Universita' di Palermo, Via Archirafi, 34, 90123 Palermo, Italy, e-mail: `maniscal@math.unipa.it`.