A NOTE ON CONGRUENCE SYSTEMS OF MS-ALGEBRAS

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Abstract. Let L be an MS-algebra with congruence permutable skeleton. We prove that solving a system of congruences $(\theta_1,\ldots,\theta_n;x_1,\ldots,x_n)$ in L can be reduced to solving the restriction of the system to the skeleton of L, plus solving the restrictions of the system to the intervals $[x_1,\bar{\bar{x}}_1],\ldots,[x_n,\bar{\bar{x}}_n]$.

Keywords: MS-algebra, permutable congruence, congruence system

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Let A be an algebra. We use $\mathrm{Con}(A)$ to denote the congruence lattice of A. We say that $\theta,\delta\in\mathrm{Con}(A)$ *permute* if $\theta\vee\delta=\{(x,y)\in A^2\colon$ there is $z\in A$ such that $(x,z)\in\theta$ and $(z,y)\in\delta\}$. The algebra A is *congruence permutable* (permutable for short) if every pair of congruences in $\mathrm{Con}(A)$ permutes. By a *system* on A we understand a $2n$-tuple $(\theta_1,\ldots,\theta_n;x_1,\ldots,x_n)$, where $\theta_1,\ldots,\theta_n\in\mathrm{Con}(A), x_1,\ldots,x_n\in A$ and $(x_i,x_j)\in\theta_i\vee\theta_j$ for every $1\leqslant i,j\leqslant n$. A *solution* of a system $(\theta_1,\ldots,\theta_n;x_1,\ldots,x_n)$ is an element $x\in A$ such that $(x,x_i)\in\theta_i$ for every $i=1,\ldots,n$. We note that if A is congruence permutable and $\mathrm{Con}(A)$ is distributive, then every system on A has a solution (folklore).

An algebra $\langle L,\wedge,\vee,{}^-,0,1\rangle$ of type $(2,2,1,0,0)$ is an *MS-algebra* if it satisfies the following conditions:

$\langle L,\wedge,\vee,0,1\rangle$ is a bounded distributive lattice

$$\begin{aligned}\overline{(x\wedge y)}&=\bar{x}\vee\bar{y},\\ \overline{(x\vee y)}&=\bar{x}\wedge\bar{y},\\ x&\leqslant\bar{\bar{x}},\\ \bar{1}&=0.\end{aligned}$$

We refer the reader to [2] for the basic properties of MS-algebras. By $\mathcal{MS}$ we denote the class of all MS-algebras. A *de Morgan algebra* is an algebra $L \in \mathcal{MS}$ satisfying the identity $\bar{\bar{x}} = x$. We write $\mathcal{M}$ to denote the class of de Morgan algebras.

Let $L \in \mathcal{M}$. An element $z \in L$ is *central* if $z \vee \bar{z} = 1$. The central elements of L are naturally identified with the factor congruences of L. For $x, y \in L$, let $x \Leftrightarrow y$ denote the greatest central u such that $u \wedge x = u \wedge y$ if such an u exists. Two basic properties of $\Leftrightarrow$ will be used without explicit mention:

$$
\begin{aligned}
&x \Leftrightarrow x = 1,\\
&x \Leftrightarrow y = \bar{x} \Leftrightarrow \bar{y}
\end{aligned}
$$

(the latter one can be checked easily). We remark that for every simple de Morgan algebra the only central elements are 0 and 1 [1], so for these algebras $\Leftrightarrow$ is the equality test. In [3] it is proved that the existence of $x \Leftrightarrow y$ is guaranteed for every $x, y \in L$ provided L is permutable.

Lemma 1 (Gramaglia and Vaggione [3])**.** *Let $L \in \mathcal{M}$. Then following conditions are equivalent:*
(1) *L is congruence permutable.*
(2) *$x \Leftrightarrow y$ exists for every $x, y \in L$, and $(x \Leftrightarrow 0) \vee (x \Leftrightarrow 1) \vee (x \Leftrightarrow \bar{x}) = 1$, $\forall x \in L$.*

Lemma 2. *Let $L \in \mathcal{M}$ be congruence permutable. Let $\theta \in \mathrm{Con}(L)$ and $x_1, x_2, y_1, y_2 \in L$ be such that $(x_1, y_1), (x_2, y_2) \in \theta$. Then $(x_1 \Leftrightarrow x_2, y_1 \Leftrightarrow y_2) \in \theta$.*

