ON SYSTEMS OF CONGRUENCES ON PRINCIPAL FILTERS OF ORTHOMODULAR IMPLICATION ALGEBRAS

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Abstract. Orthomodular implication algebras (with or without compatibility condition) are a natural generalization of Abbott's implication algebras, an implication reduct of the classical propositional logic. In the paper deductive systems (= congruence kernels) of such algebras are described by means of their restrictions to principal filters having the structure of orthomodular lattices.

Keywords: orthoimplication algebra, orthomodular lattice, p-filter

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1. Introduction

The classical two-valued propositional logic has its algebraic counterpart in the Boolean algebra. If one considers the logical connective implication of the classical logic only then the clone generated by this connective is not the clone of all Boolean functions. The algebraic counterpart of this case is the so-called implication algebra introduced and treated by Abbott [1]. Similarly, an algebraic counterpart of the fragment of intuitionistic logic containing only the intuitionistic implication and the constant 1 (which serves as a true value) was introduced by Henkin and treated by Diego under the name Hilbert algebra.

In some considerations concerning quantum mechanics another type of logic turned out to be suitable. Algebraic counterparts of these logics are either orthomodular lattices or the so-called ortomodular algebras or certain generalizations of Boolean rings. These logics are related to the Hilbert space logic of quantum mechanics.

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This motivated the study of their implication reducts. The notion of an orthologic was introduced by J. C. Abbott [2] by weakening the axioms and rules of inference of the classical propositional calculus. The resulting Lindenbaum-Tarski algebra generalized the notion of the implication algebra.

An orthologic consists of a set P of propositions closed under a binary operation $\to$ satisfying the axioms

O1: $\vdash p \to (q \to p)$

O2: $\vdash ((p \to q) \to q) \to ((q \to p) \to p)$

and the rules of inference

R1: If p and $p \to q$ then q

R2: If $p \to q$ then $(p \to (q \to r)) \to (p \to r)$

R3: If $p \to q$ then $(p \to r) \to (p \to (q \to r))$

R4: If $p \to q$ and $q \to p$ then $(r \to p) \to (r \to q)$.

It has been shown by Abbott that the operation $\to$ satisfies the axioms

1. $(p \to q) \to p = p$
2. $(p \to q) \to q = (q \to p) \to p$
3. $p \to ((q \to p) \to r) = p \to r$.

Based on these properties, he introduced *orthoimplication algebras* as algebras $(A, \bullet)$ of type (2) satisfying the identities

OI1: $(x \bullet y) \bullet x = x$

OI2: $(x \bullet y) \bullet y = (y \bullet x) \bullet x$

OI3: $x \bullet ((y \bullet x) \bullet z) = x \bullet z$.

Recall that algebras satisfying the axioms OI1, OI2 and the left-distributivity

I3: $x \bullet (y \bullet z) = (x \bullet y) \bullet (x \bullet z)$

are known as implication algebras [1]. The results of Abbott show that the Lindenbaum-Tarski algebra associated with an orthologic is an orthoimplication algebra.

The following lemma is a direct consequence of the axioms of the orthoimplication algebra [2]:

Lemma 1. *If $\mathscr{A} = (A, \bullet)$ is an orthoimplication algebra, then A has a constant 1 satisfying*

(i) $x \bullet x = 1$

(ii) $1 \bullet x = x$

(iii) $x \bullet 1 = 1$

(iv) $x \bullet y = y \bullet x$ *implies* $x = y$

(v) $x \bullet (y \bullet x) = 1$

(vi) $x \bullet y = 1$ *implies* $x \bullet (y \bullet z) = x \bullet z$

(vii) $x \bullet y = 1$ *implies* $(y \bullet z) \bullet (x \bullet z) = 1$.

Hence, every orthoimplication algebra satisfies the identity $x \bullet x = y \bullet y (= 1)$ and the element 1 is an algebraic constant in the variety $\mathscr{O}$ of orthoimplication algebras. Every orthoimplication algebra $(A, \bullet)$ is a poset with respect to a natural relation $\leqslant$ defined by

(1) $$x \leqslant y \text{ iff } x \bullet y = 1.$$

Moreover, orthoimplication algebras are very closely related to orthomodular lattices.

Recall that an *ortholattice* is an algebra $(A, \wedge, \vee, 0, 1, ^\perp)$ of type $(2, 2, 0, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice satisfying the identities
OM1: $x \vee x^\perp = 1, x \wedge x^\perp = 0$
OM2: $(x^\perp)^\perp = x$
OM3: $x \leqslant y$ implies $y^\perp \leqslant x^\perp$.

