COMMUTATIVE SEMIGROUPS THAT ARE NIL OF INDEX 2 AND HAVE NO IRREDUCIBLE ELEMENTS

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(Received March 16, 2006)

Abstract. Every commutative nil-semigroup of index 2 can be imbedded into such a semigroup without irreducible elements.

Keywords: nil-semigroup, commutative nil-semigroup

MSC 2000: 20M14

1. Introduction

(Congruence-)simple semimodules over semigroups (and/or semirings) are easily divided into four pair-wise disjoint classes. That is, if M is a simple semimodule then the additive semigroup $M(+)$ is either

(1) cancellative, or
(2) idempotent, or
(3) constant (i.e. $|M+M|=1$), or
(4) nil of index 2 and without irreducible elements (i.e., $2x+y=2x$ for all $x,y\in M$ and $M+M=M$).

Now, the last class is the most enigmatic one and was scarcely studied so far (cf. [1]). In fact, structural properties of commutative 2-nil semigroups without irreducible elements (zs-semigroups in the sequel) are not yet well understood and examples of these semigroups are rarely seen (see e.g. [2]). The aim of the present short note is to show that every commutative 2-nil semigroups can be imbedded into a commutative zs-semigroup. Consequently, there should exist many commutative zs-semigroups and then many simple semimodules of type (4) as well.

Supported by the institutional grant MSM 0021620839 and by the Grant Agency of Czech Republic, grant #GAČR-201/05/0002.

Throughout this note, the word *semigroup* will always mean a commutative semigroup, the binary operation of which will be denoted additively.

1.1 An element w of a semigroup S is called *absorbing* if $S+w=w$. There exists at most one absorbing element in S and it will be denoted by the symbol o $(=o_S)$ in the sequel. The fact that S possesses the absorbing element will be denoted by $o \in S$.

1.2 A non-empty subset I of S is an *ideal* if $S+I \subseteq I$.

1.3 Lemma.

(i) *A one-element subset $\{w\}$ is an ideal iff $w=o_S$.*

(ii) *If I is an ideal then the relation $r=(I \times I) \cup \mathrm{id}_S$ is a congruence of S and $I=o_T$, where $T=S/r$.*

(iii) *If $o \in S$ and and s is a congruence of S then the set $\{a \in S;\ (a,o) \in s\}$ is an ideal.*

1.4 Put $(Q_S(a) =)$ $Q(a)=S+a$ and $(P_S(a) =)$ $P(a)=Q(a) \cup \{a\}$ for every $a \in S$.

1.5 Lemma.

(i) *$Q(a) \subseteq P(a)$ and both these sets are ideals of S.*

(ii) *$P(a)$ is just the (principal) ideal generated by the one-element set $\{a\}$.*

1.6 Assume that $o \in S$. An element $a \in S$ is said to be *nilpotent (of index at most $m \geqslant 1$)* if $ma=o$. We denote by $N(S)$ $(N_m(S))$ the set of nilpotent (of index at most m) elements of S.

The semigroup S is said to be *nil* (*of index at most* m) if $N(S)=S$ $(N_m(S)=S)$ and *reduced* if o_S is the only nilpotent element of S.

1.7 Lemma.

(i) *$o=N_1(S) \subseteq N_2(S) \subseteq N_3(S) \subseteq \dots$ and all these sets are ideals.*

(ii) *$N(S)=\bigcup N_m(S)$ is an ideal.*

(iii) *The factor-semigroup $T=S/N(S)$ is reduced.*

1.8 Lemma. *The following conditions are equivalent:*

(i) *$o \in S$ and $2x=o$ for every $x \in S$.*

(ii) *S is nil of index at most 2.*

(iii) *$2x+y=2z$ for all $x,y,z \in S$.*

(iv) *$2x+y=2x$ for all $x,y \in S$.*

1.9 A semigroup satisfying the equivalent conditions of 1.8 will be called *zeropotent* (or, in a colourless manner, a *zp-semigroup*) in the sequel.

A zp-semigroup without irreducible elements (i.e., when $S + S = S$) will be called a *zs-semigroup.*

1.10 Define a relation $|_S$ on S by $a \,|_S\, b$ iff $b = a + u$ for some $u \in S^0$, where S^0 is the least monoid containing S and 0 denotes the neutral element of S^0.

1.11 Lemma. *The following conditions are equivalent:*

(i) $a \,|_S\, b$.
(ii) $b \in P(a)$.
(iii) $P(b) \subseteq P(a)$.

Moreover, if $a \neq b$ then these conditions are equivalent to:

(iv) $b \in Q(a)$.
(v) $P(b) \subseteq Q(a)$.

1.12 Lemma. *The relation $|_S$ is a fully invariant compatible quasiordering of the semigroup S and the equivalence $\|_S = \ker(|_S)$ is a fully invariant congruence of the semigroup S.*

1.13 Lemma. *The following conditions are equivalent:*

(i) $a \,\|_S\, b$.
(ii) $P(a) = P(b)$.

