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ON γ-LABELINGS OF ORIENTED GRAPHS

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Abstract. Let D be an oriented graph of order n and size m. A γ-labeling of D is a one-to-one function $f\colon V(D) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f'\colon E(D) \to \{\pm 1, \pm 2, \ldots, \pm m\}$ of the arcs of D defined by $f'(e) = f(v) - f(u)$ for each arc $e = (u, v)$ of D. The value of a γ-labeling f is $\mathrm{val}(f) = \sum\limits_{e \in E(G)} f'(e)$. A γ-labeling of D is balanced if the value of f is 0. An oriented graph D is balanced if D has a balanced labeling. A graph G is orientably balanced if G has a balanced orientation. It is shown that a connected graph G of order $n \geqslant 2$ is orientably balanced unless G is a tree, $n \equiv 2 \pmod 4$, and every vertex of G has odd degree.

Keywords: oriented graph, γ-labeling, balanced γ-labeling, balanced oriented graph, orientably balanced graph

MSC 2000: 05C20, 05C78

1. Introduction

Let G be a graph of order n and size m. A *γ-labeling* of G is defined in [1] as a one-to-one function $f\colon V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f'\colon E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geqslant n - 1$. Each γ-labeling f is assigned a *value* denoted by $\mathrm{val}(f)$ and defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geqslant 1$ for each edge e in G and so $\mathrm{val}(f) \geqslant m$.

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Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

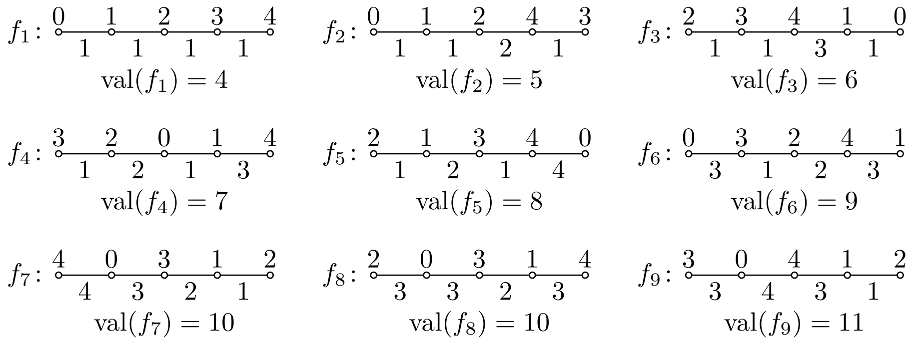

Figure 1: Some γ-labelings of P_5

For a graph G of order n and size m, the *maximum value* $\mathrm{val}_{\max}(G)$ of G is defined as the maximum value among all γ-labelings of G and the *minimum value* $\mathrm{val}_{\min}(G)$ is the minimum value among all γ-labelings of G, that is,

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\},$$
$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f)\colon\ f \text{ is a } \gamma\text{-labeling of } G\}.$$

It turns out that $\mathrm{val}_{\max}(P_5) = 11$ and $\mathrm{val}_{\min}(P_5) = 4$.

If the induced edge-labeling f' of a γ-labeling f of a graph is also one-to-one, then f is a *graceful labeling.* Among all labelings of graphs, graceful labelings are probably the best known and most studied. Graceful labelings originated with a 1967 paper of Rosa [6], who used the term β-valuations. A few years later, Golomb [5] called these labelings "graceful" and this is the terminology that has been used since then. Gallian [4] has written an extensive survey on labelings of graphs. The subject of γ-labelings of graphs was studied in [1], [2]. In this paper, we study γ-labelings of oriented graphs. We refer to the book [3] for graph-theoretical notation and terminology not described in this paper.

If a digraph D has the property that for each pair u, v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D, then D is an *oriented graph.* Thus an oriented graph D is obtained by assigning a direction to each edge of some graph G. In this case, the digraph D is also called an *orientation* of G. For an oriented graph D of order n and size m, a γ*-labeling* of D is a one-to-one function $f\colon\ V(D) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling

$$f'\colon\ E(D) \to \{\pm 1, \pm 2, \ldots, \pm m\}$$

of the arcs of D defined by $f'(e) = f(v) - f(u)$ for each arc $e = (u, v)$ of D. The *value* of a γ-labeling f is denoted by $\mathrm{val}(f : D)$ and is defined by

$$\mathrm{val}(f : D) = \sum_{e \in E(D)} f'(e).$$

If the oriented graph D under consideration is clear, we use $\mathrm{val}(f)$ to denote $\mathrm{val}\,(f : D)$. For example, γ-labelings of two oriented graphs are shown in Figure 2, where $\mathrm{val}(f_1) = 2$ and $\mathrm{val}(f_2) = 0$.

$\mathrm{val}(f_1) = 2$ $\qquad$ $\mathrm{val}(f_2) = 0$

Figure 2: Illustrating γ-labelings of digraphs

For a digraph D, the *converse* D^* of D is the digraph obtained from D by reversing the direction of every arc of D. If f is a γ-labeling of D, then f is also a γ-labeling of D^* and

$$\mathrm{val}(f, D^*) = -\,\mathrm{val}(f, D). \tag{1}$$

The following lemma will be useful.

Lemma 1.1. *Let G be a nonempty graph that is decomposed into graphs $G_1, G_2, \ldots, G_k$ $(k \geqslant 2)$. For $1 \leqslant i \leqslant k$, let D_i be an orientation of G_i and let D be the orientation of G for which $E(D) = \sum_{i=1}^{k} E(D_i)$. If f is a γ-labeling of D and f_i is the restriction of f to D_i for $1 \leqslant i \leqslant k$, then*

$$\mathrm{val}(f : D) = \sum_{i=1}^{k} \mathrm{val}(f_i : D_i).$$

A γ-labeling f of an oriented graph D is *balanced* if the value of f is 0 and an oriented graph having a balanced labeling is called a *balanced digraph.* For example, the γ-labeling f_1 in Figure 2 is not balanced, while the γ-labeling f_2 in Figure 2 is balanced and so the second digraph in Figure 2 is a balanced digraph. In fact, the first digraph of Figure 2 is not a balanced digraph. In general, if $\vec{P}_n\colon v_1, v_2, \ldots, v_n$ is a nontrivial directed path of order n and f is a γ-labeling of $\vec{P}_n$, then

$$\operatorname{val}(f) = \sum_{e \in E(\vec{P}_n)} f'(e) = \sum_{i=1}^{n-1} [f(v_{i+1}) - f(v_i)] = f(v_n) - f(v_1) \neq 0.$$

This gives the following observation.

