# ON REFLEXIVITY AND HYPERREFLEXIVITY OF SOME SPACES OF INTERTWINING OPERATORS

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*Abstract.* Let $T, T'$ be weak contractions (in the sense of Sz.-Nagy and Foiaş), $m, m'$ the minimal functions of their $C_0$ parts and let $d$ be the greatest common inner divisor of $m, m'$. It is proved that the space $I(T, T')$ of all operators intertwining $T, T'$ is reflexive if and only if the model operator $S(d)$ is reflexive. Here $S(d)$ means the compression of the unilateral shift onto the space $H^2 \ominus dH^2$. In particular, in finite-dimensional spaces the space $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of minimal polynomials of $T, T'$ are simple. The paper is concluded by an example showing that quasisimilarity does not preserve hyperreflexivity of $I(T, T')$.

*Keywords*: intertwining operator, reflexivity, $C_0$ contraction, weak contraction, hyperreflexivity

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## 1. Introduction

Let $H$, $H'$ be complex separable Hilbert spaces, let $\mathcal{B}(H, H')$ denote the space of all bounded linear operators $H \to H'$. If $H = H'$ then $\mathcal{B}(H, H) = \mathcal{B}(H)$ is the algebra of all bounded linear operators on $H$. By a subspace we mean a closed linear subspace. For a subset $A \subset H$, we denote by $\bigvee A$ the closed linear span of $A$. A subspace $L \subset H$ is called invariant for $T \in \mathcal{B}(H)$ if $TL \subset L$. As usual, $T|L$ means the restriction of the operator $T$ to $L$. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg} \mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing $\mathcal{A}$ and the identity. $\operatorname{Lat} \mathcal{A}$ denotes the set of all subspaces of $H$ that are invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces

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of $H$, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H)\colon\ \mathcal{L} \subset \operatorname{Lat} T\}$. A (unital weakly closed) subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is called reflexive if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. An operator $T \in \mathcal{B}(H)$ is called reflexive if $\operatorname{Alg}\{T\}$ is reflexive.

H. Bercovici, C. Foiaş and B. Sz.-Nagy [3] studied reflexivity of $C_0$ contractions and their commutants. They showed also that if the commutant of a $C_0$ contraction $T$ is reflexive then $T$ is also reflexive. Generally, the reflexivity of $\{T\}'$ does not imply the reflexivity of the operator $T$ [6].

The reflexivity of subalgebras was studied for the first time in [12]. The notion of reflexivity of algebras of operators was generalized to subspaces of operators by V. S. Shul'man [13]:

**Definition 1.1.** Let $\mathcal{M}$ be a subset of $\mathcal{B}(H, H')$. Then the reflexive closure of $\mathcal{M}$ is

$$\operatorname{ref}\mathcal{M} = \bigcap_{x\in H} \Big\{T \in \mathcal{B}(H,H')\colon\ Tx \in \bigvee\{Mx\colon\ M \in \mathcal{M}\}\Big\}.$$

A (closed linear) subspace $\mathcal{M} \subset \mathcal{B}(H, H')$ is called *reflexive* if $\mathcal{M} = \operatorname{ref}\mathcal{M}$.

Clearly, in Definition 1.1 the Hilbert spaces $H, H'$ can be replaced by arbitrary Banach spaces. A stronger concept of hyperreflexivity was introduced for algebras in [1] and extended to subspaces in [10].

**Definition 1.2.** Let $X, X'$ be complex Banach spaces and let $\mathcal{M}$ be a norm-closed subspace of $\mathcal{B}(X, X')$. $\mathcal{M}$ is called *hyperreflexive* if there exists $c > 0$ such that for all $T \in \mathcal{B}(X, X')$

$$\operatorname{dist}(T, \mathcal{M}) \leqslant c\alpha(T, \mathcal{M}), \text{ where } \alpha(T, M) = \sup\{\operatorname{dist}(Tx, \mathcal{M}x)\colon\ x \in H, \|x\| = 1\}.$$

$\inf\big\{c > 0\colon\ \operatorname{dist}(T, \mathcal{M}) \leqslant c\alpha(T, \mathcal{M})\big\}$ is called the hyperreflexivity constant of $\mathcal{M}$.