An element $x^\perp$ is called an orthocomplement of x. Ortholattices satisfying the orthomodular law
OM4: $x \leqslant y$ implies $x \vee (x^\perp \wedge y) = y$
are called *orthomodular lattices* (OML). These are closely related to the logic of quantum mechanics, for details we refer to standard books [3], [10].

The exact connection between orthoimplication algebras and orthomodular lattices is as follows:

Proposition 1. *Let $\mathscr{A} = (A, \bullet)$ be an orthoimplication algebra. Then $(A, \leqslant)$ is a join semilattice and for each $p \in A$ the interval $[p, 1]$ is an orthomodular lattice, where for $x, y \in [p, 1]$ we have*

$$x \vee y = (x \bullet y) \bullet y$$
$$x \wedge y = ((x \bullet p) \vee (y \bullet p)) \bullet p$$

and the orthocomplement $x^p = x \bullet p$.

Moreover, each interval satisfies the compatibility condition

(CC) $$p \leqslant x \leqslant y \text{ implies } y^x = y^p \vee x.$$

Conversely, let $(A, \vee)$ be a join semilattice where for each $p \in A$ the section $[p, 1]$ is an OML satisfying (CC)*. Then the operation $\bullet$ on A defined by*

$$x \bullet y = (x \vee y)^y$$

where $(x \vee y)^y$ is an orthocomplement of $x \vee y$ in the orthomodular lattice $[y, 1]$ determines an orthoimplication algebra.

There is a natural question if there is a similar construction if we do not require a compatibility condition. The affirmative answer to this question leads us to the following notion:

an algebra $\mathscr{A} = (A, \bullet)$ of type (2) is called an *orthomodular implication algebra* (OMIA) if it satisfies the axioms

OMI 1: $(x \bullet y) \bullet x = x$

OMI 2: $(x \bullet y) \bullet y = (y \bullet x) \bullet x$

OMI 3: $(((x \bullet y) \bullet y) \bullet z) \bullet (x \bullet z) = 1$

OMI 4: $((((((x \bullet y) \bullet y) \bullet z) \bullet x) \bullet x) \bullet z) \bullet x) \bullet x = (((x \bullet y) \bullet y) \bullet z) \bullet z$.

It is easy to show that every orthomodular implication algebra becomes a poset with respect to the ordering defined by (1).

Moreover, similarly to Proposition 1, the following description holds:

Proposition 2. *If $\mathscr{A} = (A, \bullet)$ is an OMIA and one defines*

$$x \vee y = (x \bullet y) \bullet y$$

for all $x, y \in A$, then $(A, \vee)$ is a join semilattice and for each $p \in A$ the interval $[p, 1]$ is an orthomodular lattice where for $x, y \in [p, 1]$ we have

$$\begin{aligned} x \wedge y &= ((x \bullet p) \vee (y \bullet p)) \bullet p \\ x^p &= x \bullet p. \end{aligned}$$

Conversely, if $(A, \vee)$ is a join semilattice where for each $p \in A$ the interval $[p, 1]$ is an OML, then the operation $\bullet$ on A defined by

$$x \bullet y = (x \vee y)^y$$

determines an orthomodular implication algebra.

To keep unified terminology, let us call orthoimplication algebras *orthomodular implication algebras with* (CC), briefly OMIA's with (CC).

2. Congruences on OMIA's

The aim of this paper is to describe congruences on OMIA's. We already know that OMIA's, both with or without (CC), are join semilattices whose sections (= principal filters) are orthomodular lattices. It is also well known that in OML's, congruences are completely determined by their *congruence kernels*, i.e. classes of the form $[1]_\theta$. These are in a 1-1 corespondence with the so-called *p-filters.* More precisely, a lattice filter D in an OML $\mathscr{L}$ is a congruence kernel iff

$$x^\perp \vee (i \wedge x) \in D$$

for all $x \in L$ and $i \in D$. For details see the standard books [3], [10].

