132 (2007) MATHEMATICA BOHEMICA No. 4, 369–387

ON IDEALS OF LATTICE ORDERED MONOIDS

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(Received April 18, 2006)

Abstract. In the paper the notion of an ideal of a lattice ordered monoid A is introduced and relations between ideals of A and congruence relations on A are investigated. Further, it is shown that the set of all ideals of a soft lattice ordered monoid or a negatively ordered monoid partially ordered by inclusion is an algebraic Brouwerian lattice.

Keywords: lattice ordered monoid, ideal, normal ideal, congruence relation, dually residuated lattice ordered monoid

MSC 2000: 06F05

Ideals of lattice ordered groups (notation l-groups) were investigated by Birkhoff. His well known results are that congruence relations on an l-group G and l-ideals of G are in a one-to-one correspondence and that the set of all l-ideals of G is an algebraic Brouwerian lattice [1, p. 304].

Dually residuated lattice ordered semigroups were introduced and studied by Swamy [13] as a common abstraction of Boolean rings and abelian l-groups. Ideals of dually residuated lattice ordered semigroups were investigated by Kovář [8], Hansen [5], Rachůnek [12].

The theory of non-commutative dually residuated lattice ordered semigroups (called DRl-monoids) has been developed by Kovář [8], Kühr [9], [10], [11]. Kühr [10] studied ideals of DRl-monoids and extended the above mentioned Birkhoff's results to DRl-monoids. Ideals of DRl-monoids were also dealt with by Šalounová [14].

The purpose of this paper is to extend the concept of an ideal to any lattice ordered monoid and study relations between ideals of A and congruence relations on A. Further, the set of all ideals of an l-monoid A is investigated. The results

Support of Slovak Vega Grant 1/3081/06 is acknowledged.

obtained show that many assertions concerning ideals of DRl-monoids and l-groups also hold for lattice ordered monoids.

We review some notions and notation used in the paper.

A lattice ordered monoid (notation l-monoid) is a system $(A,+,0,\vee,\wedge)$ such that

(1) $(A,+,0)$ is a monoid,

(2) $(A,\vee,\wedge)$ is a lattice and

(3) $a+(b\vee c)=(a+b)\vee(a+c)$, $(b\vee c)+a=(b+a)\vee(c+a)$,
$a+(b\wedge c)=(a+b)\wedge(a+c)$, $(b\wedge c)+a=(b+a)\wedge(c+a)$ for each a, b, $c\in A$.

The partial order induced by lattice operations $\vee$ and $\wedge$ is denoted by $\leqslant$. Clearly, if $a\leqslant b$, then $c+a\leqslant c+b$ and $a+d\leqslant b+d$ for each $a,b,c,d\in A$.

We shall denote an l-monoid $(A,+,0,\vee,\wedge)$ simply by A or by $(A,+,\leqslant)$ if there is no danger of confusion.

A subset S of a monoid $(M,+,0)$ is called a submonoid of M if $0\in S$ and $a+b\in S$ for each $a,b\in S$. A submonoid S of an l-monoid A is called an l-submonoid of A if S is also a sublattice of A.

If for elements a and b of an l-monoid A there exist a least $x\in A$ such that $b+x\geqslant a$ and a least $y\in A$ such that $y+b\geqslant a$, then the element x is denoted by $a\leftharpoondown b$ and the element y by $a\rightharpoonup b$.

A system $(B,+,0,\vee,\wedge,\leftharpoondown,\rightharpoonup)$ is called a dually residuated lattice ordered monoid (notation DRl-monoid) iff

(1) $(B,+,0,\vee,\wedge)$ is an l-monoid,

(2) for each a, b in B there exist elements $a\leftharpoondown b$ and $a\rightharpoonup b$,

(3) $b+((a\leftharpoondown b)\vee 0)\leqslant a\vee b$, $((a\rightharpoonup b)\vee 0)+b\leqslant a\vee b$ for each a, $b\in B$,

(4) $a\leftharpoondown a\geqslant 0$, $a\rightharpoonup a\geqslant 0$ for each $a\in B$.

An element x of an l-monoid A is called positive (negative) if $x\geqslant 0$ ($x\leqslant 0$, respectively). The set of all positive (negative) elements of an l-monoid A will be denoted by A^+ (A^-, respectively).

If x is an element of an l-monoid A, then $x^+=x\vee 0$ is called the positive part of x and $x^-=x\wedge 0$ is called the negative part of x.

If x is an invertible element of an l-monoid A, then the inverse of x is denoted by $-x$. The set of all invertible elements of an l-monoid A will be denoted by $\mathrm{In}(A)$.

We use $\mathbb{N}$ for the set of all positive integers. Throughout this paper 0 will denote a zero element.

We shall often need the following assertions and we shall apply them without special references.

(A_1) For each element a of an l-monoid, $a=a^++a^-=a^-+a^+$ [8, p. 16].

(A_2) Each negative element of a DRl-monoid is invertible [8, Lemma 1.2.2].

Kühr defined the absolute value of an element x of a DRl-monoid B by $|x|=(x\rightharpoonup 0)\vee(0\rightharpoonup x)$ [10, p. 99].

The following lemma shows that for the absolute value in DRl-monoids the same relation is valid as in lattice ordered groups [1, p. 295].

Lemma 1. *Let B be a DRl-monoid, $x \in B$. Then $|x| = x^+ + (-x^-) = (-x^-) + x^+$.*

P r o o f. By Lemma 4 [10], $|x| = x^+ \rightharpoonup x^-$. In view of Lemma 1 [10] we have $x^+ \rightharpoonup x^- \geqslant x^- \rightharpoonup x^- = 0$, $x^+ = x^+ \vee x^- = (x^+ \rightharpoonup x^-)^+ + x^- = (x^+ \rightharpoonup x^-) + x^-$. Therefore $(-x^-) + x^+ = x^+ + (-x^-) = x^+ \rightharpoonup x^- = |x|$. $\square$

In [10] Kühr defined an ideal of a DRl-monoid B to be a subset I of B satisfying the following conditions:
(I_1) $0 \in I$,
(I_2) if $x, y \in I$, then $x + y \in I$,
(I_3) if $x \in B$, $y \in I$ and $|x| \leqslant |y|$, then $x \in I$.

Theorem 1. *Let B be a DRl-monoid, $I \subseteq B$. Let $0 \in I$ and $u + v \in I$ for each $u, v \in I$. Then the following propositions are equivalent:*
(i) *If $x \in B$, $y \in I$ and $|x| \leqslant |y|$, then $x \in I$.*
(ii) *If $x \in B$, $a, b \in I$ and $x^+ + a^- \leqslant x^- + b^+$, then $x \in I$.*
(iii) *If $x \in B$, $a, b \in I$ and $a^- + x^+ \leqslant b^+ + x^-$, then $x \in I$.*

P r o o f. (i) $\Rightarrow$ (ii) Let $x^+ + a^- \leqslant x^- + b^+$ for some $x \in B$, $a, b \in I$. Since $|b^+| \leqslant |b|$, $|-a^-| \leqslant |a|$, we have $b^+, -a^- \in I$. Hence $b^+ + (-a^-) \in I$. Then $|x| = (-x^-) + x^+ \leqslant b^+ + (-a^-) = |b^+ + (-a^-)|$ yields $x \in I$.

(ii) $\Rightarrow$ (i) Let $|x| \leqslant |y|$ for some $x \in B$, $y \in I$. Then $x^+ + y^- \leqslant x^- + y^+$. This implies $x \in I$.

Analogously we can prove that (i) $\Leftrightarrow$ (iii). $\square$

In view of Theorem 1 we can introduce the following concept of an ideal of an l-monoid A so that if A is a DRl-monoid our definition is equivalent to Kühr's one.

Let A be an l-monoid. A subset I of A is called a left (right) ideal of A, if the conditions (I_1), (I_2) and the following condition (I_3^l) ((I_3^r), respectively) are fulfilled:
(I_3^l) If $x \in A$, $a, b \in I$ and $x^+ + a^- \leqslant x^- + b^+$, then $x \in I$.
(I_3^r) If $x \in A$, $a, b \in I$ and $a^- + x^+ \leqslant b^+ + x^-$, then $x \in I$.

A subset I of an l-monoid A is an ideal of A if I is both a left and a right ideal of A.

Clearly $\{0\}$ and A are ideals of A.

