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THE LINEARIZED MAXIMUM PRINCIPLE FOR OPTIMAL PROBLEMS WITH VARIABLE DELAYS AND CONTINUOUS INITIAL CONDITION

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Optimal problems with variable delays in phase coordinates and controls are considered. Without commensurability conditions for delays in controls (incommensurability), necessary conditions of optimality are obtained: in the form of the linearized integral maximum principle for initial function and control, in the form of equalities and inequalities for initial and final moments.

Let $\mathcal{J}=[a,b]$ be a finite interval; $\mathcal{O}\subset R^n$, $G\subset R^n$ be open sets and let the function $f:\mathcal{J}\times\mathcal{O}^s\times G^\nu\to R^n$ satisfies the following conditions:

1. for a fixed $t\in\mathcal{J}$, the function $f(t,x_1,\dots,x_s,u_1,\dots,u_\nu)$ is continuously differentiable with respect to $(x_1,\dots,x_s,u_1,\dots,u_\nu)\subset\mathcal{O}^s\times G^\nu$;
2. for a fixed $(x_1,\dots,x_s,u_1,\dots,u_\nu)\subset\mathcal{O}^s\times G^\nu$, the functions f, f_{x_i}, $i=1,\dots,s$, f_{u_j}, $j=1,\dots,\nu$, are measurable with respect to t. For arbitrary compacts $K\subset\mathcal{O}$, $V\subset G$, there exists a function $m_{K,V}(\cdot)\in L(\mathcal{J},R_0^+)$, $R_0^+=[0,\infty)$, such that for almost all $t\in\mathcal{J}$ and $\forall(x_1,\dots,x_s,u_1,\dots,u_\nu)\in K^s\times V^\nu$,
$$|f(t,x_1,\dots,x_s,u_1,\dots,u_\nu)|+\sum_{i=1}^{s}|f_{x_i}(\cdot)|+\sum_{j=1}^{\nu}|f_{u_j}(\cdot)|\le m_{K,V}(t).$$

Let now $\tau_i(t)$, $i=1,\dots,s$, $\theta_j(t)$, $j=1,\dots,\nu$, $t\in\mathcal{J}$, are absolutely continuous functions satisfying the conditions: $\tau_i(t)\le t$, $\dot\tau_i(t)>0$, $\theta_j(t)\le t$, $\dot\theta_j(t)\ge 0$; Δ be a set of continuous functions $\varphi:[\tau,b]\to M$, $\tau=\min\{\tau_1(a),\dots,\tau_s(a)\}$, $M\subset\mathcal{O}$ is a convex set; Ω be a set of measurable functions $u:\mathcal{J}_2=[\theta,b]\to U$, $\theta=\min\{\theta_1(a),\dots,\theta_\nu(a)\}$ satisfying the conditions $\operatorname{cl}\{u(t),\ t\in\mathcal{J}_2\}$ is compact lying in G, $U\subset G$ is a convex set; $q^i(t_0,t_1,x_0,x_1)$, $i=0,\dots,l$, $(t_0,t_1,x_0,x_1)\in\mathcal{J}^2\times\mathcal{O}^2$, are continuously differentiable scalar functions.

We consider the differential equation in R^n
$$\dot x(t)=f(t,x(\tau_1(t)),\dots,x(\tau_s(t)),u(\theta_1(t)),\dots,u(\theta_\nu(t))),\tag{1}$$
with the continuous condition
$$x(t)=\varphi(t),\quad t\in[\tau,t_0].\tag{2}$$

Definition 1. The function $x(t)=x(t,\sigma)\subset\mathcal{O}$, $\sigma=(t_0,t_1,\varphi(\cdot),u(\cdot))\in A=\mathcal{J}^2\times\Delta\times\Omega$, $t_0<t_1$, defined on the interval $[\tau,t_1]$, is said to be a solution corresponding to the element $\sigma\in A$ if on the interval $[\tau,t_0]$ it satisfies the condition (2), while on the interval $[t_0,t_1]$ the trajectory $x(t)$ is absolutely continuous and almost everywhere satisfies the equation (1).