Hence to describe congruence kernels on OMIA's, we may ask how the congruences behave on sections. One can expect the following:

Lemma 2. *Let $\mathscr{A} = (A, \bullet, 1)$ be an OMIA with* (CC)*, let $\theta \in \operatorname{Con} \mathscr{A}$ and $D = [1]_\theta$. Then for each $p \in A$, $D_p = D \cap [p, 1]$ is a p-filter on an OML $[p, 1]$.*

P r o o f. If $a, b \in D_p$, we have $(a, 1), (b, 1) \in \theta$ and $a \wedge b = ((a \bullet p) \vee (b \bullet p)) \bullet p = (((a \bullet p) \bullet (b \bullet p)) \bullet (b \bullet p)) \bullet p \equiv_\theta ((p \bullet p) \bullet p) \bullet p = 1$, i.e. $a \wedge b \in D_p$.

Further, if $x \in [p, 1]$ and $i \in D_p$, then $(x \wedge i) \vee x^\perp = (x^\perp \vee i^\perp)^\perp \vee x^\perp = (((x \bullet p) \vee (i \bullet p)) \bullet p) \vee (x \bullet p) \equiv_\theta ((x \bullet p) \vee p) \bullet p) \vee (x \bullet p) = ((x \bullet p) \bullet p) \vee (x \bullet p) = x \vee (x \bullet p) = x \vee x^\perp = 1$.

Altogether, D_p is a p-filter on $[p, 1]$. □

There is a more important question, namely whether also the converse statement holds:

Theorem 1. *Let $\mathscr{A} = (A, \bullet, 1)$ be an OMIA with* (CC)*. Let $D \subseteq A$ be a subset where for each $p \in A$ the set $D_p = D \cap [p, 1]$ is a p-filter on a section $[p, 1]$. Then the relation θ_D on A defined by*

$$(x, y) \in \theta_D \iff (x \vee y, y) \in \theta_{D_y} \,\&\, (x \vee y, x) \in \theta_{D_x},$$

where θ_{D_y} or θ_{D_x} is the congruence on $[y, 1]$ or $[x, 1]$ induced by the p-filter D_y or D_x, respectively, is a congruence on $\mathscr{A}$ and $[1]_{\theta_D} = D$.

P r o o f. The relation θ_D is evidently reflexive and symmetric. From the theory of OML's we know that

$$\begin{aligned}(x, y) \in \theta_{D_p} &\text{ iff } (x \wedge y) \vee ((x \bullet p) \wedge (y \bullet p)) \in D_p \\ &\text{ iff } (x \wedge y) \vee ((x \vee y) \bullet p) \in D_p.\end{aligned}$$

We will show that

$$(*)\qquad p \leqslant q \Rightarrow \theta_{D_q} = \theta_{D_p} \cap [q,1]^2.$$

Indeed, let $x, y \in [q,1]$. By (CC) we have $(x \vee y) \bullet q = ((x \vee y) \bullet p) \vee q$, thus

$$\begin{aligned}(x,y) \in \theta_{D_q} &\text{ iff } (x \wedge y) \vee ((x \vee y) \bullet q) \in D_q\\ &\text{ iff } (x \wedge y) \vee ((x \vee y) \bullet p) \vee q \in D_q\\ &\text{ iff } (x \wedge y) \vee ((x \vee y) \bullet p) \in D_q\\ &\text{ iff } (x \wedge y) \vee ((x \vee y) \bullet p) \in D_p\\ &\text{ iff } (x,y) \in \theta_{D_p}.\end{aligned}$$

Let us prove transitivity of θ_D. Assume that $(x \vee y, y) \in \theta_{D_y}$, $(x \vee y, x) \in \theta_{D_x}$, $(y \vee z, z) \in \theta_{D_z}$, $(y \vee z, y) \in \theta_{D_y}$.

Let us show that $\theta_{D_x} \cap \theta_{D_y} = \theta_{D_{x\vee y}}$.

Indeed, since $x, y \leqslant x \vee y$, by $(*)$ we deduce

$$\begin{aligned}\theta_{D_{x\vee y}} &= \theta_{D_x} \cap [x \vee y, 1]^2,\\ \theta_{D_{x\vee y}} &= \theta_{D_y} \cap [x \vee y, 1]^2, \text{ hence also}\\ \theta_{D_{x\vee y}} &= \theta_{D_x} \cap \theta_{D_y} \cap [x \vee y, 1]^2.\end{aligned}$$

But $\theta_{D_x} \cap \theta_{D_y} \subseteq [x,1]^2 \cap [y,1]^2 = [x \vee y, 1]^2$, thus

$$\theta_{D_x} \cap \theta_{D_y} = \theta_{D_{x\vee y}}.$$

Since $x \vee y \in [y,1]$ and $(y \vee z, y) \in \theta_{D_y}$, we have also $(x \vee y \vee z, x \vee y) \in \theta_{D_y}$. Similarly, $x \vee z \in [x,1]$ and $(x \vee y, x) \in \theta_{D_x}$ yield $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$.