Note that if $\mathcal{M}$ is hyperreflexive then it is reflexive. It is well-known that if both $H$ and $H'$ are finite-dimensional then reflexivity and hyperreflexivity coincide. In [11, Theorem 2.5] V. Müller and M. Ptak have shown that if $X, X'$ are arbitrary Banach spaces and $\mathcal{M}$ is a finite dimensional subspace of $\mathcal{B}(X, X')$ then $\mathcal{M}$ is reflexive if and only if it is hyperreflexive. Clearly, if $\mathcal{M}$ is a subalgebra of $\mathcal{B}(H)$ then $\operatorname{ref}\mathcal{M} = \operatorname{Alg}\operatorname{Lat} M$.

In [13] reflexivity of the space

$$I(T, T') = \{A \in \mathcal{B}(H, H')\colon\ AT = T'A\}$$

of operators intertwining $T \in \mathcal{B}(H)$ and $T' \in \mathcal{B}(H')$ was studied and a characterization of reflexive spaces $I(T, T')$ was given in the case of isometries $T, T'$. Moreover, it was stated that if $\dim H < \infty$, $\dim H' < \infty$ then $I(T, T')$ is reflexive if $T$ or $T'$

is similar to a normal operator. In [5] $\operatorname{Alg}\{T\}'$ was described if $\dim H < \infty$ and this showed that $\{T\}'$ is reflexive if and only if $T$ is similar to a normal operator or equivalently, if all roots of the minimal polynomial of $T$ are simple.

In [20] we described (using the Jordan forms of $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$) $I(T,T')$ and $\operatorname{ref} I(T,T')$ in finite-dimensional spaces and we showed that $I(T,T')$ is reflexive if all roots of the greatest common divisor of the minimal polynomials of $T$ and $T'$ are simple. The purpose of this paper is to extend this result to pairs of weak contractions. To prove our results we use the fact that quasi-similarity preserves reflexivity of $I(T,T')$. We give an example showing that quasi-similarity does not preserve hyperreflexivity of $I(T,T')$.

## 2. Compressions of the unilateral shift

We will use the terminology and results of Sz.-Nagy-Foiaş dilation theory [14]. In particular, $H^2, H^\infty$ mean the Hardy spaces of analytic functions in the unit disc, $S(\Theta)$ means the compression of the unilateral shift $S$ onto the space $H(\Theta) = H^2 \ominus \Theta H^2$. For $f, g \in H^\infty$ we write $f \mid g$ ($f$ divides $g$) if there exists $\varphi \in H^\infty$ such that $g = \varphi f$. The orthogonal projection onto a subspace $K$ of a Hilbert space $H$ is denoted by $P_K$. For $f_1, f_2 \in H^\infty$ we denote by $f_1 \wedge f_2$ the greatest common inner divisor of $f_1$ and $f_2$.

The following result is an easy consequence of [2, Theorem III.1.16].

**Theorem 2.1.** *Let $v_1$, $v_2$, $d$ be inner functions, $v_1 \wedge v_2 = 1$. Put $\Theta_1 = v_1 d$, $\Theta_2 = v_2 d$. Then*

(i) *$X \in I(S(\Theta_1), S(\Theta_2))$ if and only if there exists a function $\varphi \in H^\infty$ such that*

$$X = P_{H(\Theta_2)} u(S)|H(\Theta_1), \quad \text{where} \quad u = v_2\varphi.$$

*Moreover, $X = 0$ if and only if $d \mid \varphi$.*

(ii) *An operator $A \in \operatorname{ref} I(S(\Theta_1), S(\Theta_2))$ if and only if*

$$\begin{aligned} &A|H^2 \ominus dH^2 \in \operatorname{ref} I\big(S(d), S(\Theta_2)|v_2(H^2 \ominus dH^2)\big),\\ \text{and}\quad &A|d(H^2 \ominus v_1 H^2) = 0. \end{aligned}$$

(iii) *$I(S(\Theta_1), S(\Theta_2))$ is reflexive if and only if $S(d)$ is reflexive.*

Proof. (i) According to [2, Theorem III.1.16], $X \in I\big(S(\Theta_1), S(\Theta_2)\big)$ if and only if there exists an inner function $u$ such that $X = P_{H(\Theta_2)} u(S)\big|H(\Theta_1)$ and $\Theta_2 \mid u\Theta_1$. Since $v_1 \wedge v_2 = 1$, we have $v_2 d \mid u v_1 d \iff v_2 \mid u$ and consequently there exists

$\varphi \in H^\infty$ such that $u = \varphi v_2$. Moreover, $X = 0$ if and only if $\Theta_2 \mid u$, i.e. if and only if $d \mid \varphi$.

