Memoirs on Differential Equations and Mathematical Physics
Volume 34, 2005, 1–76

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APPROXIMATE SOLUTION OF ONE CLASS OF SINGULAR INTEGRAL EQUATIONS BY MEANS OF PROJECTIVE AND PROJECTIVE-ITERATIVE METHODS

Abstract. We consider singular integral equations when the line of integration is the segment $[-1,1]$. Equations are considered in the weight spaces.

For the indices $\varkappa=1$ and $\varkappa=-1$ there are additional conditions which are approximated additionally by other authors. For the index $\varkappa=1$ we narrow the domain of definition of the singular operator, while for the index $\varkappa=-1$ we narrow the range of values of the singular operator. Such a procedure allows one to justify approximate schemes without any difficulty.

Projective-iterative schemes are considered, their convergence is proved and the convergence order is determined. Stability of the projective-iterative schemes is defined and proved.

2000 Mathematics Subject Classification. 65R20.

Key words and phrases. Singular integral equation, projective and projective-iterative methods, convergence and order of convergence of approximate methods, stability.

რეზიუმე. განიხილება სინგულარული ინტეგრალური განტოლება, როცა ინტეგრების წირია მონაკვეთი $[-1,1]$. განტოლებები განიხილება წონიან სივრცეებში.

$\varkappa=1$ და $\varkappa=-1$ ინდექსების შემთხვევებში თავს იჩენს დამატებითი პირობები, რომლებიც სხვა ავტორების მიერ დამატებით აპროქსიმირდება. ჩვენ $\varkappa=1$ ინდექსის შემთხვევაში ვავიწროვებთ სინგულარული ოპერატორის განსაზღვრის არეს, ხოლო $\varkappa=-1$ ინდექსის შემთხვევაში ვავიწროვებთ სინგულარული ოპერატორის მნიშვნელობათა არეს. ასეთი პროცედურა საშუალებას იძლევა ადვილად დავაფუძნოთ მიახლოებითი სქემები.

განიხილება პროექციულ-იტერაციული სქემები, მტკიცდება მათი კრებადობა და დგინდება კრებადობის რიგი.

მოცემულია პროექციულ-იტერაციული სქემების მდგრადობის განმარტება და დამტკიცებულია მათი მდგრადობა.

1. Introduction

We consider the following singular integral equation [1]:

$$a\varphi(x) + \frac{b}{\pi}\int\limits_{-1}^{1}\frac{\varphi(t)dt}{t-x} + \frac{1}{\pi}\int\limits_{-1}^{1}K(x,t)\varphi(t)dt = f(x), \quad -1 < x < 1, \tag{1.1}$$

where a and b are real numbers, $K(x,t)$ and $f(x)$ are real functions, $x, t \in [-1, 1]$, $a^2+b^2 = 1$. The equation (1.1) is called the equation of the first kind for $a = 0$ and the equation of the second kind for $a \neq 0$. Singular integral equations of the first and the second kind are theoretically identical; the only difference is in the choice of weight spaces and coordinate (basis) functions in the projective method. We consider the singular integral equation (1.1) in weight spaces [2]. The index of the equation (1.1), $\varkappa = -(\alpha+\beta)$, where

$$\alpha = \frac{1}{2\pi i}\ln\Big(\frac{a-ib}{a+ib}\Big) + N, \quad \beta = -\frac{1}{2\pi i}\ln\Big(\frac{a-ib}{a+ib}\Big) + M;$$

here N and M are integers which we choose as follows:
1) $\varkappa = 1$, $\quad -1 < \alpha,\ \beta < 0$;
2) $\varkappa = -1$, $\quad 0 < \alpha,\ \beta < 1$;
3) $\varkappa = 0$, $\quad \alpha = -\beta,\ 0 < |\alpha| < 1$.
These cases cover the well-known problems of mechanics.

The use will be made of the following short notation for the equations of the first and the second kind, respectively,

$$S\varphi + K\varphi = f, \quad (a+bS)\varphi + K\varphi = f,$$

which will be considered in the weight space $L_{2,\rho}(-1,1)$. The notation $\varphi \in L_{2,\rho}(-1,1)$ means that

$$\int\limits_{-1}^{1}\varphi^2\rho dx < +\infty.$$

The problems of approximate solution of singular integral equations go back to the work of M. A. Lavrent'ev [3]. Subsequently, these problems were studied by H. Multop, V. Ivanov, M. Schleif, S. Mikhlin, Z. Prössdorf, B. Gabdulkhaev, J. Sanikidze, B. Musayev, A. Kalandiya, M. Gagua, I. Lifanov, F. Erdogan, G. Gupta, S. Krenk, M. Sheshko, G. Thamasphyros, P. Theocaris and other authors.