Observation 2.1. *No γ-labeling of a nontrivial directed path is balanced and so no nontrivial directed path is balanced.*

On the other hand, let $\vec{C}_n\colon v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ be a directed cycle of order $n \geqslant 3$ and let f be a γ-labeling of $\vec{C}_n$. Then

$$\begin{aligned} \operatorname{val}(f) &= \sum_{e \in E(\vec{C}_n)} f'(e) \\ &= \sum_{i=1}^{n} [f(v_{i+1}) - f(v_i)] = f(v_{n+1}) - f(v_1) = 0. \end{aligned}$$

This gives the following observation.

Observation 2.2. *Every γ-labeling of a directed cycle is balanced and every directed cycle is balanced.*

By Lemma 1.1 and Observations 2.1 and 2.2, we have the following.

Proposition 2.3. *Let f be a γ-labeling of an oriented graph D.*

(a) *If D can be decomposed into k directed $u_i - v_i$ paths for $1 \leqslant i \leqslant k$, then*

$$\operatorname{val}(f) = \sum_{i=1}^{k} [f(v_i) - f(u_i)].$$

(b) *If D can be decomposed into directed cycles, then f (and therefore D) is balanced.*

An oriented graph D is *Eulerian* if D possesses a directed circuit that contains all of the arcs and vertices of D. Furthermore, an oriented graph D is Eulerian if and only if D can be decomposed into directed cycles. The following is therefore an immediate consequence of Proposition 2.3(b).

Corollary 2.4. *Every Eulerian oriented graph is balanced.*

Indeed, *every* γ-labeling of an Eulerian oriented graph is balanced by Observation 2.2. In fact, Eulerian oriented graphs are the only oriented graphs having this property. In order to show this, we first state a lemma. For a vertex v in an oriented graph, the number of vertices to which a vertex v is adjacent is the *outdegree* of v and is denoted by $\operatorname{od} v$. The number of vertices from which v is adjacent is the *indegree* of v and is denoted by $\operatorname{id} v$.

Lemma 2.5. *If f is a γ-labeling of an oriented graph D, then*

$$\operatorname{val}(f) = \sum_{v \in V(D)} (\operatorname{id} v - \operatorname{od} v) f(v). \tag{2}$$

P r o o f. For each vertex v, the term $f(v)$ appears exactly $\operatorname{id} v$ times in the sum of $\operatorname{val}(f)$; while the term $-f(v)$ appears exactly $\operatorname{od} v$ times in $\operatorname{val}(f)$. Therefore, (2) holds. □

Theorem 2.6. *Let D be an oriented graph. Then every γ-labeling of D is balanced if and only if D is Eulerian.*

P r o o f. We have seen that every γ-labeling of an Eulerian oriented graph is balanced. Thus it remains to verify the converse. Assume that D is not Eulerian. We show that there is a γ-labeling of D that is not balanced. Let f be a γ-labeling of D. If $\operatorname{val}(f) \neq 0$, then we have the desired result. Thus we may assume that $\operatorname{val}(f) = 0$. Let

$$U = \{v \in V(D) \colon \operatorname{id} v > \operatorname{od} v\} \quad \text{and} \quad W = \{v \in V(D) \colon \operatorname{id} v < \operatorname{od} v\}.$$

Since D is not Eulerian, $U \neq \emptyset$ and $W \neq \emptyset$. Hence there exist some vertex $u \in U$ and some vertex $w \in W$. We define a γ-labeling g of D by

$$g(v) = \begin{cases} f(w) & \text{if } v = u, \\ f(u) & \text{if } v = w, \\ f(v) & \text{if } v \neq u, w. \end{cases}$$

Suppose that $f(w) - f(u) = c \neq 0$. It then follows by (2) that

$$\begin{aligned}
\mathrm{val}(g) &= \sum_{v \in V(D)} (\mathrm{id}\, v - \mathrm{od}\, v) g(v) \\
&= \sum_{v \in V(D) - \{u,w\}} (\mathrm{id}\, v - \mathrm{od}\, v) g(v) \\
&\quad + (\mathrm{id}\, u - \mathrm{od}\, u) g(u) + (\mathrm{id}\, w - \mathrm{od}\, w) g(w) \\
&= \sum_{v \in V(D) - \{u,w\}} (\mathrm{id}\, v - \mathrm{od}\, v) f(v) \\
&\quad + (\mathrm{id}\, u - \mathrm{od}\, u) f(w) + (\mathrm{id}\, w - \mathrm{od}\, w) f(u) \\
&= \sum_{v \in V(D)} (\mathrm{id}\, v - \mathrm{od}\, v) f(v) \\
&\quad + (\mathrm{id}\, u - \mathrm{od}\, u)(f(w) - f(u)) + (\mathrm{id}\, w - \mathrm{od}\, w)(f(u) - f(w)) \\
&= c[(\mathrm{id}\, u - \mathrm{od}\, u) + (\mathrm{od}\, w - \mathrm{id}\, w)] \neq 0.
\end{aligned}$$

Thus g is not balanced, as claimed. □

3. Orientably balanced graphs

A graph G is *orientably balanced* if G has a balanced orientation. By Observation 2.2, every cycle is orientably balanced. In fact, more can be said. A graph G is *Eulerian* if G possesses a circuit containing all of the edges and vertices of G. In particular, every cycle is Eulerian. It is well-known that a graph G has an Eulerian orientation if and only if G is Eulerian. Thus, the following is a consequence of Theorem 2.6.

Proposition 3.1. *Every Eulerian graph is orientably balanced.*

Certainly, every γ-labeling of an orientation of the path P_2 has value 1 or -1 (see Figure 3) and so P_2 is not orientably balanced. On the other hand, both P_3 and P_4 are orientably balanced as there exist γ-labelings of orientations of both paths having value 0 (see Figure 3).