Moreover, $x \vee y \vee z, x \vee y \in [x \vee y \vee z, 1]$, hence also

$$(x \vee y \vee z, x \vee y) \in \theta_{D_y} \cap [x \vee y, 1]^2 = \theta_{D_{x\vee y}} = \theta_{D_x} \cap \theta_{D_y},$$

thus $(x \vee y \vee z, x \vee y) \in \theta_{D_x}$. This and $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$ due to transitivity of θ_{D_x} imply $(x \vee y, x \vee z) \in \theta_{D_x}$, and in view of $(x \vee y, x) \in \theta_{D_x}$ we finally get $(x \vee z, x) \in \theta_{D_x}$.

Analogously, $(x \vee y, y) \in \theta_{D_y}$ gives $(x \vee y \vee z, y \vee z) \in \theta_{D_y}$, which due to $x \vee y \vee z, y \vee z \in [y \vee z, 1]$ yields also $(x \vee y \vee z, y \vee z) \in \theta_{D_z}$. Further, $(y \vee z, z) \in \theta_{D_z}$, thus $(x \vee y \vee z, x \vee z) \in \theta_{D_z}$ and hence $(y \vee z, x \vee z) \in \theta_{D_z}$.

Finally, $(x \vee z, z) \in \theta_{D_z}$ and thus θ_D is transitive.

Now let us prove compatibility of θ_D. Assume $(x, y) \in \theta_D$, i.e. $(x \vee y, y) \in \theta_{D_y}$ and $(x \vee y, x) \in \theta_{D_x}$. Since $z \vee x \in [x, 1], z \vee y \in [y, 1]$, we obtain

$$(z \vee x \vee y, z \vee x) \in \theta_{D_x} \text{ and } (z \vee x \vee y, z \vee y) \in \theta_{D_y}.$$

Further, $\theta_{D_{x \vee z}} = \theta_{D_x} \cap [x \vee z, 1]^2$ yields $(x \vee y \vee z, x \vee z) \in \theta_{D_{x \vee z}} = \theta_{D_x} \cap \theta_{D_z}$ and $(x \vee y \vee z, x \vee z) \in \theta_{D_z}$.

Similarly one can prove $(x \vee y \vee z, y \vee z) \in \theta_{D_z}$ which together with the previous property gives $(x \vee z, y \vee z) \in \theta_{D_z}$. Now

$$(x \bullet z, y \bullet z) = ((x \vee z)^z, (y \vee z)^z) \in \theta_{D_z} \text{ and } ((x \bullet z) \vee y \bullet z, y \bullet z) \in \theta_{D_z}.$$

Since $(x \bullet z) \vee (y \bullet z) \geqslant y \bullet z \geqslant z$, we have also

$$((x \bullet z) \vee (y \bullet z), y \bullet z) \in \theta_{D_{y \bullet z}}.$$

Analogously one can prove $((x \bullet z) \vee (y \bullet z), x \bullet z) \in \theta_{D_{x \bullet z}}$ and hence $(x \bullet z, y \bullet z) \in \theta_D$.

Let us prove $(z \bullet x, z \bullet y) \in \theta_D$. We already know that $(x \vee z, y \vee z) = ((x \bullet z) \bullet z, (y \bullet z) \bullet z) \in \theta_D$. We have to prove $((z \vee x)^x, (z \vee y)^y) \in \theta_D$. From $(x, y) \in \theta_D$ we deduce $(x \vee y, x) \in \theta_{D_x}$ and $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$, thus also $((x \vee y \vee z)^x, (x \vee z)^x) \in \theta_{D_x}$. Now $(z \vee x)^x \vee (z \vee x \vee y)^x = (z \vee x)^x$, hence trivially $((z \vee x)^x \vee (z \vee x \vee y)^x, (z \vee x)^x) \in \theta_{D_{(z \vee x)^x}}$.

We know that $\theta_{D_{(x \vee y \vee z)^x}} = \theta_{D_x} \cap [(x \vee y \vee z)^x, 1]^2, (x \vee z)^x, (x \vee y \vee z)^x \in [(x \vee y \vee z)^x, 1]$ and $((x \vee y \vee z)^x, (x \vee z)^x) \in \theta_{D_x}$, thus $((z \vee x)^x, (x \vee y \vee z)^x) \in \theta_{D_{(x \vee y \vee z)^x}}$.