R e m a r k 1. If I is a submonoid of an l-monoid A with the least element u and the greatest element v, then I is a left (right) ideal of A iff for each $x \in A \setminus I$ from $x^+ + u^- \leqslant x^- + v^+$ ($u^- + x^+ \leqslant v^+ + x^-$, resp.) it follows that $x \in I$.

Theorem 2. *Let A be an l-monoid, I an ideal of A.*

(i) *If $x \in A$, $a, b \in I$ and $x^+ + a^- \leqslant b^+ + x^-$, then $x \in I$.*

(ii) *If $x \in A$, $a, b \in I$ and $a^- + x^+ \leqslant x^- + b^+$, then $x \in I$.*

Proof. (i) Let $x \in A$, $a, b \in I$, $x^+ + a^- \leqslant b^+ + x^-$. Then $(x^+)^+ + a^- = x^+ + a^- \leqslant b^+ + x^- \leqslant b^+ = (x^+)^- + b^+$, $a^- + (x^-)^+ = a^- \leqslant x^+ + a^- \leqslant b^+ + x^- = b^+ + (x^-)^-$. This implies $x^+, x^- \in I$. Therefore $x = x^+ + x^- \in I$.

The argument for (ii) is similar. □

Theorem 3. *Let A be an l-monoid, I a left ideal of A, $x \in A$. Then*

(i) *$x \in I$ iff $x^+ \in I$ and $x^- \in I$,*

(ii) *I is a convex subset of A,*

(iii) *I is an l-submonoid of A.*

Proof. (i) Let $x \in I$. Then $(x^+)^+ + x^- \leqslant x^+ = (x^+)^- + x^+$ and $(x^-)^+ + x^- = x^- \leqslant (x^-)^- + x^+$ yield $x^+, x^- \in I$. The converse is obvious.

(ii) Let $x \leqslant z \leqslant y$, x, $y \in I$, $z \in A$. Then $z^+ \leqslant y^+$, $x^- \leqslant z^-$. By (i), y^+, $x^- \in I$. Since $(z^+)^+ + (y^+)^- = z^+ \leqslant y^+ = (z^+)^- + (y^+)^+$, $(z^-)^+ + (x^-)^- = x^- \leqslant z^- = (z^-)^- + (x^-)^+$, we have z^+, $z^- \in I$. Therefore $z \in I$.

(iii) We need only to show that I is a sublattice of A. Let $x, y \in I$. By (i), $x^+ + y^+$, $x^- + y^- \in I$. Then from $x^- + y^- \leqslant x^- \wedge y^- \leqslant x \wedge y \leqslant x \vee y \leqslant x^+ \vee y^+ \leqslant x^+ + y^+$ and the convexity of I we get $x \wedge y$, $x \vee y \in I$. □

Remark 2. An analogous theorem is valid for a right ideal of an l-monoid A.

Corollary 1. *If x is an element of a left ideal I of an l-monoid A, then the interval $[x^-, x^+] \subseteq I$.*

Lemma 2. *Let A be an l-monoid, I a left ideal of A.*

(i) *If $x \in A^-$, $y \in I^+$ and $0 \leqslant x + y$, then $x \in I$.*

(ii) *If $x \in A^+$, $y \in I^-$ and $x + y \leqslant 0$, then $x \in I$.*

Proof. (i) Let $x \in A^-$, $y \in I^+$ and $x + y \leqslant 0$. Then $x^+ + y^- = 0 \leqslant x + y = x^- + y^+$ implies $x \in I$.

The proof of (ii) can be obtained dually. □

Analogously we can prove the following lemma.

Lemma 3. *Let A be an l-monoid, I a right ideal of A.*
(i) *If $x \in A^-$, $y \in I^+$ and $0 \leqslant y + x$, then $x \in I$.*
(ii) *If $x \in A^+$, $y \in I^-$ and $y + x \leqslant 0$, then $x \in I$.*

Recall that the set $\mathrm{In}(A)$ of all invertible elements of an l-monoid A is an l-group and a sublattice of A and $-(x \wedge y) = (-x) \vee (-y)$, $-(x \vee y) = (-x) \wedge (-y)$ for each $x, y \in \mathrm{In}(A)$ (cf. [6, p. 103]).

Hence, if x is an invertible element of an l-monoid A, then x^+ and x^- are invertible elements of A and $-(x^+) = (-x)^-$, $-(x^-) = (-x)^+$.

Lemma 4. *Let A be an l-monoid, I a left ideal of A, $x \in I$. If x is invertible, then $-x \in I$.*

Proof. Let x be an invertible element of a left ideal I. Then $(-x)^+ + x^- = -(x^-) + x^- = 0 = -(x^+) + x^+ = (-x)^- + x^+$. This implies $-x \in I$. □

If ϱ is an equivalence relation on a set S, then we will write $a \ \varrho \ b$ instead of $(a, b) \in \varrho$.

An equivalence relation ϱ on an l-monoid A is called a congruence relation on A iff for each $a, b, c, d \in A$ the following condition is satisfied:

(C_1) If $a \ \varrho \ b$ and $c \ \varrho \ d$, then $(a \wedge c) \ \varrho \ (b \wedge d)$, $(a \vee c) \ \varrho \ (b \vee d)$ and $(a + c) \ \varrho \ (b + d)$.

For $a \in A$ we set $[a]_\varrho = \{x \in A;\ x \ \varrho \ a\}$.

Theorem 4. *Let ϱ be a congruence on an l-monoid A. Then $[0]_\varrho$ is an ideal of A.*

Proof. By Lemma 7 [4, p. 20], $[0]_\varrho$ is a convex sublattice of A. Clearly, $[0]_\varrho$ is a submonoid of A.

Let $z \in A$, $a, b \in [0]_\varrho$, $z^+ + a^- \leqslant z^- + b^+$. Then from $a^- \leqslant z^+ + a^- \leqslant z^- + b^+ \leqslant b^+$ and the convexity of $[0]_\varrho$ it follows that $(z^+ + a^-) \ \varrho \ 0$ and $(z^- + b^+) \ \varrho \ 0$. From $(a^-) \ \varrho \ 0$ and $(b^+) \ \varrho \ 0$ we get $(z^+ + a^-) \ \varrho \ (z^+)$, $(z^- + b^+) \ \varrho \ (z^-)$. Hence $(z^+) \ \varrho \ 0$, $(z^-) \ \varrho \ 0$. Therefore $z \in [0]_\varrho$. Thus $[0]_\varrho$ is a left ideal of A. Similarly we can show that $[0]_\varrho$ is a right ideal of A. □

An ideal I of an l-monoid A is normal iff $x + I = I + x$ for each $x \in A$.

Lemma 5. *An ideal I of an l-monoid A is normal iff $x + I^+ = I^+ + x$ and $x + I^- = I^- + x$ for each $x \in A$.*

Proof. Let I be a normal ideal of A. Let $x \in A$, $a \in I^+$. Then $x + a = b + x$ for some $b \in I$. Then $x + a = (x + a) \vee x = (b + x) \vee x = (b \vee 0) + x = b^+ + x$. Hence $x + I^+ \subseteq I^+ + x$. Analogously $I^+ + x \subseteq x + I^+$. Therefore $x + I^+ = I^+ + x$.

Dually we can show that $x + I^- = I^- + x$.

Let $x+I^+=I^++x$ and $x+I^-=I^-+x$ for each $x\in A$. Then for $z\in A$ and $d\in I$ we get $z+d=z+d^++d^-=g+h+z$ for some $g\in I^+$ and $h\in I^-$. Hence $z+I\subseteq I+z$. Similarly, $I+z\subseteq z+I$. $\square$

Kühr [10, p. 103] defined a normal ideal I of a DRl-monoid B as an ideal of B satisfying the following condition for each $x, y\in B$:

(N_1) $(x\rightharpoonup y)^+\in I$ iff $(x\leftharpoondown y)^+\in I$.

Further, he showed that an ideal I of a DRl-monoid B is normal iff $x+I^+=I^++x$ for each $x\in B$ [10, Proposition 20].

The next lemma shows that in the case of DRl-monoids our definition of a normal ideal coincides with the definition of Kühr.

Lemma 6. *Let I be an ideal of a DRl-monoid B. Then I is a normal ideal iff $x+I^+=I^++x$ for each $x\in B$.*

Proof. Necessity follows from Lemma 5.