My first works dealing with the projective and collocation methods for approximate solution of the singular integral equation (1.1) were published in the Journal of Computational Mathematics and Mathematical Physics in 1979 and 1981. These works suggest approximate schemes which somewhat differ from the earlier known ones for the indices $\varkappa = 1$ and $\varkappa = -1$. Justification of these schemes is also given.

In 1993 there appeared the work of D. Porter and D.S.G. Stirling in which the authors suggested the cyclic projective-iterative scheme for the equation of the first kind $(I+T)u = f$ in the Banach space. This scheme has been used in our joint work with G. Khvedelidze for the singular integral equation (1.1).

The present paper combines the results obtained by the author in the recent years. Moreover, it concerns the problem of stability of the projective-iterative scheme.

The work is not a survey. In References we indicate the works which are used in the proof of theorems.

2. Projective Method for the Singular Integral Equation of the First Kind

We consider the equation

$$S\varphi + K\varphi = f, \tag{2.1}$$

where $S\varphi \equiv \frac{1}{\pi}\int\limits_{-1}^{1} \frac{\varphi(t)dt}{t-x}$, $-1 < x < 1$,

$$K\varphi \equiv \frac{1}{\pi}\int\limits_{-1}^{1} K(x,t)\varphi(t)dt, \quad x,t \in [-1,1].$$

For the above equation there may take place three cases: 1) $\varkappa = 1$, $\alpha = \beta = -\frac{1}{2}$; 2)$\varkappa = -1$, $\alpha = \beta = -\frac{1}{2}$; 3) $\varkappa = 0$, $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, or $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$.

All these cases will be considered separately.

1) $\varkappa = 1$, $\alpha = \beta = -\frac{1}{2}$. Let us introduce the weight space $L_{2,\rho_1}[-1,1]$, where the weight $\rho_1 = (1-x^2)^{1/2}$, and the scalar product

$$[u,v] = ((1-x^2)^{1/2}u, v) = \int\limits_{-1}^{1} (1-x^2)^{1/2} u(x)v(x)dx.$$

Any function of the type $\varphi(x) = (1-x^2)^{-\frac{1}{2}}\varphi_0$, where $\varphi_0(x)$ is a bounded measurable function, belongs to the space L_{2,ρ_1}.

Let the function $K(x,t)$ satisfy the condition

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1} K^2(x,t)(1-x^2)^{\frac{1}{2}}(1-t^2)^{-\frac{1}{2}}\, dtdx < +\infty.$$

Then the operator K is completely continuous in L_{2,ρ_1}. It is known that the singular operator S is bounded in L_{2,ρ_1} [4].

The equation (2.1) will be considered in the weight space L_{2,ρ_1}; f, $\varphi \in L_{2,\rho_1}$.

Let us take the Chebyshev polynomials [5]

$$T_k(x) = \cos(k \arccos x), \quad k = 0,1,\ldots, \quad |x| \le 1.$$

The system of functions

$$\widehat{T}_0(x) = \pi^{-\frac{1}{2}} T_0, \quad \widehat{T}_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} T_k(x), \quad k = 1, 2, \ldots,$$

is complete and orthonormal with the weight $(1 - x^2)^{-\frac{1}{2}}$ in the space $L_2[-1, 1]$. Therefore the system of functions

$$\varphi_k \equiv (1 - x^2)^{-\frac{1}{2}} \widehat{T}_k(x), \quad k = 0, 1, \ldots,$$

is complete and orthonormal in the space L_{2,ρ_1}. Moreover, the system of functions

$$\psi_{k+1} \equiv \left(\frac{2}{\pi}\right)^{\frac{1}{2}} U_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (1 - x^2)^{-\frac{1}{2}} \sin((k+1) \arccos x), \ k = 0, 1, \ldots,$$

where U_k, $k = 0, 1, \ldots$, are the Chebyshev polynomials of the second kind, is also complete and orthonormal in the space L_{2,ρ_1}.