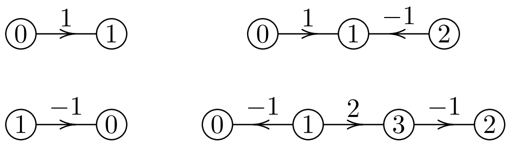

Figure 3: Illustrating γ-labelings of orientations of P_2, P_3, and P_4

In fact, every path of order 3 or more is orientably balanced.

Proposition 3.2. *For each integer $n \geqslant 3$, the path P_n is orientably balanced.*

P r o o f. Let $P_n\colon v_1, v_2, \ldots, v_n$, let D be the orientation of P_n such that $(v_1, v_2) \in E(D)$ and $(v_{i+1}, v_i) \in E(D)$ for $2 \leqslant i \leqslant n-1$, and let f be any γ-labeling of D for which $f(v_1) = 0$, $f(v_2) = 1$, and $f(v_n) = 2$. By Proposition 2.3(a), $\mathrm{val}(f) = (f(v_2) - f(v_1)) + (f(v_2) - f(v_n)) = 1 + (-1) = 0$. Thus P_n is orientably balanced. □

Since P_3 is orientably balanced, the star $K_{1,2}$ is orientably balanced. The stars $K_{1,3}$ and $K_{1,4}$ are also orientably balanced, as shown in Figure 4.

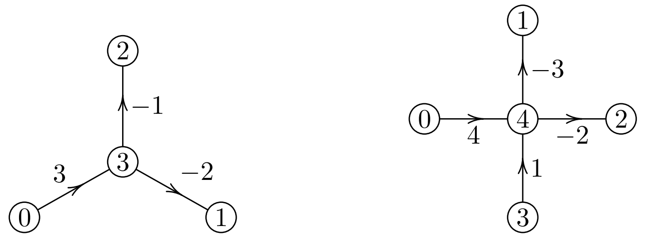

Figure 4: Illustrating γ-labelings of orientations of $K_{1,3}$ and $K_{1,4}$

Many other stars are orientably balanced.

Proposition 3.3. *For each integer $n \geqslant 3$, every star $K_{1,n-1}$, where $n \not\equiv 2 \pmod 4$, is orientably balanced.*

P r o o f. Let $V(K_{1,n-1}) = \{v, v_1, v_2, \ldots, v_{n-1}\}$, where v is the central vertex of $K_{1,n-1}$. We consider two cases, according to whether n is odd or n is even.

C a s e 1. *n is odd.* Then $n = 2k+1$ for a positive integer k. Let D be the orientation of $K_{1,n-1}$ such that $(v_i, v) \in E(D)$ for $1 \leqslant i \leqslant 2k = n-1$. Define the γ-labeling f of D by $f(v) = k$, $f(v_i) = i-1$ for $1 \leqslant i \leqslant k$, and $f(v_i) = i$ for $k+1 \leqslant i \leqslant 2k$. Figure 5 illustrates the labeling f of the orientation D of $K_{1,6}$ for $k = 3$.

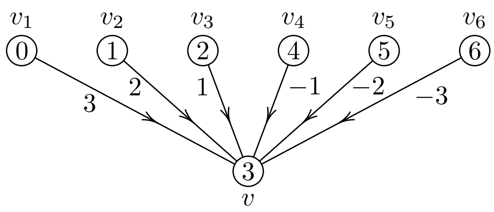

Figure 5: Illustrating the labeling f of the orientation D of $K_{1,6}$ in Proposition 3.3 for $k = 3$ and $n = 7$

Observe that

$$\begin{aligned}\operatorname{val}(f) &= \sum_{i=1}^{2k}[f(v)-f(v_i)] = \sum_{i=1}^{k}[k-(i-1)] + \sum_{i=k+1}^{2k}(k-i)\\ &= \sum_{i=1}^{k} i + \sum_{i=1}^{k}(-i) = 0.\end{aligned}$$

Therefore, $K_{1,n-1}$ is orientably balanced when n is odd.

Case 2. *n is even.* Since $n \not\equiv 2 \pmod 4$, it follows that $n \equiv 0 \pmod 4$ and so $n = 4k$ for some positive integer k. Let D' be the orientation of $K_{1,n-1}$ such that $(v, v_{k+1}) \in E(D')$ and $(v_i, v) \in E(D')$ for $1 \leqslant i \leqslant 4k-1 = n-1$ and $i \neq k+1$. Furthermore, let g be the γ-labeling of D' defined by $g(v) = 2k$, $g(v_i) = i-1$ for $1 \leqslant i \leqslant 2k$, and $g(v_i) = i$ for $2k+1 \leqslant i \leqslant 4k-1$. Figure 6 illustrates the labeling g of the orientation D' of $K_{1,11}$, where then $k = 3$ and $n = 12$.

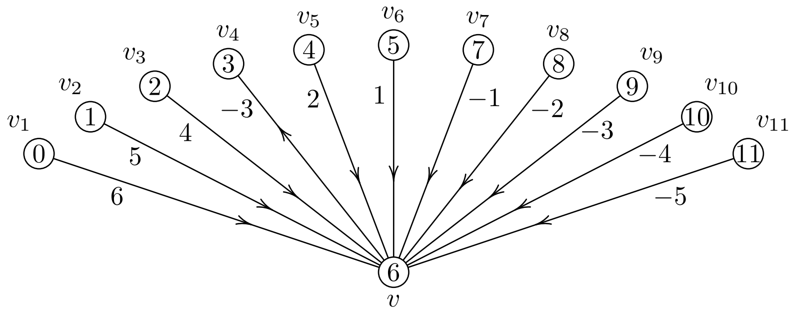

Figure 6: Illustrating the labeling g of the orientation D' of $K_{1,11}$ in Proposition 3.3 for $k = 3$ and $n = 12$

Observe that

$$\begin{aligned}\operatorname{val}(g) &= [g(v_{k+1}) - g(v)] + \sum_{\substack{1\leqslant i\leqslant 4k-1\\ i\neq k+1}} [g(v) - g(v_i)]\\ &= \left[\left(\sum_{i=1}^{2k} i\right) - 2k\right] + \left[\left(\sum_{i=1}^{2k-1}(-i)\right)\right] = \left(\sum_{i=1}^{2k-1} i\right) + \left[\left(\sum_{i=1}^{2k-1}(-i)\right)\right] = 0.\end{aligned}$$

Note that the $-2k$ occurs because the arc $(v, v_{k+1},)$ contributes $-k$ to $\operatorname{val}(g)$ rather than k. Therefore, $K_{1,n-1}$ is orientably balanced for n is even and $n \equiv 0 \pmod 4$. □

An infinite class of orientably balanced graphs can be constructed from each orientably balanced graph. By a *subdivision* of a graph G, we mean a graph obtained from G by inserting vertices (of degree 2) into one or more of the edges of G. An *elementary subdivision* of a graph G is a graph obtained from G by inserting exactly one vertex (of degree 2) into an edge of G.