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Definition 2. The element $\sigma \in A$ is said to be admissible if the corresponding solution $x(t)$ satisfies the conditions

$$q^i(t_0, t_1, x(t_0), x(t_1)) = 0, \quad i = 1, \dots, l.$$

The set of admissible elements will be denoted by A_0.

Definition 3. The element $\widetilde{\sigma} = (\widetilde{t}_0, \widetilde{t}_1, \widetilde{\varphi}(\cdot), \widetilde{u}(\cdot)) \in A_0$ is said to be optimal if for an arbitrary element $\sigma \in A_0$ the inequality

$$q^0(\widetilde{t}_0, \widetilde{t}_1, \widetilde{x}(\widetilde{t}_0), \widetilde{x}(\widetilde{t}_1)) \le q^0(t_0, t_1, x(t_0), x(t_1)), \quad \widetilde{x}(t) = x(t, \widetilde{\sigma}),$$

holds.

The problem of optimal control consists in finding an optimal element.

In order to formulate the main results, consider the following notation:

$$\omega = (t, x_1, \dots, x_s) \in \mathcal{J} \times \mathcal{O}^s, \quad \omega_0 = (\widetilde{t}_0, \widetilde{x}(\tau_1(\widetilde{t}_0)), \dots, \widetilde{x}(\tau_s(\widetilde{t}_0))),$$
$$\omega_1 = (\widetilde{t}_1, \widetilde{x}(\tau_1(\widetilde{t}_1)), \dots, \widetilde{x}(\tau_s(\widetilde{t}_1))), \quad \gamma_i(t) = \tau_i^{-1}(t),$$
$$\widetilde{f}[\omega] = f(t, x_1, \dots, x_s, \widetilde{u}(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t))),$$
$$\widetilde{f}_{x_i}[t] = f_{x_i}(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)), \widetilde{u}(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t))),$$
$$\widetilde{f}_{u_j}[t] = f_{u_j}(t, \widetilde{x}(\tau_1(t)), \dots, \widetilde{x}(\tau_s(t)), \widetilde{u}(\theta_1(t)), \dots, \widetilde{u}(\theta_\nu(t))),$$
$$R_t^- = (-\infty, t].$$

Theorem 1. *Let $\widetilde{\sigma} \in A_0$ be an optimal element, $\widetilde{t}_0 \in (a, b)$, $\widetilde{t}_1 \in (a, b]$ and the following conditions hold:*

a) *the function $\widetilde{\varphi}(t)$ is absolutely continuous in some left semi-neighborhood of the point $\widetilde{t}_0$;*

b) *there exist the finite limits:*

$$\dot{\varphi}^- = \dot{\widetilde{\varphi}}(\widetilde{t}_0^-);$$
$$\lim_{\omega \to \omega_0} \widetilde{f}[\omega] = f_0^-, \quad \omega \in R_{\widetilde{t}_0}^- \times \mathcal{O}^s; \qquad \lim_{\omega \to \omega_1} \widetilde{f}[\omega] = f_1^-, \quad \omega \in R_{\widetilde{t}_1}^- \times \mathcal{O}^s.$$

Then there exists a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$, and a solution $\psi(t)$, $t \in [\widetilde{t}_0, \gamma]$, $\gamma = \max(\gamma_1(b), \dots, \gamma_s(b))$, of the equation

$$\dot{\psi}(t) = -\sum_{i=1}^{s} \psi(\gamma_i(t)) \widetilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t), \quad t \in [\widetilde{t}_0, \widetilde{t}_1], \tag{3}$$
$$\psi(t) = 0, \quad t \in (\widetilde{t}_1, \gamma],$$