(ii) $H(\Theta_1)$ and $H(\Theta_2)$ can be written as orthogonal sums

$$H(\Theta_1) = (H^2 \ominus dH^2) \oplus d(H^2 \ominus v_1 H^2), \quad H(\Theta_2) = (H_2 \ominus v_2 H^2) \oplus v_2(H^2 \ominus dH^2).$$

It is well-known that $v_2(H^2 \ominus dH^2) \in \operatorname{Lat} S(\Theta_2)$. Using (i) we obtain for any $f \in H(\Theta_1)$

$$\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = \bigvee_{\varphi \in H^\infty} P_{H(\Theta_2)} v_2 \varphi f \subset v_2(H^2 \ominus dH^2).$$

If $f \in d(H^2 \ominus v_1 H^2)$ then $v_2 \varphi f \in d v_2 H^2 \perp H(\Theta_2)$, consequently

$$\bigvee_{X \in I(S(\Theta_1), S(\Theta_2))} Xf = 0.$$

Herefrom (ii) follows easily.

(iii) $S(\Theta_2)|v_2 H(d)$ is unitarily equivalent to $S(d)$. So the reflexivity of $I(S(\Theta_1), S(\Theta_2))$ implies that the commutant of $S(d)$ is reflexive. Since $\{S(d)\}' = \operatorname{Alg} S(d)$, this proves (iii). $\square$

## 3. General $C_0$ contractions

To prove a characterization of pairs $T$, $T'$ of $C_0$ contractions having reflexive $I(T, T')$ we need two simple lemmas.

**Lemma 3.1.** *Let $T, X \in \mathcal{B}(H)$, $T', Y \in \mathcal{B}(H')$ and $TX = XT$, $T'Y = YT'$. Put $T_X = T|(XH)^-$, $T'_Y = T'|(YH')^-$.*

*If $I(T, T')$ is reflexive then $I(T_X, T'_Y)$ is reflexive as well.*

Proof. Suppose that $A \in \operatorname{ref} I(T_X, T'_Y)$. If $B \in I(T_X, T'_Y)$ then $BX \in I(T, T')$. Therefore for all $h \in H$ we have

$$AXh \in \bigvee_{B \in I(T_X, T'_Y)} BXh \subset \bigvee_{C \in I(T, T')} Ch, \quad \text{i.e.} \quad AX \in \operatorname{ref} I(T, T')$$

and so $ATX = AXT = T'AX = T'_Y AX$, i.e. $A \in I(T_X, T'_Y)$. $\square$

**Lemma 3.2.** *Let $\vartheta_1, \Theta_1, \vartheta_2, \Theta_2$ be inner functions such that $\vartheta_1 \mid \Theta_1$ and $\vartheta_2 \mid \Theta_2$. If $I(S(\Theta_1), S(\Theta_2))$ is reflexive then $I(S(\vartheta_1), S(\vartheta_2))$ is reflexive as well.*

Proof. Put $\varphi_k = \Theta_k/\vartheta_k$, $k = 1, 2$. Since $S(\vartheta_k)$ is unitarily equivalent to $S(\Theta_i)|\bigl(\varphi_k(S(\Theta_k))H(\Theta_k)\bigr)^-$, Lemma 3.2 is a consequence of Lemma 3.1. $\square$

Now we are ready to state one of our main results.