Proposition 3.4. *Every subdivision of an orientably balanced graph is orientably balanced.*

Proof. Let G be an orientably balanced graph and D an orientation of G. We show, in fact, that if f is a γ-labeling of D and H is a subdivision of G, then there exists an orientation F of H and a γ-labeling g of F such that $\mathrm{val}(g : F) = \mathrm{val}(f : D)$, which implies that if f is a balanced γ-labeling of D, then g is a balanced γ-labeling of F. It suffices to verify this statement for an elementary subdivision of G. Suppose that H is the subdivision of G obtained by replacing the edge uv of G by the two edges uw and wv. Let D be an orientation of G with $(u, v) \in E(D)$ and let f be a γ-labeling of D. Define an orientation F of H such that $(u, w), (w, v) \in E(F)$ and $(x, y) \in E(F)$ if $(x, y) \in E(D)$ and $(x, y) \neq (u, v)$. Suppose that the size of G is m. Then the size of H is $m+1$. Define a γ-labeling g of F by $g(x) = f(x)$ if $x \neq w$ and $g(w) = m + 1$. Then

$$\begin{aligned}
\mathrm{val}(g : F) &= \sum_{e\in E(F)} g'(e) = \Bigg(\sum_{e\in E(F)-\{(u,w),(w,v)\}} g'(e) \Bigg) + g'((u, w)) + g'((w, v)) \\
&= \Bigg(\sum_{e\in E(D)-\{(u,v)\}} f'(e) \Bigg) + [g(v) - g(u)] \\
&= \Bigg(\sum_{e\in E(D)-\{(u,v)\}} f'(e) \Bigg) + [f(v) - f(u)] \\
&= \sum_{e\in E(D)} f'(e) = \mathrm{val}(f : D),
\end{aligned}$$

producing the desired result. □

By Proposition 3.4, all the graphs shown in Figure 7 are orientably balanced.

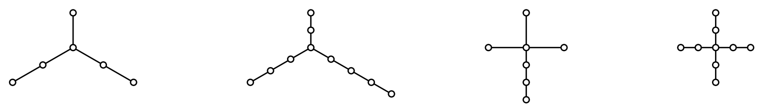

Figure 7: Orientably balanced graphs that are subdivisions of $K_{1,3}$ and $K_{1,4}$

We have already noted that the graph $K_2 = K_{1,1}$ is not orientably balanced. Actually, the stars $K_{1,5}$ and $K_{1,9}$ also fail to be orientably balanced, as is the case with the trees T_1 and T_2 in Figure 8.

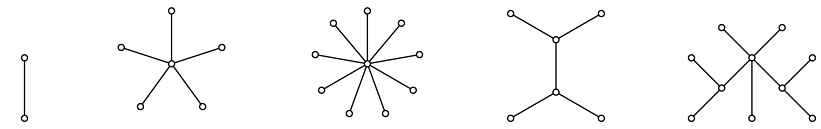

Figure 8: Trees that are not orientably balanced

Every vertex of the tree $K_{1,3}$ has odd degree and $K_{1,3}$ is orientably balanced. On the other hand, every vertex in each tree shown in Figure 8 also has odd degree and we have stated that none of these trees are orientably balanced. We now verify this statement by showing that there are many trees that fail to be orientably balanced.

Proposition 3.5. *Let T be a tree of order $n \geqslant 2$, every vertex of which has odd degree. Then T is orientably balanced if and only if $n \equiv 0 \pmod 4$.*

P r o o f. Since every vertex of T has odd degree, n is even. Thus $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. Assume that $n \equiv 2 \pmod 4$. We show that T is not orientably balanced. Let D be any orientation of T and let f be a γ-labeling of D. Then

$$\operatorname{val}(f) = \sum_{(u,v)\in E(D)} (f(v) - f(u)).$$

Observe that $\operatorname{val}(f)$ is even if and only if

$$\sum_{uv\in E(T)} (f(v) + f(u)) = \sum_{x\in V(T)} (\deg_T x) f(x)$$

is even. Since T has order $4k+2$ for some nonnegative integer k, it follows that $2k+1$ vertices of T are assigned an even label and $2k+1$ vertices of T are assigned an odd label. Because every vertex of T has odd degree,

$$\sum_{x\in V(T)} (\deg_T(x)) f(x)$$

is odd. Thus, $\operatorname{val}(f)$ is odd and so T is not orientably balanced.

Conversely, assume that $n \equiv 0 \pmod 4$. We show that T is orientably balanced. Then $n = 4k$ for some positive integer k. We proceed by induction on k. If $k = 1$, then $T = K_{1,3}$ and so T is orientably balanced by Proposition 3.3. Assume that every tree of order $4k$, each of whose vertices has odd degree, is orientably balanced. Now suppose that T is a tree of order $4k+4$ such that every vertex of T has odd degree. If T is a star, then it follows by Proposition 3.3 that T is orientably balanced. Thus, we may assume that T is not a star and so $\operatorname{diam}(T) \geqslant 3$.