Altogether, we have proved $((x \vee y \vee z)^x, (y \vee z)^x) \in \theta_D$.

Analogously one can prove $((z \vee y)^y, (x \vee y \vee z)^y) \in \theta_D$.

Since $(x \vee y \vee z)^{x \vee y} = (x \vee y \vee z)^y \vee x$ by (CC), we obtain

$$(x \vee y \vee z)^y = (x \vee y \vee z)^y \vee y \equiv_{\theta_D} (x \vee y \vee z)^y \vee x = (x \vee y \vee z)^{x \vee y}.$$

Similarly,

$$(x \vee y \vee z)^x = (x \vee y \vee z)^x \vee x \equiv_{\theta_D} (x \vee y \vee z)^x \vee y = (x \vee y \vee z)^{x \vee y},$$

thus $((x \vee y \vee z)^x, (x \vee y \vee z)^y) \in \theta_D$, which, with respect to $((z \vee y)^y, (x \vee y \vee z)^y) \in \theta_D$ and $((z \vee x)^x, (x \vee y \vee z)^x) \in \theta_D$, gives $((z \vee y)^y, (z \vee x)^x) \in \theta_D$, and we are done. $\square$

Theorem 1 allows us to show that for an OMIA with (CC) $\mathscr{A}$, the lattice $\operatorname{Con} \mathscr{A}$ is relatively pseudocomplemented (and hence distributive):

Corollary 1. *Let $\mathscr{A} = (A, \bullet)$ be an OMIA with* (CC), *let I, J be the congruence kernels of $\mathscr{A}$. Then*

$$\langle I, J\rangle := \{x \in A \mid x \vee i \in J \text{ for each } i \in I\}$$

is the relative pseudocomplement of I with respect to J in the lattice $Ck\mathscr{A}$ of congruence kernels of $\mathscr{A}$.

P r o o f. It is immediate that $I \cap \langle I, J\rangle \subseteq J$ and that for each $K \in Ck\mathscr{A}$, if $I \cap K \subseteq J$, then $K \subseteq \langle I, J\rangle$.

It is enough to show that $\langle I, J\rangle \in Ck(\mathscr{A})$. By the previous theorem it is sufficient to prove that for each $p \in A$, $\langle I, J\rangle_p = \langle I, J\rangle \cap [p, 1]$ is a p-filter on an OML $[p, 1]$.

We already know that both $I_p = I \cap [p, 1]$ and $J_p = J \cap [p, 1]$ are p-filters on $[p, 1]$. From the theory of OML's we also know that for each $p \in A$, $\langle I_p, J_p\rangle = \{x \in [p, 1] \mid x \vee i \in J_p \text{ for each } i \in I_p\}$ is the relative pseudocomplement of I_p with respect to J_p on $[p, 1]$, hence a p-filter. We will show that $\langle I_p, J_p\rangle = \langle I, J\rangle_p$.

Evidently, $\langle I_p, J_p\rangle \subseteq \langle I, J\rangle_p$, i.e. $x \vee y \in J_p$ for each $y \in I_p$. Now for each $i \in I, x \vee i \in I_p$, hence also $x \vee i = x \vee (x \vee i) \in J_p \subseteq J$, $x \in \langle I, J\rangle_p$, completing the proof. □

The situation for OMIA's without (CC) is more complicated.

Theorem 2. *Let $\mathscr{A} = (A, \bullet, 1)$ be an OMIA without* (CC), *let $\theta \in \operatorname{Con}\mathscr{A}, D \in [1]_\theta$. Then for each $p \in A$, $D_p = D \cap [p, 1]$ is a p-filter on an OML $[p, 1]$ and the following implications hold:*

(∗) *for each $x, p, q \in A$:*

$$x \geqslant q \geqslant p \Rightarrow (q \bullet p \in D_p \Rightarrow ((x \bullet q) \wedge (x \bullet p)) \vee (((x \bullet q) \bullet p) \wedge x) \in D_p)$$

(∗∗) *for each $x, y, p, q \in A$:*

$$\begin{aligned} x \geqslant y \geqslant q \geqslant p \Rightarrow ((((x \bullet q) \vee (y \bullet q) \bullet q) \vee ((x \vee y) \bullet q) \in D_q \\ \Leftrightarrow (((x \bullet p) \vee (y \bullet p) \bullet p) \vee ((x \vee y) \bullet p) \in D_p) \end{aligned}$$