**Theorem 3.3.** *Let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be $C_0$ contractions having minimal functions $m, m'$, respectively. Let $d = m \wedge m'$. Then $I(T, T')$ is reflexive if and only if the operator $S(d)$ is reflexive.*

Proof. If $T_1 \in \mathcal{B}(H_1)$ and $T_1' \in \mathcal{B}(H_1')$ are quasisimilar to $T_2 \in \mathcal{B}(H_2)$ and $T_2' \in \mathcal{B}(H_2')$, respectively, then $I(T_1, T_1')$ is reflexive if and only if $I(T_2, T_2')$ is reflexive. This was first stated (without proof which is easy) in [13, Proposition 1]. Since any $C_0$ contraction is quasisimilar to its Jordan model it is enough to prove the theorem for Jordan models

$$T = \bigoplus_\alpha S(m_\alpha), \qquad T' = \bigoplus_\beta S(m_\beta'),$$

where $\oplus$ means the direct orthogonal sum. According to [13, Proposition 2], $I(T, T')$ is reflexive if and only if each of the spaces $I(S(m_\alpha), S(m_\beta'))$ is reflexive. For all indices $\alpha, \beta$, we have $m_\alpha \mid m$, $m_\beta' \mid m'$. Therefore, by Lemma 3.2, $I(T, T')$ is reflexive if and only if $I(S(m), S(m'))$ is reflexive. According to assertion (iii) of Theorem 2.1 this completes the proof. $\square$

Theorem 3.3 generalizes [3, Theorem B]. In finite-dimensional spaces we obtain the following corollary (a generalization of [5, Theorem 3]).

**Corollary 3.4.** *Let $H$, $H'$ be finite-dimensional. Then $I(T, T')$ is reflexive if and only if all roots of the greatest common divisor of the minimal polynomials $m_T$ and $m_{T'}$ of $T$ and $T'$, respectively, are simple.*

Proof. Replacing $T$ and $T'$ by $\|T\|^{-1}T$ and $\|T'\|^{-1}T'$ we obtain a pair of contractions the minimal functions $m, m'$ of which are finite Blaschke products whose numerators are $m_T$ and $m_{T'}$, respectively. Then $d$ is also a finite Blaschke product and its numerator is the greatest common inner divisor of the minimal polynomials $m_T$ and $m_{T'}$. It is well-known (see e.g. [7]) that then $S(d)$ is reflexive if and only if all zeroes of $d$ are simple. $\square$

Note that in [20] Corollary 3.4 was proved more directly by describing $I(T, T')$ and $\operatorname{ref} I(T, T')$ for nilpotent $T$ and $T'$. In the case $T = T'$ this was done in [5].

Now, let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be weak contractions. (For the definition of weak contractions and basic results we refer to [14, Chapter VIII]). It is well-known (see, e.g., [18]) that $T$ and $T'$ can be splitted into orthogonal sums $T = T_{ac} \oplus T_{su}$, $T' = T'_{ac} \oplus T'_{su}$ of their absolutely continuous and singular unitary parts and that

$$I(T, T') = I(T_{ac}, T'_{ac}) \oplus I(T_{su}, T'_{su}).$$

It follows that

$$\operatorname{ref} I(T, T') = \operatorname{ref} I(T_{ac}, T'_{ac}) \oplus \operatorname{ref} I(T_{su}, T'_{su}).$$

Since for normal operators $A$, $B$ the space $I(A, B)$ is reflexive [13], $I(T, T')$ is reflexive if and only if so is $I(T_{ac}, T'_{ac})$. According to [17, Lemma 3] any absolutely continuous weak contraction $S$ is similar to a completely non-unitary (c.n.u.) weak contraction $S'$ and, moreover, the $C_0$ parts of $S$ and $S'$ coincide. Since similarity (even quasi-similarity [13, Proposition 1]) preserves reflexivity of $I(T, T')$, it does not restrict generality if we suppose that $T$, $T'$ are c.n.u.

**Theorem 4.1.** *Let $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$ be c.n.u. weak contractions and let $T_0 \in \mathcal{B}(H_0)$, $T'_0 \in \mathcal{B}(H'_0)$ be their $C_0$ parts and $T_1 \in \mathcal{B}(H_1)$, $T'_1 \in \mathcal{B}(H'_1)$ their $C_{11}$ parts. Then*

(i) *if $X \in I(T, T')$ then $XH_0 \subset H'_0$ and $XH_1 \subset H'_1$;*

(ii) *if $A \in \operatorname{ref} I(T, T')$ then its restrictions to subspaces $H_0$, $H_1$ satisfy $A_0 = A|H_0 \in \operatorname{ref} I(T_0, T'_0)$, $A_1 = A|H_1 \in \operatorname{ref} I(T_1, T'_1)$;*