Let u and v be two vertices of T such that $d(u,v) = \operatorname{diam}(T)$. Thus u and v are end-vertices of T. Suppose that u is adjacent to x in T and v is adjacent to y in T. Then $x \neq y$, $\deg_T x \geqslant 3$, and $\deg_T y \geqslant 3$. Let $u' \in N(x) - \{u\}$ and $v' \in N(y) - \{v\}$ be end-vertices of T. Furthermore, let $T' = T - \{u, u', v, v'\}$. Then T' is a tree of order $4k$ and every vertex of T' has odd degree. By the induction hypothesis, T' is orientably balanced. Let D' be a balanced orientation of T' and let $f'\colon V(D') \to \{0, 1, 2, \ldots, 4k-1\}$ be a balanced γ-labeling of D'. Define an orientation D of T with

$$E(D) = E(D') \cup \{(u,x), (x,u'), (v,y), (y,v')\}.$$

Now define a γ-labeling f of D by

$$f(w) = \begin{cases} 4k & \text{if } w = u, \\ 4k+1 & \text{if } w = u', \\ 4k+3 & \text{if } w = v, \\ 4k+2 & \text{if } w = v', \\ f'(w) & \text{if } w \in V(D) - \{u, u', v, v'\}. \end{cases}$$

By Lemma 1.1 and Observation 2.1, it follows that

$$\operatorname{val}(f : D) = \operatorname{val}(f' : D') + [(4k+1) - (4k)] + [(4k+2) - (4k+3)] = 0.$$

Therefore, D is a balanced orientation of T and so T is orientably balanced. $\square$

As we have just seen, if T is a tree of order n, where $n \equiv 2 \pmod 4$ and every vertex of T has odd degree, then there exists no orientation of T for which there is a γ-labeling f of D with $\operatorname{val}(f) = 0$. On the other hand, there is a related observation we can make.

Proposition 3.6. *If T is a tree of order n, where $n \equiv 2 \pmod 4$ and every vertex of T has odd degree, then there is an orientation D of T and a γ-labeling f of D such that* $\operatorname{val}(f : D) = 1$.

Proof. Let $n = 4k+2$ for some nonnegative integer k. Since the result holds for $n = 2$, we may assume that $k \geqslant 1$. Let u and v be two vertices of T such that $d(u,v) = \operatorname{diam}(T)$. Necessarily, u and v are end-vertices of T. Suppose that u is adjacent to x in T. Then $\deg_T x \geqslant 3$. Then there exists an end-vertex $u' \in N(x) - \{u\}$ of T. Then $T' = T - \{u, u'\}$ is a tree of order $4k$ and every vertex of T' has odd degree. By Proposition 3.5, T' is orientably balanced and so there is a balanced orientation D' of T'. Suppose that $f'\colon V(D') \to \{0, 1, 2, \ldots, 4k-1\}$ be a balanced γ-labeling of D'. Define an orientation D of T with

$$E(D) = E(D') \cup \{(u, x), (x, u')\}.$$

Now define a γ-labeling f of D by

$$f(w) = \begin{cases} n-2 = 4k & \text{if } w = u, \\ n-1 = 4k+1 & \text{if } w = u', \\ f'(w) & \text{if } w \in V(D) - \{u, u'\}. \end{cases}$$

By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(n-1) - (n-2)] = 1,$$

as desired. □

The following is an immediate consequence of (1) and Proposition 3.6.

Corollary 3.7. *If T is a tree of order n, where $n \equiv 2 \pmod 4$ and every vertex of T has odd degree, then there is an orientation D of T and a γ-labeling f of D such that* $\mathrm{val}(f : D) = -1$.

We are now prepared to present a characterization of all trees that are orientably balanced.

Theorem 3.8. *A tree T of order $n \geqslant 2$ is orientably balanced if and only if either $n \not\equiv 2 \pmod 4$ or T contains a vertex of even degree.*

Proof. By Proposition 3.5, it suffices to show that if T is a tree of order $n \geqslant 2$ for which either $n \not\equiv 2 \pmod 4$ or T contains a vertex of even degree, then T is orientably balanced. If every vertex of T has odd degree and $n \not\equiv 2 \pmod 4$, then $n \equiv 0 \pmod 4$. By Proposition 3.5 T is orientably balanced. Thus, we may assume that at least one vertex of T has even degree.

We proceed by induction on the order n of T. It is easy to verify that the result holds for $3 \leqslant n \leqslant 6$. Assume that every tree of order n with $2 \leqslant n < k$ (where $k \geqslant 7$) having at least one vertex of even degree is orientably balanced. Let T be a tree of order k having at least one vertex of even degree. If T is a path or T is a star, then T is orientably balanced by Propositions 3.2 and 3.3. Thus, we may assume that $T \neq P_n$ and $T \neq K_{1,n-1}$.

First, suppose that T has exactly one vertex of degree 3 or more, say $v \in V(T)$ and $\deg_T v = a \geqslant 3$. Then T is a subdivision of the star $K_{1,a}$. If $a+1 \not\equiv 2 \pmod 4$, then

$K_{1,a}$ is orientably balanced by Proposition 3.3 and so T is orientably balanced by Proposition 3.4. Thus, we may assume that $a+1 \equiv 2 \pmod 4$. Then $a = 4k+1 \geqslant 5$ for some positive integer k. Thus T contains at least five end-vertices. Let u be an end-vertex of T such that $d(u,v)$ is the maximum among all end-vertices of T. Since T is not a star, $d(u,v) \geqslant 2$. Let x_1, x_2, y_1, y_2 be end-vertices of T distinct from u, let Q_1 be the $x_1 - x_2$ path in T and Q_2 the $y_1 - y_2$ path in T. Let $\vec{Q}_1$ and $\vec{Q}_2$ be orientations of Q_1 and Q_2, respectively, such that $\vec{Q}_1$ is a directed $x_1 - x_2$ path and $\vec{Q}_2$ is a directed $y_1 - y_2$ path. Furthermore, let $X = V(Q_1) - \{v\}$ and $Y = V(Q_2) - \{v\}$. Now let $T' = T - (X \cup Y)$. Since T' contains the $v - u$ path of order at least 3, it follows that T' is a tree of order $l \geqslant 3$ having at least one vertex of even degree. By the induction hypothesis, T' is orientably balanced. Let D' be a balanced orientation of T' and let $f'\colon V(D') \to \{0,1,2,\ldots,l-1\}$ be a balanced γ-labeling of D'. Define an orientation D of T with

$$E(D) = E(D') \cup E(\vec{Q}_1) \cup E(\vec{Q}_2).$$

Now define a γ-labeling f of D such that $f(w) = f'(w)$ if $w \in V(D')$ and $f(x_1) = l$, $f(x_2) = l+1$, $f(y_1) = l+3$, and $f(y_2) = l+2$. By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(l+1) - (l)] + [(l+2) - (l+3)] = 0.$$

Thus T is orientably balanced.