P r o o f. Let θ be a maximal element of $\mathrm{Con}(L)$. We will prove that for $x, y \in L$

$$
(x \Leftrightarrow y)/\theta = \left.\begin{cases} 1/\theta & \text{if } (x, y) \in \theta \\ 0/\theta & \text{if } (x, y) \notin \theta \end{cases}\right\} = x/\theta \Leftrightarrow y/\theta.
$$

Since L/θ is simple, we have $x/\theta \in \{0/\theta, 1/\theta\}$ or $x/\theta = \bar{x}/\theta$ for all $x \in L$ (see [1] for a description of the simple algebras in $\mathcal{M}$). Also, as $(x \Leftrightarrow y)/\theta$ is central, we have $(x \Leftrightarrow y)/\theta \in \{0/\theta, 1/\theta\}$ for all $x, y \in L$. Now, the equality $x \wedge (x \Leftrightarrow y) = y \wedge (x \Leftrightarrow y)$ yields that if $(x, y) \notin \theta$ then $(x \Leftrightarrow y)/\theta$ has to be $0/\theta$. This fact in combination with (2) of Lemma 1 says that for every $x \in L$

$$
\begin{aligned}
&(x \Leftrightarrow 0)/\theta = 1 \Leftrightarrow x/\theta = 0/\theta,\\
&(x \Leftrightarrow 1)/\theta = 1 \Leftrightarrow x/\theta = 1/\theta,\\
&(x \Leftrightarrow \bar{x})/\theta = 1 \Leftrightarrow x/\theta = \bar{x}/\theta.
\end{aligned}
$$

Let $(a, b) \in \theta$; there are three cases:

Case $a/\theta = 0/\theta$. Here we have $(a \Leftrightarrow 0)/\theta = 1/\theta = (b \Leftrightarrow 0)/\theta$, and it is easy to check that $(a \Leftrightarrow 0) \wedge (b \Leftrightarrow 0) \leqslant (a \Leftrightarrow b)$. Thus $(a \Leftrightarrow b)/\theta = 1/\theta$.

Case $a/\theta = 1/\theta$. This case is analogous to the previous one.

Case $a/\theta = \bar{a}/\theta$. Since $(a \Leftrightarrow b) = (a \wedge b \Leftrightarrow a \vee b)$ and $\overline{a \wedge b}/\theta = \overline{a \vee b}/\theta$ we can assume without loss of generality that $a \leqslant b$. Also, as $a/\theta = \bar{a}/\theta$ and $b/\theta = \bar{b}/\theta$, we know that $(a \Leftrightarrow \bar{a})/\theta = 1/\theta = (b \Leftrightarrow \bar{b})/\theta$. Now,

$$\begin{aligned} a \wedge (a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}) &= b \wedge a \wedge (a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}) \\ &= \bar{b} \wedge \bar{a} \wedge (a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}) \\ &= \bar{b} \wedge (a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}) \\ &= b \wedge (a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}). \end{aligned}$$

Hence $(a \Leftrightarrow \bar{a}) \wedge (b \Leftrightarrow \bar{b}) \leqslant (a \Leftrightarrow b)$ and $(a \Leftrightarrow b)/\theta = 1/\theta$.

Finally, since every congruence in a de Morgan algebra is an intersection of maximal congruences, the lemma follows. □

For an MS-algebra L we will write $\mathrm{Sk}(L)$ to denote the *skeleton* of L, that is $\mathrm{Sk}(L) = \{\bar{x} \colon x \in L\}$. It is a well known fact that for $L \in \mathcal{MS}$, $\mathrm{Sk}(L)$ is the greatest subalgebra of L which is a de Morgan algebra. If $L \in \mathcal{MS}$ has a permutable skeleton, then the operation $\Leftrightarrow$ is defined for the elements in $\mathrm{Sk}(L)$. Furthermore, by Lemma 2, the congruences of L are compatible with this operation. We summarize this in

Corollary 3. *Let L be an MS-algebra with congruence permutable skeleton. Let $\theta \in \mathrm{Con}(L)$ and let $x_1, x_2, y_1, y_2 \in \mathrm{Sk}(L)$ be such that $(x_1, y_1), (x_2, y_2) \in \theta$. Then $(x_1 \Leftrightarrow x_2, y_1 \Leftrightarrow y_2) \in \theta$.*

In the next lemma we state a Boolean algebra identity we will need in the proof of our main theorem.