P r o o f. It is almost evident that D_p is a p-filter on $[p, 1]$. To prove (∗), assume $q \bullet p \in D_p = [1]_\theta$, i.e. $(1, q \bullet p) \in \theta$. From this we deduce

$$(p, q) = (p, p \vee q) = (1 \bullet p, (q \bullet p) \bullet p) \in \theta$$

and

$$((x \bullet q) \wedge (x \bullet p) \vee (((x \bullet q) \bullet p) \wedge x) \equiv_\theta (x \bullet p) \vee (((x \bullet p) \bullet p) \wedge x) = (x \bullet p) \vee x = 1,$$

hence also $((x \bullet q) \wedge (x \bullet p) \vee (((x \bullet q) \bullet p) \wedge x) \in D_p$.

Let us prove $(**)$. Since D_p is a p-filter on $[p,1]$, it generates a congruence θ_{D_p} on $[p,1]$ by

$$(x,y) \in \theta_{D_p} \Leftrightarrow (((x \bullet p) \vee (y \bullet p)) \bullet p) \vee ((x \vee y) \bullet p) \in D_p.$$

Thus $(**)$ is equivalent to

$$x \geqslant y \geqslant q \geqslant p \Rightarrow ((x,y) \in \theta_{D_p} \ \text{ iff } \ (x,y) \in \theta_{D_q}).$$

But this easily follows from the facts

$$\theta_{D_p} = \theta \cap [p,1] \text{ and } \theta_{D_q} = \theta \cap [q,1].$$

□

We are able to show that also the converse holds:

Theorem 3. *Let $\mathscr{A} = (A, \bullet)$ be an OMIA without* (CC) *and let $D \subseteq A$ be a subset where for each $p \in A$ the set $D_p = D \cap [p,1]$ is a p-filter on an OML $[p,1]$. Let the following conditions be satisfied:*
$(*)$ $\forall x, q \in A:$

$$x \geqslant q \geqslant p \Rightarrow (q^p \in D_p \Rightarrow (x^q \wedge x^p) \vee ((x^q)^p \wedge x) \in D_p)$$

$(**)$ $\forall x, y, p, q \in A:$

$$x \geqslant y \geqslant q \geqslant p \Rightarrow ((x,y) \in \theta_{D_p} \Leftrightarrow (x,y) \in \theta_{D_q}),$$

where θ_{D_p} or θ_{D_q} are the congruences on $[p,1]$ or $[q,1]$ induced by the p-filters D_p or D_q, respectively. Then the relation θ_D on A defined by

$$(x,y) \in \theta_D \Longleftrightarrow (x \vee y, x) \in \theta_{D_x} \,\&\, (x \vee y, y) \in \theta_{D_y}$$

is a congruence on $\mathscr{A}$ with $D = [1]_{\theta_D}$.

P r o o f. The condition $(**)$ immediately yields that for $p \leqslant q$ we have $\theta_{D_q} = \theta_{D_p} \cap [q,1]^2$. The relation θ_D is evidently reflexive and symmetric.

Further, $\theta_{D_{x\vee y}} = \theta_{D_x} \cap [x \vee y, 1]^2$,

$$\theta_{D_{x\vee y}} = \theta_{D_y} \cap [x \vee y, 1]^2,$$

hence $\theta_{D_{x\vee y}} = \theta_{D_x} \cap \theta_{D_y}$.

To prove transitivity of θ_D, assume $(x, y), (y, z) \in \theta_D$, i.e.

$$(x \vee y, y) \in \theta_{D_y}, (x \vee y, x) \in \theta_{D_x}, (y \vee z, y) \in \theta_{D_y}, (y \vee z, z) \in \theta_{D_z}.$$

From $x \vee y \in [y, 1]$ and $(y \vee z, y) \in \theta_{D_y}$ we deduce $(x \vee y \vee z, y \vee x) \in \theta_{D_y}$, similarly $x \vee z \in [x, 1]$ and $(x \vee y, x) \in \theta_{D_x}$ yield $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$.

Since $x \vee y \vee z, x \vee y \in [x \vee y, 1]$, we have also $(x \vee y \vee z, x \vee y) \in \theta_{D_y} \cap [x \vee y, 1]^2 = \theta_{D_{x\vee y}} = \theta_{D_x} \cap \theta_{D_y}$, hence also $(x \vee y \vee z, x \vee y) \in \theta_{D_x}$ and due to transitivity of θ_{D_x}, $(x \vee y, x \vee z) \in \theta_{D_x}$.