Moreover, if $a \neq b$ then these conditions are equivalent to:

(iii) $Q(a) = Q(b) = P(a) = P(b)$.

1.14 Lemma. *The following conditions are equivalent:*

(i) S *is a group.*
(ii) $|_S = S \times S$.
(iii) $\|_S = S \times S$.
(iv) $P(a) = P(b)$ *for all* $a, b \in S$.
(v) $P(a) = S$ *for every* $a \in S$.
(vi) $Q(a) = S$ *for every* $a \in S$.

1.15 Lemma. *The relation $|_S$ is a (fully invariant compatible) ordering (or, equivalently, $\|_S = \mathrm{id}_S$), provided that at least one of the following four conditions is satisfied:*

(1) *S is not a group and id_S, $S \times S$ are the only fully invariant congruences of S;*
(2) *S is cancellative and $0 \notin S$;*
(3) *S is nil;*
(4) *S is idempotent.*

Proof. (1) Combine 1.13 and 1.14.

(2) If $a \neq b$, $b = a+u$ and $a = b+v$, $a, b, u, v \in S$, then $a = a+w$, where $w = u+v$, and hence $w = 0$, a contradiction.

(3) If $a = a + w$, $a, w \in S$, then $a = a + mv$ for every $m \geqslant 1$, and hence $a = o$.

(4) If $b = a + u$, $a, b, u \in S$, then $a + b = a + a + u = a + u = b$. □

1.16 Define a relation $/_S$ on S by $a/_S b$ iff $Q(b) \subseteq Q(a)$.

1.17 Lemma. *The relation $/_S$ is an invariant compatible quasiordering of the semigroup S and the equivalence $//_S = \ker(/_S)$ is an invariant congruence of the semigroup S.*

1.18 Lemma. *The following conditions are equivalent:*

(i) $/_S = S \times S$.
(ii) $//_S = S \times S$.
(iii) *$S + a = S + b$ for all $a, b \in S$.*
(iv) *$S + S = I$ is the smallest ideal of S and I is a subgroup of S.*

2. The distractibility ordering of zp-semigroups

2.1 In this section, let S be a zp-semigroup. Put $\mathrm{Ann}(S) = \{a \in S;\ S + a = o\}$.

2.2 Lemma.

(i) *The relation $|_S$ is a fully invariant compatible ordering of the semigroup S.*
(ii) *o is the greatest element.*
(iii) *$\mathrm{Ann}(S) \setminus \{o\}$ is the set of maximal elements of $T = S \setminus \{o\}$.*
(iv) *If $|S| \geqslant 2$ then $S \setminus (S + S)$ is the set of minimal elements of S.*
(v) *If $|S| \geqslant 3$ then S has no smallest element.*

2.3 Lemma. *If S is a non-trivial zs-semigroup then S has no minimal elements, S is infinite and not finitely generated.*

P r o o f. Being nil, S is finitely generated iff it is finite. The rest is clear from 2.2(iv). $\square$

2.4 Lemma. *If $0 \in S$ then S is trivial.*

3. Every zp-semigroup is a subsemigroup of a zs-semigroup

Now, we are in position to show the main result of this note.

3.1 Proposition. *Every zp-semigroup is a subsemigroup of a zs-semigroup.*

P r o o f. Let S be a non-trivial zp-semigroup and $Q = S \setminus (S+S)$. For every $a \in Q$, put $R_a = S \setminus P(a)$; then $o \notin R_a$ and $R_a \neq \emptyset$, provided that $|S| \geqslant 3$. Further, $0 \notin S$ by 2.4 and we put $R_{a,0} = R_a \cup \{0_a\}$, where the elements 0_a, $a \in Q$, are all distinct, $V_{a,1} = R_{a,0} \times \{1\}$ and $V_{a,2} = R_{a,0} \times \{2\}$. Now, consider the disjoint union

$$T = S \cup \bigcup_{a \in Q} V_{a,1} \cup \bigcup_{a \in Q} V_{a,2}$$

and define an addition on T in the following way:

(1) $x+y$ coincides in $S(+)$ and $T(+)$ for all $x, y \in S$;

(2) $x + (y,i) = (x+y, i) = (y,i) + x$ for all $x \in S$, $(y,i) \in V_{a,i}$, $a \in Q$, $i = 1,2$, $x + y \in R_a$ (i.e., $x+y \notin P(a)$);

(3) $(x,i) + (y,j) = x + y + a$ for all $x, y \in R_{a,0}$, $a \in Q$, $i \neq j$;

(4) $\alpha + \beta = o$ if $\alpha, \beta \in T$ and the sum $\alpha + \beta$ is not defined by (1), (2) or (3).

Clearly, $\alpha+\beta = \beta+\alpha$, $\alpha+\alpha = o$, $\alpha+o = o$ and $o+\alpha = o$ for every $\alpha \in T$. Next, we check that $\alpha + (\beta+\gamma) = (\alpha+\beta)+\gamma$ for all $\alpha, \beta, \gamma \in T$.