Hence, we may assume that T has at least two vertices of degree 3 or more. Among all vertices of degree 3 or more in T, let u and v be two such vertices for which $d(u,v)$ is maximum. Let u_1 and u_2 be end-vertices of T such that the only vertex of degree 3 or more on the $u_1 - u_2$ path Q_u in T is u. Similarly, let v_1 and v_2 be end-vertices of T such that the only vertex of degree 3 or more on the $v_1 - v_2$ path Q_v in T is v. Let

$$S_1 = V(Q_u) - \{u\} \quad \text{and} \quad S_2 = V(Q_v) - \{v\}.$$

Then $|S_1 \cup S_2| \geqslant 4$. Furthermore, if $S_1 \cup S_2 - \{u_1, u_2, v_1, v_2\} \neq \emptyset$, then every vertex in $S_1 \cup S_2 - \{u_1, u_2, v_1, v_2\}$ has degree 2. Let $T' = T - (S_1 \cup S_2)$. Then T' is a tree of order $l = k - |S_1 \cup S_2| \geqslant 2$. Let $\vec{Q}_u$ and $\vec{Q}_v$ be orientations of Q_u and Q_v, respectively, such that $\vec{Q}_u$ is a directed $u_1 - u_2$ path and $\vec{Q}_v$ is a directed $v_1 - v_2$ path. We consider two cases.

Case 1. *T' has a vertex of even degree or T' contains only vertices of odd degree and $l \equiv 0 \pmod 4$.* By the induction hypothesis or by Proposition 3.5, T' is orientably balanced. Let D' be a balanced orientation of T' and let $f'\colon V(D') \to \{0,1,2,\ldots,l-1\}$ be a balanced γ-labeling of D'. Define an orientation D of T with

$$E(D) = E(D') \cup E(\vec{Q}_u) \cup E(\vec{Q}_v).$$

Now define a γ-labeling f of D such that $f(w) = f'(w)$ if $w \in V(D')$ and $f(u_1) = l$, $f(u_2) = l+1$, $f(v_1) = l+3$, and $f(v_2) = l+2$. By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(l+1) - (l)] + [(l+2) - (l+3)] = 0.$$

Case 2. *T' contains only vertices of odd degree and $l \equiv 2 \pmod 4$.* By Proposition 3.6, there is an orientation D' of T' and a γ-labeling $f'\colon V(D') \to \{0, 1, 2, \ldots, l-1\}$ of D' such that $\mathrm{val}(f' : D') = 1$. In this case, $S_1 \cup S_2$ contains at least one vertex of degree 2 and so $|S_1 \cup S_2| \geqslant 5$. Thus the order of T is $n \geqslant l+5$. Again, define an orientation D of T with

$$E(D) = E(D') \cup E(\vec{Q}_u) \cup E(\vec{Q}_v).$$

Now define a γ-labeling f of D such that $f(w) = f'(w)$ if $w \in V(D')$ and $f(u_1) = l$, $f(u_2) = l+1$, $f(v_1) = l+4$, and $f(v_2) = l+2$. By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(l+1) - (l)] + [(l+2) - (l+4)] = 1 + 1 + (-2) = 0.$$

In each case, there is a balanced orientation D of T and so T is orientably balanced. □

Next, we determine all connected orientably balanced graphs. In order to do this, we need an additional information. If we allow ourselves to work with "modified" γ-labelings, then there is a result related to Proposition 3.6 that holds for all nontrivial trees.

Proposition 3.9. *If T is a tree of order $n \geqslant 2$, then there is an orientation D of T and an injective function $f\colon V(D) \to \{0, 1, 2, \ldots, n\}$ with $\mathrm{val}(f : D) = 1$.*

Proof. First, assume that $T = P_n\colon v_1, v_2, \ldots, v_n$. Let D be the orientation of P_n that is a directed $v_1 - v_n$ path and let f be any γ-labeling of D for which $f(v_1) = 0$, $f(v_n) = 1$. By Proposition 2.3(a), $\mathrm{val}(f) = f(v_n) - f(v_1) = 1$.

Next, assume that $T \neq P_n$. Then T contains a vertex of degree 3 or more. Let w be a vertex of degree 3 or more lying on a $u - v$ path Q in T such that u and v are end-vertices of T and for which every vertex in $V(Q) - \{w\}$ has degree at most 2. Let $T' = T - (V(Q) - \{w\})$. Then T' is a tree of order $l \geqslant 2$. Let $\vec{Q}$ be an orientation of Q that is a directed $u - v$ path. We consider two cases.

Case 1. *T' is orientably balanced.* Let D' be a balanced orientation of T' and let $f'\colon V(D') \to \{0, 1, 2, \ldots, l-1\}$ be a balanced γ-labeling of D'. Define an orientation D of T by

$$E(D) = E(D') \cup E(\vec{Q}).$$

Now define a γ-labeling f of D such that $f(x) = f'(x)$ if $x \in V(D')$, $f(u) = l$, and $f(v) = l + 1$. By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(l + 1) - (l)] = 0 + 1 = 1.$$

C a s e 2. *T' is not orientably balanced.* By Theorem 3.8 and Corollary 3.7, there is an orientation D' of T' and a γ-labeling $f'\colon V(D') \to \{0, 1, 2, \dots, l-1\}$ of D' such that $\mathrm{val}(f' : D) = -1$. Again, define an orientation D of T with

$$E(D) = E(D') \cup E(\vec{Q}).$$

Since $n \geqslant l + 2$, we can define an injective function $f\colon V(D) \to \{0, 1, 2, \dots, n\}$ such that $f(x) = f'(x)$ if $x \in V(D')$, $f(u) = l$, and $f(v) = l + 2$. Then

$$\mathrm{val}(f : D) = \mathrm{val}(f' : D') + [(l + 2) - (l)] = -1 + 2 = 1,$$

as desired. □

Again, the following is an immediate consequence of (1) and Proposition 3.9.

Corollary 3.10. *If T is a tree of order $n \geqslant 2$, then there is an orientation D of T and an injective function $f\colon V(D) \to \{0, 1, 2, \dots, n\}$ with $\mathrm{val}(f : D) = -1$.*

With the aid of Theorem 3.8, Proposition 3.9, and Corollary 3.10, we can now determine all connected orientably balanced graphs.