This and $(x \vee y, x) \in \theta_{D_x}$ give $(x \vee z, x) \in \theta_{D_x}$.

Analogously we can show that $(x \vee z, z) \in \theta_{D_z}$ verifying that θ_D is transitive.

Assume further that $(x, y) \in \theta_D$ and let us prove that $(x \bullet z, y \bullet z) \in \theta_D$. Using the same arguments as in the proof of transitivity of θ_D we obtain $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$, $(x \vee y \vee z, y \vee z) \in \theta_{D_y}$, thus

$$(x \vee y \vee z, x \vee z) \in \theta_{D_x} \cap [x \vee z, 1]^2 = \theta_{D_{x\vee z}} = \theta_{D_x} \cap \theta_{D_z},$$
$$(x \vee y \vee z, y \vee z) \in \theta_{D_y} \cap [y \vee z, 1]^2 = \theta_{D_{y\vee z}} = \theta_{D_y} \cap \theta_{D_z},$$

hence $(x \vee y \vee z, x \vee z) \in \theta_{D_z}$, $(x \vee y \vee z, y \vee z) \in \theta_{D_z}$ and $(x \vee z, y \vee z) \in \theta_{D_z}$.

Further, due to the compatibility of orthocomplementation in $[z, 1]$ we derive $((x \vee z)^z, (y \vee z)^z) = (x \bullet z, y \bullet z) \in \theta_{D_z}$, and

$$((x \bullet z) \vee (y \bullet z), y \bullet z) \in \theta_{D_z}.$$

Since $(x \bullet z) \vee (y \bullet z) \geqslant y \bullet z \geqslant z$, also

$$((x \bullet z) \vee (y \bullet z), y \bullet z) \in \theta_{D_{y\bullet z}}.$$

Analogously we prove $((x \bullet z) \vee (y \bullet z), y \bullet z) \in \theta_{D_{x\bullet z}}$, and finally

$$(x \bullet z, y \bullet z) \in \theta_D.$$

Let us prove that $(z \bullet x, z \bullet y) \in \theta_D$, i.e. $((z \vee x)^x, (z \vee y)^y) \in \theta_D$.

First, $(x, y) \in \theta_D$ implies $(x \vee z, y \vee z) = ((x \bullet z) \bullet z, (y \bullet z) \bullet z) \in \theta_D$. Further, $(x \vee y, x) \in \theta_{D_x}$ yields $(x \vee y \vee z, x \vee z) \in \theta_{D_x}$ and $((x \vee y \vee z)^x, (x \vee z)^x) \in \theta_{D_x}$.

We have $(z \vee x)^x = (z \vee x)^x \vee (z \vee x \vee y)^x, (x \vee y \vee z)^x \geqslant x$, thus $((z \vee x)^x, (x \vee y \vee z)^x) \in \theta_{D_x}$ gives also $((z \vee x)^x, (x \vee y \vee z)^x) = ((z \vee x)^x \vee (x \vee y \vee z)^x, (x \vee y \vee z)^x) \in \theta_{D_{(x \vee y \vee z)^x}}$.

Further, $((z \vee x)^x \vee (z \vee x \vee y)^x, (z \vee x)^x) = ((z \vee x)^x, (z \vee x)^x) \in \theta_{D_{(x \vee x)^x}}$, proving that $((z \vee x)^x, (x \vee y \vee z)^x) \in \theta_D$.

Analogously we can prove $((x \vee y \vee z)^y, (z \vee y)^y) \in \theta_D$. Hence to prove $((z \vee x)^x, (z \vee y)^y) \in \theta_D$, due to transitivity of θ_D it is enough to show

$$((x \vee y \vee z)^x, (x \vee y \vee z)^y) \in \theta_D.$$

Due to $(x, y) \in \theta_D$, it suffices to show

$$(x \vee y \vee z)^x \vee y \equiv_{\theta_D} (x \vee y \vee z)^y.$$

Let us denote $a = x \vee y \vee z$ and prove

$$a^x \vee x \vee y \equiv_{\theta_D} a^y \vee x \vee y.$$

This is equivalent with

$$(a^x \vee x \vee y, a^x \vee a^y \vee x \vee y) \in \theta_{D_{a^x \vee y \vee x}},$$
$$(a^y \vee x \vee y, a^x \vee a^y \vee x \vee y) \in \theta_{D_{a^y \vee x \vee y}}.$$

Since $a^y \vee x \vee y, a^x \vee x \vee y, a^x \vee a^y \vee x \vee y \geqslant x, y$, we have by $(**)$

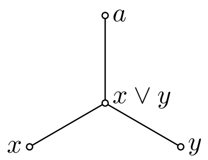
$$(a^x \vee x \vee y, a^x \vee a^y \vee x \vee y) \in \theta_{D_{a^x \vee x \vee y}} \Leftrightarrow (a^x \vee x \vee y, a^x \vee a^y \vee x \vee y) \in \theta_{D_{x \vee y}},$$

and the same when interchanging the elements x, y.