(iii) *$I(T, T')$ is reflexive if and only if $I(T_0, T'_0)$ is reflexive.*

Proof. (i) According to [14, Chapters II.4 and VIII.2]

$$\begin{aligned} H_0 &= \{h \in H\colon\ T^n h \to 0\}, & H'_0 &= \{h' \in H'\colon\ {T'}^n h' \to 0\} \\ \text{and}\quad H_1^{\perp} &= \{h \in H\colon\ T^{*n} h \to 0\}, & {H'_1}^{\perp} &= \{h' \in H'\colon\ {T'}^{*n} h' \to 0\}. \end{aligned}$$

$XT = T'X$ implies $XT^n = {T'}^n X$ for all positive integers $n$. Therefore $h_0 \in H_0 \Longrightarrow \lim {T'}^n X h_0 = \lim X T^n h_0 = 0$, i.e. $Xh_0 \in H'_0$. By taking adjoints we obtain $XT = T'X \Longrightarrow T^* X^* = X^* {T'}^*$ and so $X^* {H'_1}^{\perp} \subset H_1^{\perp}$, which is equivalent to $XH_1 \subset H'_1$.

(ii) This is an obvious consequence of (i).

(iii) There are operators $R, S \in \{T\}''$, $R', S' \in \{T'\}''$ such that

$$\begin{aligned} H_0 &= \ker R = (SH)^-, & H_1 &= (RH)^- = \ker S, \\ H'_0 &= \ker R' = (S'H)^-, & H_1 &= (R'H)^- = \ker S' \end{aligned}$$

([14], [15], [16, Theorem 1]). Suppose that $I(T,T')$ is reflexive. Then, by Lemma 3.1, $I(T_0,T_0')$ is reflexive. Conversely, if $I(T_0,T_0')$ is reflexive and $A \in \operatorname{ref} I(T,T')$ then by (ii) $A|H_0 \in \operatorname{ref} I(T_0,T_0')$ and $A|H_1 \in \operatorname{ref} I(T_1,T_1')$. The operators $T_1$, $T_1'$ are quasi-similar to unitary operators and so $I(T_1,T_1')$ is reflexive. Therefore $A|H_0 \in I(T_0,T_0')$ and $A|H_1 \in I(T_1,T_1')$. Since $H_0 \vee H_1 = H$, this shows that $I(T,T')$ is reflexive. $\square$

**Theorem 4.2.** *Let $T$, $T'$ be weak contractions and let their $C_0$ parts $T_0$, $T_0'$ have minimal functions $m$, $m'$, respectively. Let $d = m \wedge m'$ be the greatest common inner divisor of $m$, $m'$. Then the space $I(T,T')$ is reflexive if and only if the operator $S(d)$ is reflexive.*

Proof. This is an obvious consequence of Theorems 3.3 and 4.1. $\square$

Remarks.

1. Theorems 4.1 and 4.2 are generalizations of [19, Theorem 5.1].
2. Inner functions $m$ for which $S(m)$ is a reflexive operator were characterized in [7, Theorem 3.1].

## 5. Quasisimilarity does not preserve hyperreflexivity

First, let us recall the definition of quasisimilarity:

**Definition 5.1.** $T \in B(H)$, $S \in B(K)$ are *quasi-similar* (we write $T \overset{\text{q.s.}}{\sim} S$) if there are quasi-affinities (injective operators with dense range) $X \in I(T,S)$, $Y \in I(S,T)$.

Example 5.2. Put $H_n = H_n' = C^2$, $H = H' = \bigoplus_{n=1}^{\infty} H_n$,

$$T_n = \frac{1}{n}\begin{pmatrix} 2n & n \\ 0 & 2n+1 \end{pmatrix}, \quad T_n' = \frac{1}{n}\begin{pmatrix} 2n & 0 \\ -n & 2n+1 \end{pmatrix},$$
$$S_n = S_n' = \frac{1}{n}\begin{pmatrix} 2n+1 & 0 \\ 0 & 2n \end{pmatrix},$$
$$T = \bigoplus_{n=1}^{\infty} T_n, \quad T' = \bigoplus_{n=1}^{\infty} T_n', \quad S = S' = \bigoplus_{n=1}^{\infty} S_n.$$

Then, obviously, $T \in \mathcal{B}(H)$, $T' \in \mathcal{B}(H')$, $S = S' \in \mathcal{B}(H)$.