The following formulas are well-known [6]:

$$\left.\begin{aligned} &S[T_k(t)(1 - t^2)^{-\frac{1}{2}}] = U_{k-1}(x), && k = 1, 2, \ldots, \\ &S[U_{k-1}(t)(1 - t^2)^{\frac{1}{2}}] = -T_k(x), && k = 1, 2, \ldots, \\ &S(1 - t^2)^{-\frac{1}{2}} = 0. && \end{aligned}\right\} \tag{2.2}$$

The first and the third relations from (2.2) provide us with

$$\left.\begin{aligned} &S\varphi_0 = 0, \\ &S\varphi_k = \psi_k, \quad k = 1, 2, \ldots. \end{aligned}\right\} \tag{2.3}$$

We decompose the space L_{2,ρ_1} as the orthogonal sum $L_{2,\rho_1} = L^{(1)}_{2,\rho_1} \oplus L^{(2)}_{2,\rho_1}$, where $L^{(1)}_{2,\rho_1}$ is the linear span of the function $\varphi_0 = (\pi(1 - x^2))^{-\frac{1}{2}}$. The null-space of the operator S is one-dimensional. The conjugate operator $S^* = -(1 - t^2)^{-\frac{1}{2}} S(1 - x^2)^{\frac{1}{2}}$. The equation $S^*\varphi = 0$ in the space L_{2,ρ_1} has only zero solution which corresponds to the index $\varkappa = 1$. Now we restrict the domain of definition of the operator S and consider it not in $L^{(2)}_{2,\rho_1}$ but only in $L^{(2)}_{2,\rho_1}$. We denote the restricted operator by S. Then the operator S transforms the orthonormal basis $\varphi_1, \varphi_2, \ldots$ of the space $L^{(2)}_{2,\rho_1}$ into the orthonormal basis $\psi_1, \psi_2, \ldots$ of the space L_{2,ρ_1}. Therefore the operator S is isometric, $S(L^{(2)}_{2,\rho_1}) = L_{2,\rho_1}$. There exists an inverse operator $S^{-1}(L_{2,\rho_1}) = L^{(2)}_{2,\rho_1}$.

To single out a unique solution of the equation (2.1), in applied problems for $\varkappa = 1$ the additional condition

$$\int_{-1}^{1} \varphi(t) dt = p \tag{2.4}$$

is imposed, where p is a given number.

Let us introduce a new unknown function

$$\phi(t) \equiv \varphi(t) - p\pi^{-1}(1-t^2)^{-\frac{1}{2}}.$$

Then the equation (2.1) with the condition (2.4) can be written in the form

$$S\phi + K\phi = f_1, \tag{2.5}$$

where $f_1 \equiv f - p\pi^{-1}K(1-t^2)^{-\frac{1}{2}}$,

$$\int_{-1}^{1} \phi(t)\,dt = 0. \tag{2.6}$$

The condition (2.6) is obtained from (2.4) if we take into account that $\int_{-1}^{1}(1-t^2)^{-\frac{1}{2}}\,dt = \pi$. The condition (2.6) implies that

$$[\phi, \varphi_0] = 0,$$

i.e., $\phi \in L^{(2)}_{2,\rho_1}$. Therefore the operator $S+K$ can be considered from $L^{(2)}_{2,\rho_1}$ into L_{2,ρ_1}. Thus we have

$$S\phi + K\phi = f_1, \quad \phi \subset L^{(2)}_{2,\rho_1}, \quad f_1 \in L_{2,\rho_1}. \tag{2.7}$$

The term f_1 involves the integral $K(1-t^2)^{-\frac{1}{2}}$ with a weak singularity. The corresponding approximate formulas are known [7].

Let there exist the inverse operator $(S+K)^{-1}$ mapping L_{2,ρ_1} onto $L^{(2)}_{2,\rho_1}$; this is equivalent to the existence of the inverse operator $(I+KS^{-1})^{-1}$ transforming L_{2,ρ_1} onto itself.

An approximate solution of the equation (2.7) is sought in the form

$$\phi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k.$$

The residual $(S+K)\phi^{(n)} - f_1$ is required to be orthogonal to the functions $\psi_1, \psi_2, \ldots, \psi_n$,

$$[S\phi^{(n)} + K\phi^{(n)} - f_1, \psi_i] = 0, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

Taking into account (2.3) and the fact that the system $\psi_1, \psi_2, \ldots$ is orthonormal, we obtain the following algebraic system

$$a_i + \sum_{k=1}^{n} a_k [K\varphi_k, \psi_i] = [f_1, \psi_i], \quad i = 1, 2, \ldots, n. \tag{2.9}$$