Let I be an ideal of B, $x\in B$ and $x+I^+=I^++x$. Let $g\in I^-$. Hence $x+(-g)=h+x$ for some $h\in I^+$. By the Representation Theorem of Kovář [8, p. 25–27], B is the direct product of $\mathrm{In}(B)$ and a DRl-monoid S with the least element 0. Let x_{In}, x_S be components of x in the direct factors $\mathrm{In}(B)$ and S, respectively. Then we have $x+(-g)=x_{\mathrm{In}}+x_S+(-g)=x_{\mathrm{In}}+(-g)+x_S=x_{\mathrm{In}}+(-g)+(-x_{\mathrm{In}})+x$. Let $h'=x_{\mathrm{In}}+(-g)+(-x_{\mathrm{In}})$. Hence $h'\in(\mathrm{In}(B))^+$. Then $(h\wedge h')+x=(h+x)\wedge(h'+x)=x+(-g)$. Since $0\leqslant h\wedge h'\leqslant h$, h', from the convexity of I and $\mathrm{In}(B)$ we obtain $h\wedge h'\in I^+\cap(\mathrm{In}(B))^+$. Then $x+(-g)=(h\wedge h')+x$ implies $(-(h\wedge h'))+x=x+g$. Hence $x+I^-\subseteq I^-+x$. Analogously, $I^-+x\subseteq x+I^-$. Then Lemma 5 implies that I is a normal ideal of B. $\square$

Kühr imposed the condition (C_1) and the condition

(C_2) if $a\,\varrho\,b$ and $c\,\varrho\,d$, then $(a\rightharpoonup c)\,\varrho\,(b\rightharpoonup d)$, $(a\leftharpoondown c)\,\varrho\,(b\leftharpoondown d)$

on an equivalence relation ϱ on a DRl-monoid B to be a congruence relation on B and showed that $[0]_\varrho$ is a normal ideal of B for each congruence relation ϱ on a DRl-monoid B [10, Theorem 28].

The following example shows that $[0]_\varrho$ need not be a normal ideal for any congruence ϱ on an l-monoid A.

Example 1. Let $A=\{0,a,b,c,d\}$. The binary operation $+$ on A is defined by Table 1. The partial order $\leqslant$ on A is defined by the diagram in Figure 1. Then $(A,+,\leqslant)$ is a non-commutative l-monoid and the relation $\varrho=\{(0,0),(a,a),(b,b),(c,c),(d,d),(c,0),(d,b),(a,b),(a,d),(0,c),(b,d),(b,a),(d,a)\}$ is a congruence on the l-monoid A. The ideal $[0]_\varrho=\{0,c\}$ is not a normal ideal, since $d+[0]_\varrho=\{d,a\}\neq\{d\}=[0]_\varrho+d$.

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	a	a
c	c	a	d	c	d
d	d	a	d	a	a

Table 1

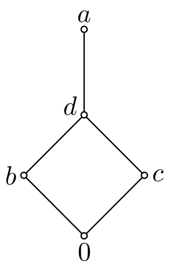

Figure 1

Let A be an l-monoid, I a left ideal of A and J a right ideal of A. We define two binary relations ϱ^1_I and ϱ^2_J on A:

$x\ \varrho^1_I\ y$ iff there exist elements $g_1, h_1 \in I$ such that $x \leqslant g_1 + y$ and $y \leqslant h_1 + x$,
$x\ \varrho^2_J\ y$ iff there exist elements $g_2, h_2 \in J$ such that $x \leqslant y + g_2$ and $y \leqslant x + h_2$

for each $x, y \in A$.

In [10, p. 106] for each ideal I of a DRl-monoid B two binary relation $\Theta_1(I)$ and $\Theta_2(I)$ were defined on B by

$x\Theta_1(I)y$ iff $(x \rightharpoonup y) \vee (y \rightharpoonup x) \in I$,
$x\Theta_2(I)y$ iff $(x \leftharpoondown y) \vee (y \leftharpoondown x) \in I$

for each $x, y \in B$, and it was proved that $\Theta_1(I)$ and $\Theta_2(I)$ are congruence relations on the lattice $(B, \vee, \wedge)$ [10, Theorem 18].

Further, it was shown that if I is a normal ideal of a DRl-monoid B, then $\Theta_1(I) = \Theta_2(I)$, $[0]_{\Theta_1(I)} = I$ and if ϱ is a congruence on B, then $\Theta_1([0]_\varrho) = \varrho$ [10, p. 108–109].

We will show that in the case that I is an ideal of a DRl-monoid B, ϱ^1_I coincides with $\Theta_1(I)$ and ϱ^2_I coincides with $\Theta_2(I)$.

Lemma 7. *Let B be a DRl-monoid.*

(i) *If $x, y \in B$, then $0 \rightharpoonup (x \rightharpoonup y) \leqslant y \rightharpoonup x$, $0 \leftharpoondown (x \leftharpoondown y) \leqslant y \leftharpoondown x$.*
(ii) *If I is an ideal of B, then $\varrho^1_I = \Theta_1(I)$ and $\varrho^2_I = \Theta_2(I)$.*

Proof. (i) Let $x, y \in B$. By Lemma 6(2) [10] and Lemma 1(2) [10], $(y \rightharpoonup x) + (x \rightharpoonup y) \geqslant y \rightharpoonup y = 0$ and $(x \leftharpoondown y) + (y \leftharpoondown x) \geqslant y \leftharpoondown y = 0$. This yields $0 \rightharpoonup (x \rightharpoonup y) \leqslant y \rightharpoonup x$, $0 \leftharpoondown (x \leftharpoondown y) \leqslant y \leftharpoondown x$.

(ii) Let I be an ideal of B, $x, y \in B$, $x\ \varrho^1_I\ y$. Hence $x \leqslant g + y$, $y \leqslant h + x$ for some $g, h \in I$. Thus $x \rightharpoonup y \leqslant g$, $y \rightharpoonup x \leqslant h$. In view of Proposition 8(6) [10] we have $0 \leqslant (x \rightharpoonup y) \vee (y \rightharpoonup x) \leqslant g \vee h$. From the convexity of I we obtain $(x \rightharpoonup y) \vee (y \rightharpoonup x) \in I$. Therefore $x\Theta_1(I)y$.

Let $z, t \in B$, $z\Theta_1(I)t$. Thus $(z \rightharpoonup t) \vee (t \rightharpoonup z) \in I$. In view of (i) we have $|z \rightharpoonup t| = (z \rightharpoonup t) \vee (0 \rightharpoonup (z \rightharpoonup t)) \leqslant (z \rightharpoonup t) \vee (t \rightharpoonup z) = |(z \rightharpoonup t) \vee (t \rightharpoonup z)|$. This yields $z \rightharpoonup t \in I$. Analogously, $t \rightharpoonup z \in I$. Since $z \leqslant (z \rightharpoonup t) + t$, $t \leqslant (t \rightharpoonup z) + z$, we have $z \ \varrho_I^1 \ t$. Hence $\varrho_I^1 = \Theta_1(I)$.

Similarly we can show that $\varrho_I^2 = \Theta_2(I)$. □

Theorem 5. *Let A be an l-monoid, I a left ideal of A and J a right ideal of A. Then ϱ_I^1 and ϱ_J^2 are congruence relations on the lattice $(A, \vee, \wedge)$.*

P r o o f. It is clear that the relation ϱ_I^1 is reflexive and symetric. Let $a, b, c \in A$, $a \ \varrho_I^1 \ b$ and $b \ \varrho_I^1 \ c$. Hence $a \leqslant g + b$, $b \leqslant h + a$, $b \leqslant u + c$, $c \leqslant v + b$ for some $g, h, u, v \in I$. Then $a \leqslant g + u + c$, $c \leqslant v + h + a$. Since $g + u, v + h \in I$, we have $a \ \varrho_I^1 \ c$.