such that the following conditions are fulfilled

$$\sum_{i=1}^{s} \int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0} \psi(t) \widetilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t) \widetilde{\varphi}(t)\, dt \ge$$
$$\ge \sum_{i=1}^{s} \int_{\tau_i(\widetilde{t}_0)}^{\widetilde{t}_0} \psi(t) \widetilde{f}_{x_i}[\gamma_i(t)] \dot{\gamma}_i(t) \varphi(t)\, dt, \quad \forall \varphi(\cdot) \in \Delta; \tag{4}$$
$$\big(\pi \widetilde{Q}_{x_0} + \psi(\widetilde{t}_0)\big) \widetilde{\varphi}(\widetilde{t}_0) \ge \big(\pi \widetilde{Q}_{x_0} + \psi(\widetilde{t}_0)\big) \varphi, \quad \forall \varphi \in M; \tag{5}$$
$$\sum_{j=1}^{\nu} \int_{\widetilde{t}_0}^{\widetilde{t}_1} \psi(t) \widetilde{f}_{u_j}[t] \widetilde{u}(\theta_j(t))\, dt \ge \sum_{j=1}^{\nu} \int_{\widetilde{t}_0}^{\widetilde{t}_1} \psi(t) \widetilde{f}_{u_j}[t] u(\theta_j(t))\, dt, \quad \forall u(\cdot) \in \Omega; \tag{6}$$
$$\pi \widetilde{Q}_{t_1} \ge -\psi(\widetilde{t}_1) f_1^-;$$
$$\pi(\widetilde{Q}_{t_0} + \widetilde{Q}_{x_0} \dot{\varphi}^-) \ge \psi(\widetilde{t}_0)(f_0^- - \dot{\varphi}^-).$$

Here the tilde over $Q=(q^0,\dots,q^l)^\top$ *means the the corresponding gradient is calculated at the point* $(\widetilde{t}_0,\widetilde{t}_1,\widetilde{x}(\widetilde{t}_0),\widetilde{x}(\widetilde{t}_1))$.

Theorem 2. *Let* $\widetilde{\sigma}\in A_0$ *be an optimal element,* $\widetilde{t}_0\in[a,b)$, $\widetilde{t}_1\in(a,b)$ *and the following conditions hold:*

c) *the function* $\widetilde{\varphi}(t)$ *is absolutely continuous in some right semi-neighborhood of the point* $\widetilde{t}_0$*;*

d) *there exist the finite limits:*

$$\dot{\varphi}^+=\dot{\widetilde{\varphi}}(\widetilde{t}_0^+);$$

$$\lim_{\omega\to\omega_0}\widetilde{f}[\omega]=f_0^+,\quad \omega\in R^+_{\widetilde{t}_0}\times\mathcal{O}^s;\qquad \lim_{\omega\to\omega_1}\widetilde{f}[\omega]=f_1^+,\quad \omega\in R^+_{\widetilde{t}_1}\times\mathcal{O}^s.$$

Then there exists a non-zero vector $\pi=(\pi_0,\dots,\pi_l)$, $\pi_0\le 0$, *and a solution* $\psi(t)$ *of the equation* (3) *such that* (4)–(6) *are fulfilled. Moreover,*

$$\pi\widetilde{Q}_{t_1}\le-\psi(\widetilde{t}_1)f_1^+;\qquad \pi(\widetilde{Q}_{t_0}+\widetilde{Q}_{x_0}\dot{\varphi}^+)\le\psi(\widetilde{t}_0)(f_0^+-\dot{\varphi}^+).$$

Theorem 3. *Let* $\widetilde{\sigma}\in A_0$ *be an optimal element,* $\widetilde{t}_0,\widetilde{t}_1\in(a,b)$ *and the assumptions of Theorems* 1, 2 *hold. Let, besides,*

e) $\dot{\varphi}^-=\dot{\varphi}^+=\dot{\varphi}$, $f_0^-=f_0^+=f_0$;

f) $f_1^-=f_1^+=f_1$.

Then there exists a non-zero vector $\pi=(\pi_0,\dots,\pi_l)$, $\pi_0\le 0$, *and a solution* $\psi(t)$ *of the equation* (3) *such that* (4)–(6) *are fulfilled. Moreover,*

$$\pi\widetilde{Q}_{t_1}=-\psi(\widetilde{t}_1)f_1;\qquad \pi(\widetilde{Q}_{t_0}+\widetilde{Q}_{x_0}\dot{\varphi})=\psi(\widetilde{t}_0)(f_0-\dot{\varphi}).$$

We would note that if $\operatorname{rank}(\widetilde{Q}_{t_0},\widetilde{Q}_{t_1},\widetilde{Q}_{x_0},\widetilde{Q}_{x_1})=1+l$, then in Theorem 3 $\psi(t)\not\equiv 0$.

Optimal problems of various classes with commensurable and incommensurable delays in control are considered in [1]–[6].

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