To compose this system, we have to calculate the integrals

$$[K\varphi_k, \psi_i] = \frac{2}{\pi^2} \int_{-1}^{1}\int_{-1}^{1} K(x,t)\Big(\frac{1-x^2}{1-t^2}\Big)^{\frac{1}{2}} \cos(k \arccos t) \sin(\arccos x) dt dx$$

and

$$[f_1,\psi_i]=\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\int\limits_{-1}^{1} f_1(x)\sin(i\arccos x)dx,\quad k,i=1,2,\dots,n.$$

Theorem 2.1. *If there exists the inverse operator $(I+KS^{-1})^{-1}$ mapping L_{2,ρ_1} onto itself, then the algebraic system (2.9) for sufficiently large n has a unique solution $(a_1,a_2,\dots,a_n)$, and the sequence of approximate solutions $\{\varphi^{(n)}\}$ converges to the exact solution of the initial problem in the space L_{2,ρ_1}.*

Proof. The inverse operator $S^{-1}(L_{2,\rho_1})=L^{(2)}_{2,\rho_1}$ exists and therefore the equation (2.7) can formally be rewritten in the form

$$(I+KS^{-1})S\phi=f.$$

Denote $w\equiv S\phi$. Then we have the equation

$$(I+KS^{-1})w=f_1,\quad w,f_1\in L_{2,\rho_1}. \tag{2.10}$$

Let us seek an approximate solution of the equation (2.10) in the form

$$w^{(n)}=\sum_{k=1}^{n} b_k\psi_k$$

by the Bubnov–Galerkin method

$$[(I+KS^{-1})w^{(n)}-f_1,\psi_i]=0,\quad i=1,2,\dots,n. \tag{2.11}$$

Using the orthoprojector

$$p_n v=\sum_{k=1}^{n}[v,\psi_k]\psi_k,\quad v\in L_{2,\rho_1},$$

we write the equation (2.11) as follows:

$$w^{(n)}+P_nKS^{-1}w^{(n)}=P_nf_1,\quad w^{(n)}\in \overline{L}^{(n)}_{2,\rho_1}, \tag{2.12}$$

where $\overline{L}^{(n)}_{2,\rho_1}$ is the linear span of the functions $\psi_1,\psi_2,\dots,\psi_n$.

The operator KS^{-1} is completely continuous in L_{2,ρ_1}, the inverse operator $(I+KS^{-1})^{-1}$ exists, and the system $\psi_1,\psi_2,\dots$ is a basis in L_{2,ρ_1}. Thus, since as is known ([8], [9]) $\|P^{(n)}KS^{-1}\|\to 0$ as $n\to\infty$ ($P^{(n)}\equiv I-P_n$), for sufficiently large n the equation (2.12) has a unique solution $w^{(n)}$, and the sequence of approximate solutions of the equation (2.10) converges to the exact solution:

$$\|w^{(n)}-w\|\to 0 \quad\text{as}\quad n\to\infty. \tag{2.13}$$

Then the algebraic system (2.11) has a unique solution $(b_1,b_2,\dots,b_n)$, since $\psi_1,\psi_2,\dots,\psi_n$ are linearly independent. It remains to note that the algebraic systems (2.11) and (2.9) coincide.

Next,

$$\phi^{(n)}-\phi=S^{-1}S(\phi^{(n)}-\phi)=S^{-1}(w^{(n)}-w),$$

and hence taking into account (2.13), we have

$$\|\phi^{(n)} - \phi\|_{L^{(2)}_{2,\rho_1}} = \|w^{(n)} - w\|_{L_{2,\rho_1}} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.14}$$

An approximate solution of the initial problem will be

$$\varphi^{(n)} = \phi^{(n)} + \pi^{-1}P(1-t^2)^{-\frac{1}{2}}.$$

We have

$$\varphi^{(n)} - \varphi = \phi^{(n)} - \phi.$$

As a result, we obtain the convergence in the weight space

$$\int\limits_{-1}^{1} (1-x^2)^{\frac{1}{2}} (\varphi^{(n)} - \varphi)^2 dx \to 0 \quad \text{as} \quad n \to \infty.$$

The theorem is proved. □

If the operator $K = 0$, i.e., if we have a purely characteristic singular equation

$$S\varphi = f \tag{2.15}$$

with the condition

$$\int\limits_{-1}^{1} \varphi(x)dx = P,$$

then $f_1 = f$, $\phi = \varphi - \pi^{-1}P(1-t^2)^{-\frac{1}{2}}$ and hence we have the equation

$$S\phi = f, \quad \phi \in L^{(2)}_{2,\rho_1}, \quad f \in L_{2,\rho_1}. \tag{2.16}$$