Lemma 4. *Let B be a Boolean algebra, and let $a_1, \dots, a_n \in B$. Then*

$$\bigvee_{U \subseteq \{1,\dots,n\}} \left(\bigwedge_{k \in U} a_k \wedge \bigwedge_{k \in \{1,\dots,n\} - U} \bar{a}_k \right) = 1.$$

Let $(\theta_1, \dots, \theta_n; x_1, \dots, x_n)$ be a system on L, and suppose s is a solution for it. Then the systems $(\theta_1, \dots, \theta_n; (x_1 \vee x_k) \wedge \bar{\bar{x}}_k, \dots, (x_n \vee x_k) \wedge \bar{\bar{x}}_k)$, $k = 1, \dots, n$, all have a solution (namely $s_k = (s \vee x_k) \wedge \bar{\bar{x}}_k$). Also, $\bar{s}$ is a solution for $(\theta_1, \dots, \theta_n; \bar{x}_1, \dots, \bar{x}_n)$. We prove in the next theorem that, when $\mathrm{Sk}(L)$ is permutable, the existence of solutions to these new systems is sufficient to find a solution for the original system.

Theorem 5. *Let L be an MS-algebra with congruence permutable skeleton. Take $(\theta_1,\dots,\theta_n;x_1,\dots,x_n)$ to be a system on L, and let $z\in\mathrm{Sk}(L)$ be a solution for $(\theta_1,\dots,\theta_n;\bar{x}_1,\dots,\bar{x}_n)$. Suppose there are $s_1,\dots,s_n\in L$ such that s_k is a solution for $(\theta_1,\dots,\theta_n;(x_1\vee x_k)\wedge\bar{\bar{x}}_k,\dots,(x_n\vee x_k)\wedge\bar{\bar{x}}_k)$, $k=1,\dots,n$. Then*

$$s=\bigvee_{U\subseteq\{1,\dots,n\}}\left(\left(\bigwedge_{k\in U}\bar{x}_k\Leftrightarrow z\right)\wedge\left(\bigwedge_{k\in\{1,\dots,n\}-U}\overline{\bar{x}_k\Leftrightarrow z}\right)\wedge\left(\bigwedge_{k\in U}s_k\right)\right)$$

is a solution for $(\theta_1,\dots,\theta_n;x_1,\dots,x_n)$.

Proof. In order to make this proof easier to read we will use the notation $x\equiv_\theta y$ for equality modulo θ. Let $1\leqslant l\leqslant n$; we will prove that $(s,x_l)\in\theta_l$. For $U\subseteq\{1,\dots,n\}$ define

$$t_U=\left(\bigwedge_{k\in U}\bar{x}_k\Leftrightarrow z\right)\wedge\left(\bigwedge_{k\in\{1,\dots,n\}-U}\overline{\bar{x}_k\Leftrightarrow z}\right)\wedge\left(\bigwedge_{k\in U}s_k\right).$$

Note that if $l\notin U$ then

$$t_U\leqslant\left(\bigwedge_{k\in\{1,\dots,n\}-U}\overline{\bar{x}_k\Leftrightarrow z}\right)\equiv_{\theta_l}\left(\bigwedge_{k\in\{1,\dots,n\}-U}\overline{\bar{x}_k\Leftrightarrow\bar{x}_l}\right)\leqslant\overline{\bar{x}_l\Leftrightarrow\bar{x}_l}=0.$$

Now if $l\in U$ we have

$$\begin{aligned}
&\left(\bigwedge_{k\in U}\bar{x}_k\Leftrightarrow z\right)\wedge\left(\bigwedge_{k\in U}s_k\right)\\
&\qquad\equiv_{\theta_l}\left(\bigwedge_{k\in U}\bar{x}_k\Leftrightarrow\bar{x}_l\right)\wedge\left(\bigwedge_{k\in U}(x_l\vee x_k)\wedge\bar{\bar{x}}_k\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}(\bar{x}_k\Leftrightarrow\bar{x}_l)\wedge(x_l\vee x_k)\wedge\bar{\bar{x}}_k\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}(\bar{x}_k\Leftrightarrow\bar{x}_l)\wedge\bar{\bar{x}}_k\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}(\bar{\bar{x}}_k\Leftrightarrow\bar{\bar{x}}_l)\wedge\bar{\bar{x}}_k\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}(\bar{\bar{x}}_k\Leftrightarrow\bar{\bar{x}}_l)\wedge\bar{\bar{x}}_l\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}\bar{\bar{x}}_k\Leftrightarrow\bar{\bar{x}}_l\right)\\
&\qquad=x_l\wedge\left(\bigwedge_{k\in U-\{l\}}\bar{x}_k\Leftrightarrow\bar{x}_l\right).
\end{aligned}$$