The following assertions hold.

(a) $T \overset{\text{q.s.}}{\sim} S = S' \overset{\text{q.s.}}{\sim} T'$,
(b) all $I(T_m,T_n')$ are hyperreflexive,
(c) $I(T,T')$ is not hyperreflexive,
(d) $I(S,S')$ is hyperreflexive.

P r o o f. The common minimal polynomial $(\lambda-2n)(\lambda-2n-1)$ of $T_n$, $T_n'$, $S_n$ has simple roots, which implies that all $I(T_m,T_n')$ are reflexive. In finite dimension this implies that they are also hyperreflexive and this proves (b).

Putting $A_n=\begin{pmatrix}0&n\\0&1\end{pmatrix}$, $B_n=\begin{pmatrix}0&0\\-n&1\end{pmatrix}$, $C_n=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ we obtain

$$T_n=\frac{1}{n}(2nI+A_n),\ T_n'=\frac{1}{n}(2nI+B_n),\ S_n=S_n'=\frac{1}{n}(2nI+C_n)$$

and if $P_n=\begin{pmatrix}n&1\\1&0\end{pmatrix}$ then $P_n^{-1}=\begin{pmatrix}0&1\\1&-n\end{pmatrix}$ and $A_n=P_nC_nP_n^{-1}$.

Hence $A_nP_n=P_nC_n$, $P_n^{-1}A_n=C_nP_n^{-1}$ and after perturbation by $2nI$, $T_nP_n=P_nS_n$, $P_n^{-1}T_n=S_nP_n^{-1}$.

Now, it is easy to compute $\|P_n\|$ and $\|P_n^{-1}\|$:

$$P_n^{\top}=P_n\Longrightarrow\|P_n\|=\varrho(P_n)=\frac{n+\sqrt{n^2+4}}{2}=\varrho(P_n^{-1})=\|P_n^{-1}\|.$$

Putting $Y=\bigoplus\limits_{n=1}^{\infty}n^{-1}P_n$, $X=\bigoplus\limits_{n=1}^{\infty}n^{-1}P_n^{-1}$ we obtain quasiaffinities $X\in I(T,S)$, $Y\in I(S,T)$, i.e., $T\overset{\text{q.s.}}{\sim}S$. Similarly, it can be proved that $T'\overset{\text{q.s.}}{\sim}S$. This completes the proof of (a).

(c): $m\neq n\Longrightarrow I(T_n,T_m')=\{0\}$ because their minimal polynomials are relatively prime. Therefore $I(T,T')=\bigoplus\limits_{n=1}^{\infty}I(T_n,T_n')$ and similarly $I(S,S')=\bigoplus\limits_{n=1}^{\infty}I(S_n,S_n')$ By a simple direct computation we obtain $X_n\in I(T_n,T_n')=I(A_n,B_n)$ if and only if $X_n=\begin{pmatrix}0&\alpha\\\beta&-n(\alpha+\beta)\end{pmatrix}$ for some $\alpha,\beta\in C$. So $I(T_n,T_n')=\mathcal{S}_n$ from an example due to Kraus and Larson [9] (see also [4, Example 58.9]) who proved that $\mathcal{S}_n$ is hyperreflexive with $\kappa_{\mathcal{S}_n}\geqslant\frac{1}{3}n$. So $I(T,T')=\bigoplus\limits_{n=1}^{\infty}\mathcal{S}_n$ is not hyperreflexive.

(d): Observe that $I(S_n,S_n)=I\left(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)$ for all $n$, i.e. its hyperreflexivity constant does not depend on $n$. Using a recent result of K. Kliś and M. Ptak [8, Theorem 5.1] we obtain that $I(S,S')$ is hyperreflexive. □

It easy to show that if $T=\bigoplus\limits_{n=1}^{\infty}T_n$, $T'=\bigoplus\limits_{n=1}^{\infty}T_n'$ and $I(T,T')$ is hyperreflexive, then all $I(T_n,T_m')$ are hyperreflexive. From Example 5.2 it follows that the converse implication does not hold.

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