Put $\delta = \alpha + (\beta + \gamma)$, $\varepsilon = (\alpha+\beta)+\gamma$ and consider the following cases:

(a) $\alpha, \beta, \gamma \in S$. Then $\delta = \varepsilon$ by (1).

(b) $\alpha, \beta \in S$ and $\gamma = (x,i) \in V_{a,i}$. Assume first that $\alpha+\beta+x \in R_a$. Then $\varepsilon = (\alpha+\beta+x, i)$ by (2). Moreover, $\beta + x \in R_a$, and hence $\beta+\gamma = (\beta+x, i)$ and $\delta = \alpha + (\beta+x, i) = (\alpha+\beta+x, i) = \varepsilon$.

Assume next that $\alpha+\beta+x \notin R_a$. Then $\varepsilon = o$ by (4). Moreover, either $\beta+x \notin R_a$, $\beta+\gamma = o$ and $\delta = \alpha + o = o = \varepsilon$, or $\beta + x \in R_a$, $\beta+\gamma = (\beta+x, i)$ and $\delta = \alpha + (\beta+x, i) = o = \varepsilon$.

(c) $\alpha, \gamma \in S$, $\beta \in V_{a,i}$ (or $\beta, \gamma \in S$, $\alpha \in V_{a,i}$). These cases are similar and/or dual to (b).

(d) $\alpha = (x, i) \in V_{a,i}$, $\beta = (y, i) \in V_{a,i}$ and $\gamma \in S$. Then $\alpha + \beta = o$ by (4), and so $\varepsilon = o + \gamma = o$. Assume first that $y + \gamma \in R_a$. Then $\beta + \gamma = (y + \gamma, i)$ by (2) and $\delta = (x, i) + (y + \gamma, i) = o$ by (4). Thus $\varepsilon = \delta$.

Assume next that $y + \gamma \notin R_a$. Then $\beta + \gamma = o$ by (4) and $\delta = (x, i) + o = o = \varepsilon$.

(e) $\alpha, \gamma \in V_{a,i}$, $\beta \in S$ (or $\beta, \gamma \in V_{a,i}$, $\alpha \in S$). These cases are similar to (d).

(f) $\alpha = (x, i) \in V_{a,i}$, $\beta = (y, j) \in V_{a,j}$, $i \neq j$, $\gamma \in S$. Then $\alpha + \beta = x + y + a$ by (3), and hence $\varepsilon = x + y + a + \gamma$ by (1). Assume first that $y + \gamma \in R_a$. Then $\beta + \gamma = (y + \gamma, j)$ by (2) and $\delta = (x, i) + (y + \gamma, j) = x + y + \gamma + a = \varepsilon$.

Assume next that $y + \gamma \notin R_a$. Then $\beta + \gamma = o$ by (4), and hence $\delta = (x, i) + o = o$. However, $y + \gamma \notin R_a$ means $y + \gamma \in P(a)$ and then $a + y + \gamma = o$, since S is nil of index at most 2. Thus $\varepsilon = x + a + y + \gamma = x + o = o = \delta$.

(g) $\alpha \in V_{a,i}$, $\gamma \in V_{a,j}$, $\beta \in S$ (or $\beta \in V_{a,i}$, $\gamma \in V_{a,j}$, $\alpha \in S$). These cases are similar to (f).

(h) $\alpha, \beta, \gamma \in V_{a,i}$. Then $\beta + \gamma = o = \alpha + \beta$, and hence $\delta = a + o = o = o + \gamma = \varepsilon$.

(i) $\alpha = (x, i) \in V_{a,i}$, $\beta = (y, i) \in V_{a,i}$ and $\gamma = (z, j) \in V_{a,j}$, $i \neq j$. Then $\alpha + \beta = o$ by (4), and hence $\varepsilon = o + (z, j) = o$. Further, $\beta + \gamma = y + z + a$ by (3). Now, $x + y + z + a \in P(a)$ and $\delta = (x, i) + y + z + a = o$ by (4). Thus $\delta = \varepsilon$.

(j) $\alpha, \gamma \in V_{a,i}$, $\beta \in V_{a,j}$ (or $\beta, \gamma \in V_{a,i}$, $\alpha \in V_{a,j}$). These cases are similar to (i).

(k) In all the remaining cases we get $\delta = o = \varepsilon$ due to (4).

We have shown that $T = T(+)$ is a zp-semigroup and S is a subsemigroup of T. Clearly,

$$T + T = S \cup \bigcup_{a \in Q} (R_a \times \{1\}) \cup \bigcup_{a \in Q} (R_a \times \{2\}).$$

Thus $S \subseteq T + T$ and

$$T \setminus (T + T) = \bigcup_{a \in Q} \{(0_a, 1), (0_a, 2)\}.$$

Finally, put $T_0 = S$, $T_1 = T$ and consider a sequence

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

of zp-semigroups such that T_i is a subsemigroup of T_{i+1} and $T_i \subseteq T_{i+1} + T_{i+1}$. Then $\bigcup T_i$ is a zs-semigroup. $\square$

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