Theorem 3.11. *A connected graph G of order $n \geqslant 2$ is orientably balanced unless G is a tree, $n \equiv 2 \pmod 4$, and every vertex of G has odd degree.*

P r o o f. By Theorem 3.8, we may assume that G is connected graph of order $n \geqslant 2$ that is not a tree. Since G is not a tree, $G_0 = G$ contains at least one cycle. Let C_1 be a cycle in G and let $G_1 = G_0 - E(C_1)$. If G_1 is a forest, then let $F = G_1$. If G_1 is not a forest, then G_1 contains a cycle C_2. Let $G_2 = G_1 - E(C_2)$. If G_2 is a forest, then let $F = G_2$. Otherwise, we continue this procedure until a spanning forest $F = G_{k-1} - E(C_k)$ of G is produced, where $k \geqslant 1$. If all components of F are trivial, then G is Eulerian and so G is orientably balanced by Proposition 3.1. Hence we may assume that F has $p \geqslant 1$ nontrivial components $T_1, T_2, \dots, T_p$, where each tree T_i has order $n_i \geqslant 2$ for $1 \leqslant i \leqslant p$. Then the order of G is

$$n \geqslant n_1 + n_2 + \ldots + n_p.$$

Since G is not a tree, the size m of G is at least n. For each cycle C_j removed from G, where $1 \leqslant j \leqslant k$, let $\vec{C}_j$ be an orientation of C_j that is a directed cycle. We consider two cases.

Case 1. *Every tree T_i $(1 \leqslant i \leqslant p)$ is orientably balanced.* For each integer i with $1 \leqslant i \leqslant p$, let D_i be a balanced orientation of T_i and let $f_i\colon V(D_i) \to \{0, 1, 2, \ldots, n_i - 1\}$ be a balanced γ-labeling of D_i. Define an orientation D of G with

$$E(D) = \Big(\bigcup_{i=1}^{p} E(D_i)\Big) \cup \Big(\bigcup_{j=1}^{k} E(\vec{C}_j)\Big). \tag{3}$$

Now define a γ-labeling $g\colon V(D) \to \{0, 1, 2, \ldots, m\}$ of D such that

$$g(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1), \\ f_2(v) + n_1 & \text{if } v \in V(T_2), \\ f_3(v) + n_1 + n_2 & \text{if } v \in V(T_3), \\ \quad\vdots & \qquad\vdots \\ f_p(v) + n_1 + n_2 + \ldots + n_{p-1} & \text{if } v \in V(T_p). \end{cases} \tag{4}$$

Observe that the restriction g_1 of g to D_1 is f_1 and, for $2 \leqslant i \leqslant p$, the restriction g_i of g to D_i defined by $g_i(v) = f_i(v) + n_1 + n_2 + \ldots + n_{i-1}$ for $v \in V(D_i)$. Thus $\mathrm{val}(g_i : D_i) = \mathrm{val}(f_i : D_i) = 0$ for $1 \leqslant i \leqslant p$. Furthermore, each directed cycle $\vec{C}_j$ is Eulerian for $1 \leqslant j \leqslant k$ and so the restriction g'_j of g to each $\vec{C}_j$ is balanced. By Lemma 1.1,

$$\mathrm{val}(f, D) = \Big(\sum_{i=1}^{p} \mathrm{val}(f_i : D_i)\Big) + \Big(\sum_{j=1}^{k} \mathrm{val}(g'_j : \vec{C}_j)\Big) = 0.$$

Case 2. *Some tree T_i $(1 \leqslant i \leqslant p)$ is not orientably balanced.* Assume, without loss of generality, that $T_1, T_2, \ldots, T_a$ are not orientably balanced, where $1 \leqslant a \leqslant p$, and $T_{a+1}, T_{a+2}, \ldots, T_p$ are orientably balanced if $a < p$. By Theorem 3.8, for each integer i with $1 \leqslant i \leqslant a$, T_i is a tree of order n_i where $n_i \equiv 2 \pmod 4$ and every vertex of T_i has odd degree. We consider two subcases, according to whether a is even or a is odd.

Subcase 2.1. *a is even.* Then $a = 2b$ for some positive integer b. For each integer i with $1 \leqslant i \leqslant p$, let D_i be an orientation of T_i and let $f_i\colon V(D_i) \to \{0, 1, 2, \ldots, n_i - 1\}$ be a γ-labeling of D_i such that (i) $\mathrm{val}(f_i : D_i) = 1$ for $1 \leqslant i \leqslant b$, (ii) $\mathrm{val}(f_i : D_i) = -1$ for $b + 1 \leqslant i \leqslant a$, and (iii) $\mathrm{val}(f_i : D_i) = 0$ for any integers i with $a+1 \leqslant i \leqslant p$. Define the orientation D of G as described in (3) and a γ-labeling

g of D that satisfies (4). An argument similar to the one used in Case 1 shows that g is a balanced γ-labeling of D.

Subcase 2.2. *a is odd.* Then $a = 2b+1$ for some nonnegative integer b. Assume, without loss of generality, that $n_1 \geqslant n_i$ for every integer i with $1 \leqslant i \leqslant a$, that is, T_1 has the largest order among all trees T_i for $1 \leqslant i \leqslant a$. We consider two subcases.