Hence it suffices to show

$$a^x \vee x \vee y \equiv_{\theta_{D_{x \vee y}}} a^y \vee x \vee y.$$

The condition $(*)$ for D_p says that if $x \geqslant q \geqslant p$, then $q^p \in D_p$ (i.e. $(p, q) \in \theta_{D_p}$) implies $(x^q, x^p) \in \theta_{D_p}$. Let us apply this condition to the configuration



Evidently, $(x, x \vee y) \in \theta_{D_x}, (y, x \vee y) \in \theta_{D_y}$, thus by $(*)$

$$(a^x, a^{x \vee y}) \in \theta_{D_x},$$
$$(a^y, a^{x \vee y}) \in \theta_{D_y},$$

and

$$(a^x \vee x \vee y, a^{x \vee y}) \in \theta_{D_x},$$
$$(a^y \vee x \vee y, a^{x \vee y}) \in \theta_{D_y}.$$

Since $a^x \vee x \vee y, a^{x \vee y} \geqslant x \vee y$, by $(**)$ also

$$(a^x \vee x \vee y, a^{x \vee y}) \in \theta_{D_{x \vee y}},$$
$$(a^y \vee x \vee y, a^{x \vee y}) \in \theta_{D_{x \vee y}},$$

which due to transitivity of $\theta_{D_{x \vee y}}$ gives $(a^x \vee x \vee y, a^y \vee x \vee y) \in \theta_{D_{x \vee y}}$, as desired.

This completes the proof of $\theta_D \in \operatorname{Con} \mathscr{A}$. $\square$

Example. The following example shows that the condition $(*)$ in the previous theorem is not superfluous. Let $\mathscr{A} = (A, \bullet, 1)$ be an OMIA with the Hasse diagram in Fig. 1.

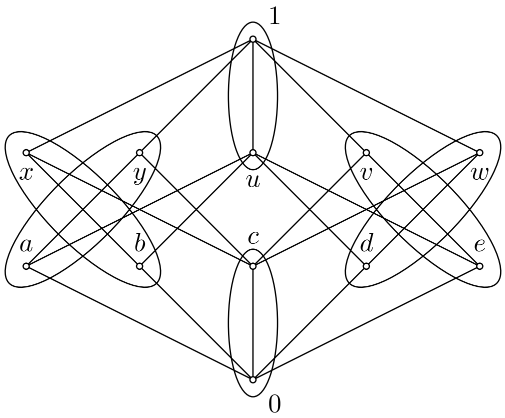


Fig. 1

All sections here are OML's; only two of them, $[0, 1]$ and $[c, 1]$ are not Boolean algebras. Let the involutions on Boolean sections be defined as usual on Boolean algebras, and for the section $[0, 1]$ we have $x^0 = a$, $a^0 = x$, $y^0 = b$, $b^0 = y$, $v^0 = d$, $d^0 = v$, $w^0 = e$, $e^0 = w$, $u^0 = c$, $c^0 = u$.

For the section $[c, 1]$ put

$$x \bullet c = x^c = w, w \bullet c = w^c = x, y \bullet c = y^c = v, v \bullet c = v^c = y.$$

Then the operation $\bullet$ on A is defined by

$$x \bullet y = (x \vee y)^y,$$

and $(A, \bullet, 1)$ is an OMIA (without (CC)).

Consider the equivalence θ on A as visualized in Fig.1. One can easily verify that for $D = [1]_\theta = \{1, u\}$, the filters $D_p = D \cap [p, 1]$ satisfy the condition $(**)$, but not the condition $(*)$: we have $y \geqslant c \geqslant 0, c^0 = c \bullet 0 = u \in D_0$ but

$$(y^c \wedge y^0) \vee ((y^c)^0 \wedge y) = (v \wedge b) \vee (v^0 \wedge y) = 0 \vee (d \wedge y) = 0 \notin D_0.$$

Thus $\theta \notin \mathrm{Con}(\mathscr{A})$.

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