ON SUFFICIENT CONDITIONS OF EXISTENCE AND UNIQUENESS OF PERIODIC IN A STRIP SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS

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Let \( b > 0 \) and \( \mathcal{D}_b = \mathbb{R} \times [0, b] \). In the strip \( \mathcal{D}_b \), consider the hyperbolic equation

\[
\frac{\partial^2 u}{\partial x \partial y} - f(x, y, u, \frac{\partial u}{\partial y}),
\]

where \( f : \mathcal{D}_b \times \mathbb{R}^2 \to \mathbb{R} \) satisfies the local Carathéodory conditions, i.e., \( f(x, \cdot, z_0, z_1) : \mathcal{D}_b \to \mathbb{R} \) is measurable for every \((z_0, z_1) \in \mathbb{R}^2\), \( f(x, y, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R} \) is continuous for almost every \((x, y) \in \mathcal{D}_b\) and the function

\[
\sup\{|f(x, \cdot, z_0, z_1)| : |z_0| + |z_1| \leq \rho\}
\]
is summable on the rectangle \([-a, a] \times [0, b]\) for any \( \rho > 0 \). \( a > 0 \).

By solution of the equation (1) we understand a locally absolutely continuous function \( u : \mathcal{D}_b \to \mathbb{R} \) (see [2]) satisfying the equation (1) almost everywhere in \( \mathcal{D}_b \).

We study the case where \( f \) is \( \omega \)-periodic in the first argument for some \( \omega > 0 \), i.e.,

\[
f(x + \omega, y, z_0, z_1) = f(x, y, z_0, z_1) \quad \text{for} \quad (x, y) \in \mathcal{D}_b, (z_0, z_1) \in \mathbb{R}^2.
\]

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a locally absolutely continuous \( \omega \)-periodic function. Below we formulate sufficient conditions of existence and uniqueness of the solution of the equation (1) satisfying

\[
u(x, 0) = \varphi(x), \quad u(x + \omega, y) = u(x, y) \quad \text{for} \quad (x, y) \in \mathcal{D}_b.
\]

Note that earlier the problem (1), (2) has been investigated when \( f \) is linear or quasi-linear with respect to the last two arguments (see [1], [4]-[10]).

**Theorem 1.** Let the inequalities

\[
[f(x, y, z_0, 0)] \leq p_0(x, y)(1 + |z_0|),
\]

\[
\sigma(y)|f(x, y, z_0, z_1) - f(x, y, z_0, z_1')| \leq p(x, y, z_0)z_1 - z_1',
\]

\( p_0 : [0, \omega] \times [0, b] \to \mathbb{R}_+ \) is summable, \( \sigma : [0, b] \to \{-1, 1\} \) is measurable and \( p : [0, \omega] \times [0, b] \to \mathbb{R}_+ \) is a measurable function such that \( p(x, \cdot) : [0, b] \to \mathbb{R} \) is continuous almost for every \( x \), max\{|p(\cdot, t)| : t \in [0, b]\} \to [0, \omega] \to \mathbb{R}_+ \) is summable and

\[
\int_0^\omega p(s, y)ds < 0 \quad \text{for} \quad y \in [0, b].
\]

Then the problem (1), (2) is solvable.

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Remark 1. The conditions (3) and (5) are essential and cannot be weakened. Violation of the condition (3) may result in the loss of global solvability in the whole strip \( D_b \). As an example, consider the problem

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y} - u^{1+\varepsilon} - 1, \quad u(x, 0) = 0, \quad u(x + \omega, y) = u(x, y),
\]

in the strip \( D_b \) where \( b = 2 \int_0^{+\infty} e^{-t + \varepsilon} \, dt \). \( \varepsilon > 0 \) is an arbitrary constant. The given problem may have at most one solution (see Theorem 2 below). Therefore if \( u(x, y) \) is a solution of the given problem, then \( u(x, y) = u(y) \) and it is simultaneously the solution of the Cauchy problem

\[
\frac{du}{dy} = u^{1+\varepsilon} + 1, \quad u(0) = 0,
\]

defined on the segment \([0, b]\). But it is impossible since \( \lim_{y \to b} u(y) = +\infty \).

On the other hand, violation of the condition (5) may result in the loss of solvability of the problem (1), (2). To convince ourselves that is so, consider the problem

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial y} + 1, \quad u(x, 0) = 0, \quad u(x + \omega, y) = u(x, y)
\]

for which all conditions of Theorem 1, except of (5), are fulfilled. Nevertheless, the above problem has no solution. In fact, otherwise we should have

\[
\frac{\partial u(x, y)}{\partial y} = \frac{1}{y} \quad \text{for} \quad 0 < y < b.
\]

But this contradicts the absolute continuity of \( u \).

**Theorem 2.** Let the conditions (4) and (5) hold and let there exist nonnegative summable functions \( c_0 : [0, \omega] \times [0, b] \to \mathbb{R}_+ \) and \( c_1 : [0, b] \to \mathbb{R}_+ \) such that the inequality

\[
|f(x, y, z_0, z_1) - f(x, y, z_0, z_1)| \leq c_0(x, y)|z_0 - z_1| + c_1(x)|z_1 - z_0|
\]

holds on \( D_b \times \mathbb{R}^2 \). Then the problem (1), (2) has at most one solution.

Finally, consider the case where \( f(x, y, z_0, z_1) \equiv f(x, y, z_0) \), i.e., where the equation (1) has the form

\[
\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u).
\]

In addition, assume that \( f \) has partial derivatives in the second and the third arguments satisfying local Carathéodory conditions. Put

\[
g_0(x, y, z) = \frac{\partial f(x, y, z)}{\partial y}, \quad g_1(x, y, z) = \frac{\partial f(x, y, z)}{\partial z}.
\]

**Theorem 3.** Let there exist a positive constant \( l \) and a measurable function \( \sigma : [0, b] \to [-1, 1] \) such that the inequalities

\[
|g_0(x, y, z)| \leq l(1 + |z|), \quad |g_1(x, y, z)| \leq l
\]

and

\[
\int_0^b g(x) \, dx > 0
\]

(7)
hold on $[0, a] \times [0, b] \times \mathbb{R}$, where $g(x) = \text{ess inf} \{e(y)g: (x, t, z) : t \in [0, b], z \in \mathbb{R}\}$. Then the problem (6), (2) is solvable if and only if

$$\int_0^\omega \int_0 f(s, 0, \varphi(s)) \, ds = 0.$$ 

If, besides, $g_1 (i = 0, 1)$ are locally Lipschitz continuous in $z$, then the problem (6), (2) is uniquely solvable.

Remark 2. The condition (7) in Theorem 3 is essential and cannot be neglected. For example, it is obvious that the problem

$$\frac{\partial^2 u}{\partial x \partial y} = (1 + u^2)^{\frac{1}{2}} - \sin \left( \frac{2\pi x}{\omega} \right)$$

$$u(x, 0) = 0, \quad u(x + \omega, y) = u(x, y)$$

has no solution, although all conditions of Theorem 3, except (7), hold since

$$g_1 (x, y, z) = (1 + z^2)^{\frac{1}{2}} - \frac{1}{2} z.$$

References


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