Let $x, y, s, z \in A$, $x \ \varrho_I^1 \ y$ and $s \ \varrho_I^1 \ z$. Hence $x \leqslant g' + y$, $y \leqslant h' + x$, $s \leqslant u' + z$, $z \leqslant v' + s$ for some $g', h', u', v' \in I$. Then $(g' \vee u') + (y \vee z) = (g' + y) \vee (g' + z) \vee (u' + y) \vee (u' + z) \geqslant (g' + y) \vee (u' + z) \geqslant x \vee s$. Analogously, $(h' \vee v') + (x \vee s) \geqslant y \vee z$. Further, we have $(g' \vee u') + (y \wedge z) = [(g' + y) \vee (u' + y)] \wedge [(g' + z) \vee (u' + z)] \geqslant (g' + y) \wedge (u' + z) \geqslant x \wedge s$. Similarly, $(h' \vee v') + (x \wedge s) \geqslant y \wedge z$. Since $g' \vee u', h' \vee v' \in I$, we have $(x \vee s) \ \varrho_I^1 \ (y \vee z)$, $(x \wedge s) \ \varrho_I^1 \ (y \wedge z)$. Therefore ϱ_I^1 is a congruence relation on the lattice $(A, \vee, \wedge)$.

Similarly we can show that ϱ_J^2 is a congruence relation on the lattice $(A, \vee, \wedge)$. □

Let A be an l-monoid and I a left ideal of A. Then $(A, \vee, \wedge)/\varrho_I^1$ is a lattice and for the partial order relation $\leqslant$ of this factor lattice the following assertion is valid.

Theorem 6. *Let A be an l-monoid, I a left ideal of A, $x, y \in A$. Then the following conditions are equivalent:*

(i) $[x]_{\varrho_I^1} \leqslant [y]_{\varrho_I^1}$,

(ii) $x \leqslant g + (x \wedge y)$ *for some* $g \in I$,

(iii) $x \vee y \leqslant h + y$ *for some* $h \in I$.

P r o o f. (i) ⇒ (ii). Let $x, y \in A$, $[x]_{\varrho_I^1} \leqslant [y]_{\varrho_I^1}$. Then $[x \wedge y]_{\varrho_I^1} = [x]_{\varrho_I^1}$. Hence $(x \wedge y) \ \varrho_I^1 \ x$. Therefore $x \leqslant g + (x \wedge y)$ for some $g \in I$.

(ii) ⇒ (iii). Let $x, y \in A$ and $x \leqslant g + (x \wedge y)$ for some $g \in I$. Let $h = g^+$. Then $h \in I$. Since $x \leqslant h + (x \wedge y) \leqslant (h + x) \wedge (h + y)$, we have $x \leqslant h + y$. Clearly, $y \leqslant h + y$. Therefore $x \vee y \leqslant h + y$.

(iii) ⇒ (i). Let $x, y \in A$, $x \vee y \leqslant h + y$ for some $h \in I$. Since $y \leqslant 0 + (x \vee y)$, we conclude that $y \ \varrho_I^1 \ (x \vee y)$. Thus $[y]_{\varrho_I^1} = [x]_{\varrho_I^1} \vee [y]_{\varrho_I^1}$ and hence $[x]_{\varrho_I^1} \leqslant [y]_{\varrho_I^1}$. □

R e m a r k 3. An analogous theorem is valid for a right ideal of an l-monoid.

Lemma 8. *If ϱ is a congruence relation on an l-monoid A, then $\varrho^1_{[0]_\varrho} \subseteq \varrho$, $\varrho^2_{[0]_\varrho} \subseteq \varrho$.*

Proof. Let $x, y \in A$, $x \;\varrho^1_{[0]_\varrho}\; y$. Then $x \leqslant g+y$ and $y \leqslant h+x$ for some $g, h \in [0]_\varrho$. Thus $g \;\varrho\; 0$, $h \;\varrho\; 0$. Then $y \;\varrho\; (g+y)$, $x \;\varrho\; (h+x)$ and hence $(x \wedge y) \;\varrho\; (x \wedge (g+y))$, $(y \wedge x) \;\varrho\; (y \wedge (h+x))$. Thus $(x \wedge y) \;\varrho\; x$, $(x \wedge y) \;\varrho\; y$. This yields $x \;\varrho\; y$.

Analogously we can show that $\varrho^2_{[0]_\varrho} \subseteq \varrho$. □

Unlike in a DRl-monoid, in an l-monoid A the relations $\varrho^1_{[0]_\varrho} = \varrho$, $\varrho^2_{[0]_\varrho} = \varrho$ need not be valid for a congruence relation ϱ on A. For the congruence ϱ from Example 1 we have $\varrho^1_{[0]_\varrho} = \{(0,0), (b,b), (c,c), (d,d), (a,a), (0,c), (c,0), (b,d), (d,b)\} \neq \varrho$.

Lemma 9. *If I is an ideal of A, then $[0]_{\varrho^1_I} \subseteq I$, $[0]_{\varrho^2_I} \subseteq I$.*

Proof. Let I be an ideal of A, $p \in [0]_{\varrho^1_I}$. By Theorems 4 and 5, $[0]_{\varrho^1_I}$ is an ideal of A and hence $(p^+) \;\varrho^1_I\; 0$, $(p^-) \;\varrho^1_I\; 0$. Then $0 \leqslant p^+ \leqslant q$, $0 \leqslant r + p^- \leqslant r$ for some $q, r \in I$. From the convexity of I it follows that $p^+ \in I$. In view of Lemma 3 (i) from $0 \leqslant r + p^-$ we obtain $p^- \in I$. Then $p = p^+ + p^- \in I$. Thus $[0]_{\varrho^1_I} \subseteq I$. Similarly we can show that $[0]_{\varrho^2_I} \subseteq I$. □

For a normal ideal I of an l-monoid the relations $[0]_{\varrho^1_I} = I$, $[0]_{\varrho^2_I} = I$ need not be valid. The set $A = \{0, a, b, c, d, e, f\}$ with the binary operation $+$ on A defined by Table 2 and the partial order $\leqslant$ on A defined by the diagram in Figure 2 is a commutative l-monoid [7, Example 2]. The ideal $I = \{0, a, c, e, \}$ of the l-monoid A is normal, $\varrho^1_I = \varrho^2_I = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (0,e), (e,0), (0,a), (a,0), (e,a), (a,e), (d,f), (f,d)\}$, $[0]_{\varrho^1_I} = [0]_{\varrho^2_I} = \{0, a, e\} \neq I$.

$+$	0	a	b	c	d	e	f
0	0	a	b	c	d	e	f
a	a	a	b	c	f	a	f
b	b	b	b	b	b	b	b
c	c	c	b	c	b	c	b
d	d	f	b	b	d	d	f
e	e	a	b	c	d	e	f
f	f	f	b	b	f	f	f

Table 2

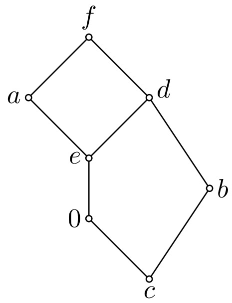

Figure 2

An ideal I of an l-monoid A is called tall iff for each $x \in I$ there exist $b, c \in I$ such that $0 \leqslant x + b$, $0 \leqslant c + x$.

Theorem 7. *Let I be a tall ideal of an l-monoid A. Then $[0]_{\varrho_I^1} = [0]_{\varrho_I^2} = I$.*

P r o o f. Let I be a tall ideal of A, $x \in I$. Then $0 \leqslant b + x$, $0 \leqslant x + c$ for some $b, c \in I$. Since $x \leqslant x^+ + 0$, $x \leqslant 0 + x^+$, we have $x \ \varrho_I^1 \ 0$ and $x \ \varrho_I^2 \ 0$. Hence $I \subseteq [0]_{\varrho_I^1}$, $I \subseteq [0]_{\varrho_I^2}$. Then Lemma 8 completes the proof. $\square$

A soft l-monoid is an l-monoid in which every negative element is invertible.

Every DRl-monoid is a soft l-monoid.

We give a simple construction of a soft l-monoid which is not a DRl-monoid from a nontrivial l-group.

E x a m p l e 2. Let $(G, +, \leqslant)$ be a nontrivial l-group. Let $G^\infty = G \cup \{\infty\}$. Let $\infty \oplus x = x \oplus \infty = \infty$ for each $x \in G^\infty$, $y \oplus z = y + z$ for each y, $z \in G$. Let $x \leqslant' \infty$ for each $x \in G^\infty$, and let $y \leqslant' z$ iff $y \leqslant z$ for each $y, z \in G$. Then $(G^\infty, \oplus, \leqslant')$ is a soft l-monoid, but not a DRl-monoid since $\infty \rightharpoonup \infty$, $\infty \leftharpoondown \infty$ do not exist in G^∞.

An l-monoid A is called a positively (negatively) ordered l-monoid iff each element of A is positive (negative, respectively).