Subcase 2.2.1. $n_1 \geqslant 3$. Thus $n_1 \geqslant 6$ and T_1 contains at least one vertex of odd degree 3 or more by Theorem 3.8. Let u be a vertex of degree 3 or more such that there exists a $u_1 - u_2$ path Q_u in T_1 for which u_1 and u_2 are end-vertices of T_1 and u is the only vertex on Q_u whose degree is 3 or more. Since T_1 contains no vertex of even degree, Q_u is a path of length 2. Let $\vec{Q}_u$ be an orientation of Q_u that is a $u_1 - u_2$ directed path. Let $T' = T_1 - (V(Q_u) - \{u\}) = T_1 - \{u_1, u_2\}$. Then T' is a tree of order $l = n_1 - 2$. By Proposition 3.9, there is an orientation D' of T' and an injective function $f'\colon V(D') \to \{0, 1, 2, \ldots, l\}$ with $\mathrm{val}(f' : D') = -1$. Define an orientation D_1 of T_1 with

$$E(D_1) = E(D') \cup E(\vec{Q}_u).$$

Since $n_1 = l + 2$, we can define an injective function $f_1\colon V(D_1) \to \{0, 1, \ldots, l+2\}$ such that $f_1(w) = f'(w)$ if $w \in V(D')$ and $f_1(u_1) = l + 1$, and $f(u_2) = l + 2$. By Lemma 1.1 and Observation 2.1, it follows that

$$\mathrm{val}(f_1 : D_1) = \mathrm{val}(f' : D') + [(l+2) - (l+1)] = (-1) + 1 = 0.$$

For each integer i with $2 \leqslant i \leqslant p$, let D_i be an orientation of T_i and let $f_i\colon V(D_i) \to \{0, 1, 2, \ldots, n_i - 1\}$ be a γ-labeling of D_i such that (i) $\mathrm{val}(f_i : D_i) = 1$ for $2 \leqslant i \leqslant b+1$, (ii) $\mathrm{val}(f_i : D_i) = -1$ for $b + 2 \leqslant i \leqslant a$, and (iii) $\mathrm{val}(f_i : D_i) = 0$ for $a + 1 \leqslant i \leqslant p$. Define an orientation D of G as described in (3) and a γ-labeling g of D such that

$$g(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1), \\ f_2(v) + n_1 + 1 & \text{if } v \in V(T_2), \\ f_3(v) + n_1 + n_2 + 1 & \text{if } v \in V(T_3), \\ \quad\vdots & \quad\vdots \\ f_p(v) + n_1 + n_2 + \ldots + n_{p-1} + 1 & \text{if } v \in V(T_p). \end{cases}$$

An argument similar to the one used in Case 1 shows that g is a balanced γ-labeling of D.

Subcase 2.2.2. $n_1 = 2$. Thus $T_i = K_2$ for all integers i with $1 \leqslant i \leqslant a$. Suppose first that $a \geqslant 3$. For each integer i with $1 \leqslant i \leqslant 3$, let $V(T_i) = \{u_i, v_i\}$ and let D_i be a directed $u_i - v_i$ path whose single arc is (u_i, v_i). Define an injective function $f'\colon V(D_1)\cup V(D_2)\cup V(D_3)\to\{0,1,\dots,6\}$ by $f'(u_1)=0$, $f'(v_1)=1$, $f'(u_2)=2$, $f'(v_2)=3$, $f'(u_3)=6$, and $f'(v_3)=4$. Let $D'=D_1\cup D_2\cup D_3$. Then $\mathrm{val}(f':D')=0$. For each integer i with $4\leqslant i\leqslant p$, let D_i be an orientation of T_i and let $f_i\colon V(D_i)\to\{0,1,2,\dots,n_i-1\}$ be a γ-labeling of D_i such that (i) $\mathrm{val}(f_i:D_i)=1$ for $4\leqslant i\leqslant b+2$, (ii) $\mathrm{val}(f_i:D_i)=-1$ for $b+3\leqslant i\leqslant a$, and (iii) $\mathrm{val}(f_i:D_i)=0$ for $a+1\leqslant i\leqslant p$. Define an orientation D of G as described in (3) and a γ-labeling g of D such that

$$g(v)=\begin{cases} f'(v) & \text{if } v\in V(D'),\\ f_4(v)+7 & \text{if } v\in V(T_4),\\ f_5(v)+n_4+7 & \text{if } v\in V(T_5),\\ \quad\vdots & \qquad\vdots\\ f_p(v)+n_4+n_5+\ldots+n_{p-1}+7 & \text{if } v\in V(T_p).\end{cases}$$

An argument similar to the one used in Case 1 shows that g is a balanced γ-labeling of D.

Hence we may assume that $a=1$. Let $T_1\colon u,v$ and let D_1 be a $u-v$ directed path whose single arc is (u,v). We consider two subcases.

Subcase 2.2.2.1. $p\geqslant 2$. Define f_1 on $V(D_1)$ by $f(u)=1$ and $f(v)=0$. Thus $\mathrm{val}(f_1:D_1)=-1$. Let D_2 be an orientation of T_2 and let $f_2\colon V(D_2)\to\{0,1,2,\dots,n_2\}$ be an injective function such that $\mathrm{val}(f_2:D_2)=1$. For each integer i with $3\leqslant i\leqslant p$, let D_i be a balanced orientation of T_i and let $f_i\colon V(D_i)\to\{0,1,2,\dots,n_i-1\}$ be a balanced γ-labeling of D_i. Define an orientation D of G as described in (3) and a γ-labeling g of D such that

$$g(v)=\begin{cases} f_1(v) & \text{if } v\in V(T_1),\\ f_2(v)+2 & \text{if } v\in V(T_2),\\ f_3(v)+n_2+2 & \text{if } v\in V(T_3),\\ \quad\vdots & \qquad\vdots\\ f_p(v)+n_2+\ldots+n_{p-1}+2 & \text{if } v\in V(T_p).\end{cases}$$

Then g is a balanced γ-labeling of D.

Subcase 2.2.2.2. $p=1$. The cycle C_1 may contain both, exactly one of, or neither of u and v but surely C_1 does not contain the edge uv. Furthermore, if C_1 contains both u and v, then the order of C_1 is 4 or more. In any case, C_1 contains at least two vertices of $V(G)-\{u,v\}$. Let $x,y\in V(C_1)-\{u,v\}$.

Consider any injective function $g\colon V(G)\to\{0,1,\dots,n\}$ such that $g(u)=2$, $g(v)=0$, $g(x)=3$, and $g(y)=4$. Now let S_1 be the orientation of C_1 that creates two directed $x - y$ paths. Define an orientation D of G by

$$E(D) = E(D_1) \cup E(S_1) \cup \left(\bigcup_{j=2}^{k} E(\vec{C}_j) \right).$$

Thus g is a γ-labeling on D and

$$\mathrm{val}(g : D) = \mathrm{val}(g : D_1) + \mathrm{val}(g : S_1) + \sum_{j=2}^{k} \mathrm{val}(g : \vec{C}_j) = -2 + 2 + 0 = 0.$$

Hence g is a balanced γ-labeling of D.

This completes the proof. □

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