Hence

$$\begin{cases} t_U \equiv_{\theta_l} 0 \text{ for } l \notin U \\ t_U \equiv_{\theta_l} x_l \wedge \left(\bigwedge\limits_{k \in U - \{l\}} (\bar{x}_k \Leftrightarrow \bar{x}_l) \right) \wedge \left(\bigwedge\limits_{k \in \{1,\dots,n\} - U} \overline{\bar{x}_k \Leftrightarrow \bar{x}_l} \right) \text{ for } l \in U. \end{cases}$$

Thus,

$$\begin{aligned} s &\equiv_{\theta_l} \bigvee_{\substack{U \subseteq \{1,\dots,n\} \\ l \in U}} x_l \wedge \left(\bigwedge_{k \in U - \{l\}} (\bar{x}_k \Leftrightarrow \bar{x}_l) \right) \wedge \left(\bigwedge_{k \in \{1,\dots,n\} - U} \overline{\bar{x}_k \Leftrightarrow \bar{x}_l} \right) \\ &= x_l \wedge \left(\bigvee_{\substack{U \subseteq \{1,\dots,n\} \\ l \in U}} \left(\bigwedge_{k \in U - \{l\}} (\bar{x}_k \Leftrightarrow \bar{x}_l) \right) \wedge \left(\bigwedge_{k \in \{1,\dots,n\} - U} \overline{\bar{x}_k \Leftrightarrow \bar{x}_l} \right) \right) \\ &= x_l \wedge 1 \end{aligned}$$

(use Lemma 4 to obtain the last equality). $\square$

For $\theta \in \mathrm{Con}(L)$ and S a subalgebra (or sublattice of L) we will write θ^S to denote the restriction of θ to S, that is $\theta^S = \theta \cap (S \times S)$. Obviously $\theta^S \in \mathrm{Con}(S)$. Let $a, b \in L$ be such that $a \leqslant b$, and let $[a, b] = \{z \in L \colon a \leqslant z \leqslant b\}$. Note that if $(\theta_1, \dots, \theta_n; x_1, \dots, x_n)$ is a system on L, then $(\theta_1^{[a,b]}, \dots, \theta_n^{[a,b]}; (x_1 \vee a) \wedge b, \dots, (x_n \vee a) \wedge b)$ is a system on the lattice $[a, b]$. Also, $(\theta_1^{\mathrm{Sk}(L)}, \dots, \theta_n^{\mathrm{Sk}(L)}; \bar{x}_1, \dots, \bar{x}_n)$ is a system on the de Morgan algebra $\mathrm{Sk}(L)$. Further, note that $(\theta_1, \dots, \theta_n; (x_1 \vee a) \wedge b, \dots, (x_n \vee a) \wedge b)$ has a solution in L iff $(\theta_1^{[a,b]}, \dots, \theta_n^{[a,b]}; (x_1 \vee a) \wedge b, \dots, (x_n \vee a) \wedge b)$ has a solution in $[a, b]$. Also, $(\theta_1, \dots, \theta_n; \bar{x}_1, \dots, \bar{x}_n)$ has a solution in L iff $(\theta_1^{\mathrm{Sk}(L)}, \dots, \theta_n^{\mathrm{Sk}(L)}; \bar{x}_1, \dots, \bar{x}_n)$ has a solution in $\mathrm{Sk}(L)$. In the light of these observations we can restate Theorem 5 in the following manner:

Theorem 6. *Let L be an MS-algebra with congruence permutable skeleton. Take $(\theta_1, \dots, \theta_n; x_1, \dots, x_n)$ to be a system on L, and let z be a solution for $(\theta_1^{\mathrm{Sk}(L)}, \dots, \theta_n^{\mathrm{Sk}(L)}; \bar{x}_1, \dots, \bar{x}_n)$. Suppose there are $s_1, \dots, s_n$ such that s_k is a solution for $(\theta_1^{[x_k, \bar{\bar{x}}_k]}, \dots, \theta_n^{[x_k, \bar{\bar{x}}_k]}; (x_1 \vee x_k) \wedge \bar{\bar{x}}_k, \dots, (x_n \vee x_k) \wedge \bar{\bar{x}}_k)$, $k = 1, \dots, n$. Then*

$$s = \bigvee_{U \subseteq \{1,\dots,n\}} \left(\left(\bigwedge_{k \in U} \bar{x}_k \Leftrightarrow z \right) \wedge \left(\bigwedge_{k \in \{1,\dots,n\} - U} \overline{\bar{x}_k \Leftrightarrow z} \right) \wedge \left(\bigwedge_{k \in U} s_k \right) \right)$$

is a solution for $(\theta_1, \dots, \theta_n; x_1, \dots, x_n)$.

Corollary 7. *Let L be an MS-algebra with congruence permutable skeleton. A system $(\theta_1, \ldots, \theta_n; x_1, \ldots, x_n)$ on L has a solution iff each of the systems*

$$(\theta_1^{[x_k, \bar{\bar{x}}_k]}, \ldots, \theta_n^{[x_k, \bar{\bar{x}}_k]}; (x_1 \vee x_k) \wedge \bar{\bar{x}}_k, \ldots, (x_n \vee x_k) \wedge \bar{\bar{x}}_k), \; k = 1, \ldots, n$$

has a solution.

We conclude our work with an example that shows that the hypothesis of permutability of the skeleton cannot be dropped in Theorems 5 and 6.

Example. Let L be the MS-algebra described in Figure 1. Let $(\theta, \delta; 1, y)$ be the system where θ, δ and y are shown in Figure 2. It is easy to check that $(\theta^{\mathrm{Sk}(L)}, \delta^{\mathrm{Sk}(L)}; \bar{1}, \bar{y})$ has a solution. Also, the intervals $[1, \bar{\bar{1}}]$ and $[y, \bar{\bar{y}}]$ have 1 and 2 elements respectively, thus the restrictions of the system to these intervals clearly have solutions. Finally, note that the system has no solution in L.

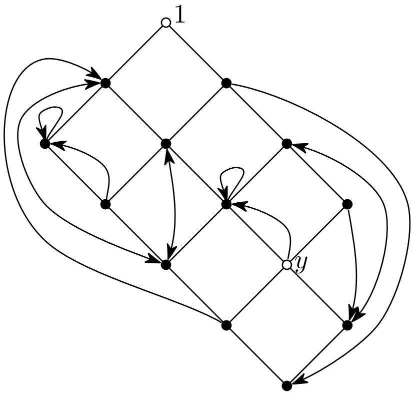

Figure 1. L

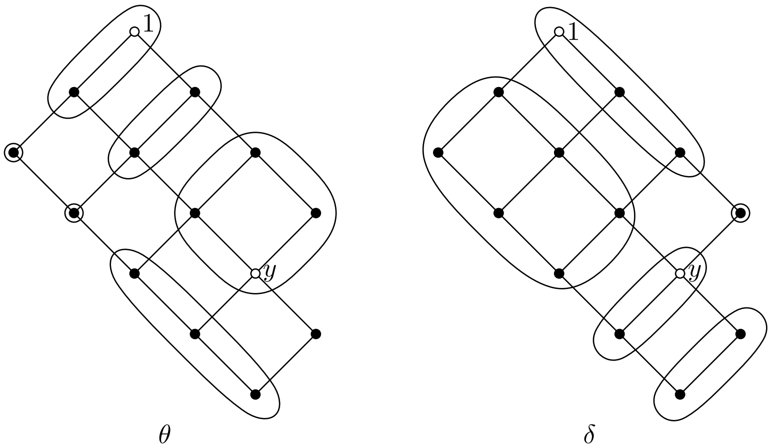

Figure 2.

References

[1] *R. Balbes, P. Dwinger*: Distributive Lattices. University of Missouri Press, Columbia, Missouri, 1974.

[2] *T. S. Blyth, J. C. Varlet*: Ockham Algebras. Oxford University Press, 1994.

[3] *H. Gramaglia, D. Vaggione*: Birkhoff-like sheaf representation for varieties of lattice expansions. Studia Logica *56* (1996), 111–131.

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