Each positively ordered l-monoid is a soft l-monoid. Each pseudo MV-algebra is a positively ordered l-monoid. For the definition of a pseudo MV-algebra (denoted also as a noncommutative MV-algebra) we refer to [2] or [3].

The l-monoid A from Example 1 is a finite positively ordered l-monoid which is not a DRl-monoid, since $(b \rightharpoonup c) \vee 0 + c = a > d = b \vee c$.

The l-monoids $(B, \oplus, \leqslant_1)$ and $(G^\infty, \oplus, \leqslant_1)$ from Examples in [6, p. 106–107] are infinite positively ordered l-monoids, but not DRl-monoids.

E x a m p l e 3. Let $(G, \oplus, \leqslant')$ be an l-group, $(H, +, \leqslant)$ a positively ordered l-monoid which is not a DRl-monoid. Then the direct product of $(G, \oplus, \leqslant')$ and $(H, +, \leqslant)$ is a soft l-monoid, but not a DRl-monoid.

Lemma 10. *Each ideal of a soft l-monoid A is a tall ideal.*

P r o o f. Let I be an ideal of A, $x \in I$. By Theorem 3 and Lemma 4, $-x^- \in I$. Since $x + (-x^-) = (-x^-) + x = x^+ \geqslant 0$, I is a tall ideal of A. $\square$

From Theorem 7 and Lemma 10 we obtain the following corollary.

Corollary 2. *If I is an ideal of a soft l-monoid A, then $[0]_{\varrho_I^1} = [0]_{\varrho_I^2} = I$.*

Theorem 8. *Let A be an l-monoid and I a normal ideal of A. Then*

(i) $\varrho_I^1 = \varrho_I^2$ *(in the sequel it will be denoted by ϱ_I),*

(ii) ϱ_I *is a congruence relation on the l-monoid A.*

Proof. (i) The proof is obvious.

(ii) Let $x, y, s, z \in A$, $x\ \varrho_I\ y$, $s\ \varrho_I\ z$. Then $x \leqslant g+y$, $y \leqslant h+x$, $s \leqslant u+z$, $z \leqslant v+s$ for some $g, h, u, v \in I$. This yields $x + s \leqslant g + y + u + z$, $y + z \leqslant h + x + v + s$. Then $x + s \leqslant g + u_1 + y + z$, $y + z \leqslant h + v_1 + x + s$ for some $u_1, v_1 \in I$. Since $g + u_1, h + v_1 \in I$, we have $(x + s)\ \varrho_I\ (y + z)$. In view of Theorem 5 we conclude that ϱ_I is a congruence relation on the l-monoid A. □

The next example shows that $\mathrm{In}(A)$ need not be a convex subset of an l-monoid A.

Example 4. Let $(Z, +, \leqslant)$ be the additive group of all integers with the natural order. We define a new operation $\oplus$ on the linearly ordered set $(Z, \leqslant)$ as follows: $(2k) \oplus (2l) = (2l) \oplus (2k) = 2k+2l$, $(2k+1) \oplus (2l+1) = (2l+1) \oplus (2k+1) = 2k+2l+1$, $(2k) \oplus (2l + 1) = (2l + 1) \oplus (2k) = 2k + 2l + 1$ for all $k, l \in Z$. Then $(Z, \oplus, \leqslant)$ is a commutative linearly ordered monoid. $\mathrm{In}(Z)$ is the set of all even numbers, which is not a convex subset of $(Z, \oplus, \leqslant)$.

Theorem 9. *Let A be an l-monoid. Then $\mathrm{In}(A)$ is an ideal of A iff $\mathrm{In}(A)$ is a convex subset of A.*

Proof. Necessity follows from Theorem 3.

Let $\mathrm{In}(A)$ be a convex subset of A. Let $x \in A, y, z \in \mathrm{In}(A)$ and $x^+ + y^- \leqslant x^- + z^+$. From $y^- \leqslant x^+ + y^- \leqslant x^- + z^+ \leqslant z^+$ and the convexity of $\mathrm{In}(A)$ it follows that $x^+ + y^-, x^- + z^+ \in \mathrm{In}(A)$. Then $x^+ = x^+ + y^- + (-y^-), x^- = x^- + z^+ + (-z^+) \in \mathrm{In}(A)$. Hence $x \in \mathrm{In}(A)$. Thus $\mathrm{In}(A)$ is a left ideal of A. Analogously we can show that $\mathrm{In}(A)$ is a right ideal of A. □

Lemma 11. *Let A be a soft l-monoid. Then $\mathrm{In}(A)$ is a convex subset of A.*

Proof. Let $x, y \in \mathrm{In}(A)$, $z \in A$, $x \leqslant z \leqslant y$. Since $z + (-y) \leqslant 0$, there exists $u \in \mathrm{In}(A)$ such that $z + (-y) + u = 0$. Then $(-y) + u + z + (-y) + u + (-u) + y = 0$ yields $z \in \mathrm{In}(A)$. □

Corollary 3. *For each DRl-monoid B, $\mathrm{In}(B)$ is an ideal of B.*

Proof. It follows from Theorem 9 and Lemma 11. □

It is clear that the intersection of any family of ideals of an l-monoid A is again an ideal of A. Thus for any family of ideals there exists the least ideal containing this family of ideals. It is the intersection of all ideals containing this family of ideals.

Hence the following theorem holds for the set $\mathrm{Id}(A)$ of all ideals of an l-monoid A partially ordered by inclusion.

Theorem 10. *For each l-monoid A, $\mathrm{Id}(A)$ is a complete lattice.*

Remark 4. An analogous assertion holds for left ideals and right ideals of an l-monoid.

Theorem 11. *Let A be a linearly ordered monoid. Then $\mathrm{Id}(A)$ is a linearly ordered set.*

Proof. Let I and J be ideals of A, $I \neq J$. Then there exists $x \in A$ such that $x \in I \setminus J$ or $x \in J \setminus I$.

Suppose that $x \in I \setminus J$, $0 \leqslant x$. Let $y \in J$. If $x \leqslant y$, then from the convexity of J we get $x \in J$, a contradiction. Thus $y \leqslant x$. If $0 \leqslant y$ then from the convexity of I we obtain $y \in I$. If $y \leqslant 0$ and $x+y \leqslant 0$, then Lemma 2(ii) yields $x \in J$, a contradiction. Hence, if $y \leqslant 0$, then $0 \leqslant x+y$. In this case Lemma 3(i) yields $y \in I$. Therefore $J \subseteq I$. For $x \leqslant 0$, analogously we can obtain that $J \subseteq I$.

It is clear that if $x \in J \setminus I$, than $I \subseteq J$. $\square$

Let A be an l-monoid. For any subset M of A, the smallest ideal containing M, i.e. the intersection of all ideals I of A such that $M \subseteq I$, is called the ideal generated by M. It will be denoted by $I(M)$. If $M = \{a\}$, we will write $I(a)$ instead of $I(\{a\})$.

Theorem 12. *Let A be a commutative l-monoid, $\emptyset \neq M \subseteq A$. Then $I(M) = \{x \in A;\ x^+ + a_1^- + \ldots + a_m^- \leqslant x^- + b_1^+ + \ldots + b_n^+$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in M, m, n \in \mathbb{N}\}$.*

Proof. Let $S = \{x \in A;\ x^+ + a_1^- + \ldots + a_m^- \leqslant x^- + b_1^+ + \ldots + b_n^+$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in M, m, n \in \mathbb{N}\}$. Clearly, $0 \in S$. Let $x, y \in S$. Then $x^+ + a_1^- + \ldots + a_m^- \leqslant x^- + b_1^+ + \ldots + b_n^+$, $y^+ + c_1^- + \ldots + c_k^- \leqslant y^- + d_1^+ + \ldots + d_l^+$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_k, d_1, \ldots, d_l \in M$, $m, n, k, l \in \mathbb{N}$. Then $(x+y)^+ + a_1^- + \ldots + a_m^- + c_1^- + \ldots + c_k^- \leqslant x^+ + y^+ + a_1^- + \ldots + a_m^- + c_1^- + \ldots + c_k^- \leqslant x^- + y^- + b_1^+ + \ldots + b_n^+ + d_1^+ + \ldots + d_l^+ \leqslant (x+y)^- + b_1^+ + \ldots + b_n^+ + d_1^+ + \ldots + d_l^+$. Hence $x + y \in S$.

Let $t \in A$, $u, v \in S$, $t^+ + u^- \leqslant t^- + v^+$. Thus $u^+ + p_1^- + \ldots + p_{m'}^- \leqslant u^- + q_1^+ + \ldots + q_{n'}^+$, $v^+ + r_1^- + \ldots + r_{k'}^- \leqslant v^- + s_1^+ + \ldots + s_{l'}^+$ for some $p_1, \ldots, p_{m'}, q_1, \ldots, q_{n'}, r_1, \ldots, r_{k'}$, $s_1, \ldots, s_{l'} \in M, m', n', k', l' \in \mathbb{N}$. Then we have $t^+ + p_1^- + \ldots + p_{m'}^- + r_1^- + \ldots + r_{k'}^- \leqslant t^+ + u^+ + p_1^- + \ldots + p_{m'}^- + r_1^- + \ldots + r_{k'}^- \leqslant t^+ + u^- + q_1^+ + \ldots + q_{n'}^+ + r_1^- + \ldots + r_{k'}^- \leqslant t^- + v^+ + q_1^+ + \ldots + q_{n'}^+ + r_1^- + \ldots + r_{k'}^- \leqslant t^- + q_1^+ + \ldots + q_{n'}^+ + v^- + s_1^+ + \ldots + s_{l'}^+ \leqslant t^- + q_1^+ + \ldots + q_{n'}^+ + s_1^+ + \ldots + s_{l'}^+$. Therefore $t \in S$. Hence S is a left ideal of A. Clearly, S is also a right ideal of A and $M \subseteq S$.

Let J be an ideal of A such that $M \subseteq J$, $z \in S$. Then $z^+ + e_1^- + \ldots + e_{m_1}^- \leqslant z^- + f_1^+ + \ldots + f_{n_1}^+$ for some $e_1, \ldots, e_{m_1}, f_1, \ldots, f_{n_1} \in M$, $m_1, n_1 \in \mathbb{N}$. Since $e_1^- + \ldots + e_{m_1}^-, f_1^+ + \ldots + f_{n_1}^+ \in J$, we have $z \in J$. Therefore $I(M) = S$. $\square$

Remark 5. Birkhoff [1, p. 294] proved that if a, b, c are elements of a lattice ordered group, $a \wedge b = 0$ and $a \wedge c = 0$, then $a \wedge (b+c) = 0$. From the proof it follows that this assertion is also valid in the case that a, b, c are elements of an l-monoid.

Lemma 12. *Let A be a soft l-monoid, $x \in A$, $n \in \mathbb{N}$. Then*

(i) $(nx^+) \wedge (n(-x^-)) = 0$,

(ii) $nx^+ + n(-x^-) = (nx^+) \vee (n(-x^-))$,

(iii) $nx^- = (nx)^-$, $nx^+ = (nx)^+$.

Proof. (i) Since $x^+ = x + (-x^-)$, we have $x^+ \wedge (-x^-) = [x + (-x^-)] \wedge (-x^-) = x \wedge 0 + (-x^-) = 0$. Then it follows from Remark 5 that $(nx^+) \wedge (n(-x^-)) = 0$.

(ii) Since nx^+ and nx^- commute, in view of (i) from $nx^+ + n(-x^-) \leqslant [(2nx^+) \vee (nx^+ + n(-x^-))] \wedge [(n(-x^-) + nx^+) \vee 2n(-x^-)] = [(nx^+) \wedge (n(-x^-))] + [(nx^+) \vee (n(-x^-))] = [(2nx^+) \wedge (n(-x^-) + nx^+)] \vee [((nx^+) + n(-x^-)) \wedge (2n(-x^-))] \leqslant nx^+ + n(-x^-)$ we get $nx^+ + n(-x^-) = (nx^+) \vee (n(-x^-))$.

(iii) In view of (i) we have $(nx)^- = (nx) \wedge 0 = (nx^+ + nx^-) \wedge [n(-x^-) + nx^-] = [(nx^+) \wedge (n(-x^-))] + nx^- = nx^-$.

By (ii), $(nx)^+ = (nx) \vee 0 = (nx^+ + nx^-) \vee [n(-x^-) + nx^-] = [nx^+ \vee n(-x^-)] + nx^- = nx^+ + n(-x^-) + nx^- = nx^+$. $\square$

The relation for the absolute value in a DRl-monoid from Lemma 1 can be used for the definition of the absolute value of an element x of a soft l-monoid A. Hence we define: $|x| = x^+ + (-x^-)$ for each $x \in A$.

This absolute value has properties analogous to the absolute value in a DRl-monoid.

Theorem 13. *Let A be a soft l-monoid, $x, y \in A$, $n \in \mathbb{N}$. Then*

(i) $|x| \geqslant 0$ *and* $|x| = 0$ *iff* $x = 0$,

(ii) $|x| = x$ *iff* $x \geqslant 0$,

(iii) $|x| = -x^-$ *iff* $x \leqslant 0$,

(iv) $|x| = (-x^-) + x^+ = x^+ \vee (-x^-)$,

(v) $|x| \leqslant |y|$ *iff* $x^+ + y^- \leqslant x^- + y^+$ *iff* $y^- + x^+ \leqslant y^+ + x^-$,

(vi) $n|x| = |nx|$,

(vii) $|x + y| \leqslant |x| + |y| + |x|$; *if A is commutative, then* $|x + y| \leqslant |x| + |y|$,

(viii) $|x| + |y| \geqslant |x| \vee |y| \geqslant |x \vee y|$, $|x| \vee |y| \geqslant |x \wedge y|$.

Proof. (i) Let $|x| = 0$. Then $0 \leqslant x^+ = x^- \leqslant 0$ yields $x = 0$. The rest is obvious.

(ii) If $x \geqslant 0$, then $x^- = 0$. Hence $|x| = x$. The rest is obviuos.

(iii) If $x \leqslant 0$, then $x^+ = 0$. Thus $|x| = -x^-$. If $|x| = -x^-$, then $x^+ = 0$. Therefore $x = x^- \leqslant 0$.

(iv) It follows from Lemma 12(ii).

(v) It is obvious.

(vi) In view of Lemma 12(iii) we get $n|x| = nx^+ + n(-x^-) = nx^+ + [-(nx^-)] = (nx)^+ + [-((nx)^-)] = |nx|$.

(vii) Obviously, $|x|+|y|+|x| \geqslant x^+ + y^+ + (-y^-) + (-x^-) \geqslant (x+y)^+ + [-(x^- + y^-)] \geqslant (x+y)^+ + [-((x+y)^-)] = |x+y|$. The rest is obvious.

(viii) Clearly $|x| + |y| \geqslant |x| \vee |y|$. In view of (iv) we have $|x| \vee |y| = x^+ \vee y^+ \vee (-x^-) \vee (-y^-) = (x \vee y)^+ \vee [-((x \wedge y)^-)] \geqslant (x \vee y)^+ \vee [-((x \vee y)^-)] = |x \vee y|$. $\square$

Corollary 4. *Let A be a soft l-monoid, $I \subseteq A$. Let $0 \in I$ and $u + v \in I$ for each u, $v \in I$. Then I is an ideal of A iff $x \in A$, $y \in I$ and $|x| \leqslant |y|$ implies $x \in I$.*

Proof. It follows from Theorem 13(v). $\square$

Corollary 5. *Let A be a soft l-monoid, I an ideal of A, $a \in A$. Then $a \in I$ iff $|a| \in I$.*

Theorem 14. *Let A be a soft l-monoid, $\emptyset \neq M \subseteq A$. Then $I(M) = \{x \in A;\ |x| \leqslant |a_1| + \ldots + |a_n|$ for some $a_1, \ldots, a_n \in M, n \in \mathbb{N}\}$.*

Proof. Let $S = \{x \in A;\ |x| \leqslant |a_1| + \ldots + |a_n|$ for some $a_1, \ldots, a_n \in M, n \in \mathbb{N}\}$. Clearly $0 \in S$. Let x, $y \in S$. Thus $|x| \leqslant |a_1| + \ldots + |a_m|$, $|y| \leqslant |b_1| + \ldots + |b_n|$ for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in M$, $m, n \in \mathbb{N}$. In view of Theorem 13 (vii) we have $|x + y| \leqslant |x| + |y| + |x| \leqslant |a_1| + \ldots + |a_m| + |b_1| + \ldots + |b_n| + |a_1| + \ldots + |a_m|$. Hence $x + y \in S$.

Let $z \in A$, $t \in S$ and $|z| \leqslant |t|$. Then $|z| \leqslant |c_1| + \ldots + |c_k|$ for some $c_1, \ldots, c_k \in M, k \in \mathbb{N}$. Hence $z \in S$. By Corollary 4, S is an ideal of A. Clearly, $M \subseteq S$.

Let J be an ideal of A containing M, $u \in S$. Then $|u| \leqslant |d_1| + \ldots + |d_l|$, for some $d_1, \ldots, d_l \in M, l \in \mathbb{N}$. Since $|d_1| + \ldots + |d_l| \in J$, we have $u \in J$. Hence $S \subseteq J$. Therefore $S = I(M)$. $\square$

Corollary 6. *Let A be a soft l-monoid.*

(i) *If I and J are ideals of A, then $I \vee J = \{x \in A;\ |x| \leqslant |a_1| + \ldots + |a_n|$ for some $a_1, \ldots, a_n \in I \cup J, n \in \mathbb{N}\}$.*

(ii) *If I and J are normal ideals of A, then $I \vee J = \{x \in A;\ |x| \leqslant c + d$ for some $c \in I^+, d \in J^+\}$.*

R e m a r k 6. Kühr [10, Lemma 11] showed that for all positive elements x, y, z of a DRl-monoid the following proposition is valid:

(P$_1$) $x \wedge (y+z) \leqslant (x \wedge y) + (x \wedge z)$.

From the proof it follows that the assertion (P$_1$) also holds for any positive elements x, y, z of an l-monoid A. The dual assertion is also valid. Hence for any negative elements x, y, z of an l-monoid A the following proposition is valid:

(P$_2$) $x \vee (y+z) \geqslant (x \vee y) + (x \vee z)$.

The following two theorems generalize Theorem 14 and Proposition 15 of Kühr [10].

Theorem 15. *For any soft l-monoid A, the lattice* $\mathrm{Id}(A)$ *is algebraic and Brouwerian.*

P r o o f. First we prove that $\mathrm{Id}(A)$ is an algebraic lattice. Let J be an ideal of A. Clearly, $J = \bigvee\limits_{a \in J} I(a)$.

Let $a \in J$ and let K_γ be ideals of A, where $\gamma \in \Gamma$, such that $I(a) \subseteq \bigvee\limits_{\gamma \in \Gamma} K_\gamma$. In view of Theorem 14 from $a \in \bigvee\limits_{\gamma \in \Gamma} K_\gamma$ we get $|a| \leqslant |a_1| + \ldots + |a_n|$, where $n \in \mathbb{N}$ and $a_i \in K_{\gamma_i}$ for some $\gamma_i \in \Gamma$, $i = 1, \ldots, n$. This yields $a \in \bigvee\limits_{i=1}^{n} K_{\gamma_i}$ and hence $I(a) \subseteq \bigvee\limits_{i=1}^{n} K_{\gamma_i}$. Thus $I(a)$ is a compact element of the lattice $\mathrm{Id}(A)$ and hence $\mathrm{Id}(A)$ is an algebraic lattice.

Since every algebraic distributive lattice is Brouwerian, it suffices to show that $\mathrm{Id}(A)$ is a distributive lattice.

Let I, J, K be ideals of A. Let $x \in I \cap (J \vee K)$. By Corollary 6, $|x| \leqslant |b_1| + \ldots + |b_k|$, where $b_1, \ldots, b_k$ are elements of $J \cup K$, $k \in \mathbb{N}$. In view of (P$_1$) we obtain $|x| = |x| \wedge (|b_1| + \ldots + |b_k|) \leqslant (|x| \wedge |b_1|) + \ldots + (|x| \wedge |b_k|)$. Since $|x| \wedge |b_i| \in (I \cap J) \vee (I \cap K)$ for $i = 1, \ldots, k$, we obtain $(|x| \wedge |b_1|) + \ldots + (|x| \wedge |b_k|) \in (I \cap J) \vee (I \cap K)$. This implies $x \in (I \cap J) \vee (I \cap K)$. Therefore $I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K)$. Hence $\mathrm{Id}(A)$ is a distributive lattice. $\square$

Since $\mathrm{Id}(A)$ is a Brouwerian lattice for each soft l-monoid A, for any ideals J and K of A the relative pseudocomplement of J with respect to K in $\mathrm{Id}(A)$ exists which is described in the following theorem.

Theorem 16. *For any ideals J and K of a soft l-monoid A, the relative pseudocomplement of J with respect to K in* $\mathrm{Id}(A)$ *is given by $J * K = \{x \in A;\ |x| \wedge |a| \in K$ for any $a \in J\}$.*

The proof is the same as the proof of Proposition 15 in [10], only instead of Proposition 9(3), Lemma 11 and Theorem 14 from [10] it is necessary to use Theorem 13(vii), (P_1) and Theorem 15, respectively.

Corollary 7. *Let A be a positively ordered l-monoid. Then* $\mathrm{Id}(A)$ *is an algebraic and Brouwerian lattice.*

Theorem 17. *Let A be a negatively ordered l-monoid.*

(i) *If M is a nonempty subset of A, then $I(M)=\{x\in A;\ a_1+\ldots+a_n\leqslant x$ for some $a_1,\ldots,a_n\in M, n\in\mathbb{N}\}$.*

(ii) $\mathrm{Id}(A)$ *is an algebraic and Brouwerian lattice.*

Proof. (i) Let $S=\{x\in A;\ a_1+\ldots+a_n\leqslant x$ for some $a_1,\ldots,a_n\in M, n\in\mathbb{N}\}$. Clearly, $0\in S$. Let x, $y\in S$. Then $a_1+\ldots+a_m\leqslant x$, $b_1+\ldots+b_n\leqslant y$ for some $a_1,\ldots,a_m$, $b_1,\ldots,b_n\in M$, $m,n\in\mathbb{N}$. Then $a_1+\ldots+a_m+b_1+\ldots+b_n\leqslant x+y$. Hence $x+y\in S$.

Let $z\in A$, $u,v\in S$, $z^++u^-\leqslant z^-+v^+$. Then $c_1+\ldots+c_k\leqslant u\leqslant z$ for some $c_1,\ldots,c_k\in M, k\in\mathbb{N}$. This yields $z\in S$. Therefore S is an ideal and $M\subseteq S$.

It is easy to see that if K is an ideal containing M, then $S\subseteq K$. Hence $I(M)=S$.

(ii) The proof is similar to the proof of Theorem 15, only instead of Theorem 14, Corollary 6 and (P_1) we have to use (i) and (P_2). $\square$

Theorem 18. *Let I and J be normal ideals of an l-monoid A. Then $I\vee J=\{x\in A;\ x^++a_1^-+a_2^-\leqslant x^-+b_1^++b_2^+$ for some $a_1,b_1\in I, a_2,b_2\in J\}$.*

Proof. Let I and J be normal ideals of A, $S=\{x\in A;\ x^++a_1^-+a_2^-\leqslant x^-+b_1^++b_2^+$ for some $a_1,b_1\in I,a_2,b_2\in J\}$. Clearly $0\in S$. Let x, $y\in S$. Hence $x^++a_1^-+a_2^-\leqslant x^-+b_1^++b_2^+$, $y^++c_1^-+c_2^-\leqslant y^-+d_1^++d_2^+$ for some a_1, b_1, c_1, $d_1\in I$, a_2, b_2, c_2, $d_2\in J$. Thus $x^++a_1^-+a_2^-+y^++c_1^-+c_2^-\leqslant x^-+b_1^++b_2^++y^-+d_1^++d_2^+$. Then $x^++y^++e_1^-+e_2^-+c_1^-+c_2^-\leqslant x^-+y^-+f_1^++f_2^++d_1^++d_2^+$ for some e_1, $f_1\in I$, e_2, $f_2\in J$. This yields $(x+y)^++(e_1^-+(c_1')^-)^-+(e_2^-+c_2^-)^-\leqslant (x+y)^-+(f_1^++(d_1')^+)^++(f_2^++d_2^+)^+$ for some $c_1',d_1'\in I$. Hence $x+y\in S$.

Let $z\in A$, u, $v\in S$, $z^++u^-\leqslant z^-+v^+$. Hence $u^++p_1^-+p_2^-\leqslant u^-+q_1^++q_2^+$, $v^++r_1^-+r_2^-\leqslant v^-+s_1^++s_2^+$ for some p_1, q_1, r_1, $s_1\in I$, p_2, q_2, r_2, $s_2\in J$. Then $z^++p_1^-+p_2^-+r_1^-+r_2^-\leqslant z^++u^++p_1^-+p_2^-+r_1^-+r_2^-\leqslant z^++u^-+q_1^++q_2^++r_1^-+r_2^-\leqslant z^-+v^++q_1^++q_2^++r_1^-+r_2^-=z^-+v^++r_1^-+r_2^-+h_1^++h_2^+\leqslant z^-+v^-+s_1^++s_2^++h_1^++h_2^+\leqslant z^-+s_1^++s_2^++h_1^++h_2^+$ for some $h_1\in I$, $h_2\in J$. Thus $z^++(p_1^-+((r_1')^-)^-+(p_2^-+r_2^-)^-\leqslant z^-+(s_1^++(h_1')^+)^++(s_2^++h_2^+)^+$ for some $r_1',h_1'\in I$. This yields $z\in S$. Therefore S is a left ideal of A. Analogously we can show that S is a right ideal of A.

For $c \in I$, $d \in J$ we have $c^+ + c^- + 0 \leqslant c^- + c^+ + 0$, $d^+ + 0 + d^- \leqslant d^- + 0 + d^+$. This implies $c, d \in S$. Hence $I \subseteq S$, $J \subseteq S$.

Clearly, if K is an ideal containing ideals I and J, then $S \subseteq K$. Therefore $S = I \vee J$. $\square$

Theorem 19. *Let A be an l-monoid,* $\mathrm{In}(A)$ *a convex subset of A, $g \in \mathrm{In}(A)$.*

(i) *If $0 \leqslant g \leqslant a_1 + \ldots + a_n$ for some $a_1, \ldots, a_n \in A^+$, $n \in \mathbb{N}$, then there exist elements $b_1, \ldots, b_n \in \mathrm{In}(A)$ such that $g = b_1 + \ldots + b_n$, $0 \leqslant b_i \leqslant a_i$, $i = 1, \ldots, n$.*

(ii) *If $a_1 + \ldots + a_n \leqslant g \leqslant 0$ for some $a_1, \ldots, a_n \in A^-, n \in \mathbb{N}$, then there exist elements $b_1, \ldots, b_n \in \mathrm{In}(A)$ such that $g = b_1 + \ldots + b_n$, $a_i \leqslant b_i \leqslant 0$, $i = 1, \ldots, n$.*

Proof. (i) We prove this statement by induction on n. The statement is valid for $n = 1$. Assume that the statement holds for $n = k$, $k \in \mathbb{N}$.

Let $0 \leqslant g \leqslant a_1 + \ldots + a_k + a_{k+1}$, where $a_1, \ldots, a_{k+1} \in A^+$. Let $a = a_1 + \ldots + a_k$, $b_{k+1} = g \wedge a_{k+1}$. Hence $b_{k+1} \leqslant a_{k+1}$. Since $0 \leqslant b_{k+1} \leqslant g$, from the convexity of $\mathrm{In}(A)$ it follows that $b_{k+1} \in (\mathrm{In}(A))^+$. Further, $g \leqslant a + g$ and $g \leqslant a + a_{k+1}$ implies $g \leqslant (a + g) \wedge (a + a_{k+1}) = a + g \wedge a_{k+1} = a + b_{k+1}$. Let $h = g + (-b_{k+1})$. Hence $g = h + b_{k+1}$, $h \in (\mathrm{In}(A))^+$. From $0 \leqslant h = g + (-b_{k+1}) \leqslant a = a_1 + \ldots + a_k$ we obtain that $h = b_1 + \ldots + b_k$ where $b_1, \ldots, b_k \in \mathrm{In}(A)$, $0 \leqslant b_i \leqslant a_i$, $i = 1, \ldots, k$. Then we have $g = b_1 + \ldots + b_k + b_{k+1}$, $0 \leqslant b_i \leqslant a_i$, $i = 1, \ldots, k+1$, where $b_1, \ldots, b_{k+1} \in \mathrm{In}(A)$.

The proof of (ii) can be obtained dually. $\square$

An l-monoid A is called a weak divisibility l-monoid if for each $a, b \in A$ such that $a \leqslant b$ there exist $x, y \in A$ such that $a + x = b$, $y + a = b$.

Remark 7. Birkhoff [1, p. 320] defined a divisibility monoid as a partially ordered monoid M in which $a \leqslant b$ is equivalent to $a + x = b$, $y + a = b$ for some $x, y \in M$.

Each weak divisibility l-monoid is a soft l-monoid. From Lemma 5 [10] it follows that if a, b are elements of a DRl-monoid A such that $a \leqslant b$, then $(b \rightharpoonup a) + a = b$, $a + (b \leftharpoonup a) = b$. Hence each DRl-monoid is a weak divisibility l-monoid.

The l-monoid $(G^\infty, \oplus, \leqslant_1)$ from Example in [6, p. 107] is a weak divisibility l-monoid, but not a DRl-monoid.

Lemma 13. *If J and K are normal ideals of an l-monoid A, then $J \cap K$ is a normal ideal of A.*

Proof. Let $x \in A$, $c \in (J \cap K)^+$. Then $x + c = c_1 + x = c_2 + x$ for some $c_1 \in J^+, c_2 \in K^+$. Since $0 \leqslant c_1 \wedge c_2 \leqslant c_1, c_2$, from the convexity of J and K it follows that $c_1 \wedge c_2 \in (J \cap K)^+$. Then $c_1 \wedge c_2 + x = (c_1 + x) \wedge (c_2 + x) = x + c$. Hence $x + (J \cap K)^+ \subseteq (J \cap K)^+ + x$. Analogously, $(J \cap K)^+ + x \subseteq x + (J \cap K)^+$.

Dually we obtain that $(J \cap K)^- + x = x + (J \cap K)^-$. Then Lemma 5 implies that $J \cap K$ is a normal ideal of A. $\square$

Theorem 20. *Let A be a weak divisibility l-monoid, I and J normal ideals of A. Then $I \vee J$ is a normal ideal of A.*

Proof. Let $x \in A$. Suppose that $c \in (I \vee J)^+$. By Corollary 6(ii), $c \leqslant a + b$ for some $a \in I^+$, $b \in J^+$. Since $x \leqslant x + c$, there exists $d \in A^+$ such that $d + x = x + c$. Hence $d + x \leqslant x + a + b = a' + b' + x$ for some $a' \in I^+$, $b' \in J^+$. Then $[d \wedge (a' + b')] + x = (d + x) \wedge (a' + b' + x) = d + x = x + c$. Since $0 \leqslant d \wedge (a' + b') \leqslant a' + b'$, from the convexity of $I \vee J$ we get $d \wedge (a' + b') \in (I \vee J)^+$. Hence $x + (I \vee J)^+ \subseteq (I \vee J)^+ + x$. Analogously $(I \vee J)^+ + x \subseteq x + (I \vee J)^+$.

Let $g \in (I \vee J)^-$, $h = -g$. Then $g, h \in \mathrm{In}(A)$, $h \in (I \vee J)^+$. In view of Corollary 6(ii) we have $h = |h| \leqslant c + d$ for some $c \in I^+$, $d \in J^+$. By Theorem 19 and Lemma 11, there exist elements $h_1, h_2 \in \mathrm{In}(A)$ such that $h = h_1 + h_2$, $0 \leqslant h_1 \leqslant c$, $0 \leqslant h_2 \leqslant d$. From the convexity of I and J we have $h_1 \in I^+$, $h_2 \in J^+$.

Since $-h_1 \in I$, $-h_2 \in J$, we get $x + g = x + (-h_2) + (-h_1) = h_1' + h_2' + x$, where $h_1' \in I^-$, $h_2' \in J^-$. Clearly $h_1' + h_2' \in (I \vee J)^-$. Hence $x + (I \vee J)^- \subseteq (I \vee J)^- + x$. Similarly $x + (I \vee J)^- \subseteq x + (I \vee J)^-$. In view of Lemma 5 we have $x + (I \vee J) = (I \vee J) + x$. $\square$

Corollary 8. *The set of all normal ideals of a weak divisibility l-monoid A is a sublattice of* $\mathrm{Id}(A)$.

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