The algebraic system (2.9) takes the form

$$a_i = [f, \psi_i], \quad i = 1, 2, \dots, n; \tag{2.17}$$

it remains to calculate the Fourier coefficients

$$[f, \psi_i] = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \int\limits_{-1}^{1} f(x)(1-x^2)^{\frac{1}{2}} U_{i-1}(x)dx, \quad i = 1, 2, \dots.$$

2) $\varkappa = -1$, $\alpha = \beta = \frac{1}{2}$. Introduce the space $L_{2,\rho_2}[-1,1]$ with the weight $\rho_2 = (1-x^2)^{-\frac{1}{2}}$. Any function of the type $\varphi(x) = (1-x^2)^{\frac{1}{2}}\varphi_0(x)$, where $\varphi_0(x)$ is a bounded measurable function, belongs to the space L_{2,ρ_2}. Let

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1} K^2(x,t)(1-t^2)^{\frac{1}{2}}(1-x^2)^{-\frac{1}{2}}\, dtdx < +\infty.$$

Then the operator K is completely continuous in L_{2,ρ_2} and the operator S is bounded. The equation $S\varphi = 0$ in the space L_{2,ρ_2} has only zero solution, because $(1-x^2)^{-\frac{1}{2}} \overline{\in} L_{2,\rho_2}$. The conjugate operator has the form

$$S^* = -(1-t^2)^{\frac{1}{2}} S (1-x^2)^{-\frac{1}{2}}.$$

The equation $S^*\varphi = 0$ in the space L_{2,ρ_2} has the nonzero solution $\varphi = 1$, $1 \in L_{2,\rho_2}$. If the equation

$$S\varphi + K\varphi = f \tag{2.18}$$

has a solution φ, then

$$[K\varphi - f, 1] = 0. \tag{2.19}$$

This condition will be fulfilled if $K(L_{2,\rho_2}) \perp 1$ and $[f, 1] = 0$. The condition $K(L_{2,\rho_2}) \perp 1$ means that

$$\int_{-1}^{1} K(x,t)(1-x^2)^{-\frac{1}{2}}\, dx = 0. \tag{2.20}$$

Here we assume that the above-given conditions are fulfilled. Later on, these restrictions will be removed.

In the space L_{2,ρ_2} the system of functions

$$\varphi_{k+1} \equiv \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} U_k(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \sin[(k+1)\arccos x], \quad k=0,1,\dots,$$

is complete and orthonormal [10].

Denote

$$\psi_{k+1} \equiv -\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \cos[(k+1)\arccos x] = -\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} T_{k+1}(x), \quad k = 0,1,\dots.$$

The second relation from (2.2) provides us with

$$S\varphi_k = \psi_k, \quad k = 1,2,\dots \tag{2.21}$$

We now decompose the space L_{2,ρ_2} into the orthogonal sum $L_{2,\rho_2} = L^{(1)}_{2,\rho_2} \oplus L^{(2)}_{2,\rho_2}$, where $L^{(1)}_{2,\rho_2}$ is the linear span of the function $\psi_0 = -\pi^{-\frac{1}{2}}$ and $L^{(2)}_{2,\rho_2}$ is the orthogonal complement of the space $L^{(1)}_{2,\rho_2}$. The systems of functions $\varphi_1, \varphi_2, \dots$ and $\psi_1, \psi_2, \dots$ are orthonormal and complete in the spaces L_{2,ρ_2} and $L^{(2)}_{2,\rho_2}$, respectively. As the relation (2.21) shows, the operator S transforms an orthonormal basis of the space L_{2,ρ_2} onto an orthonormal basis of the space $L^{(2)}_{2,\rho_2}$. Therefore the operator S is isometric and there exists S^{-1} which maps $L^{(2)}_{2,\rho_2}$ onto L_{2,ρ_2}. The function $f \in L^{(2)}_{2,\rho}$, and therefore the equation (2.18) will be considered from L_{2,ρ_2} to $L^{(2)}_{2,\rho_2}$,

$$S\varphi + K\varphi = f, \quad \varphi \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}. \tag{2.22}$$

Assume that there exists $(S+K)^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto L_{2,ρ_2}, which is equivalent to the existence of the inverse operator $(I + KS^{-1})^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto itself.

We seek an approximate solution of the equation (2.22) in the form

$$\varphi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k$$

and compose the algebraic system from the following conditions:

$$[(S+K)\varphi^{(n)} - f, \psi_i] = 0, \quad i = 1, 2, \dots, n.$$

This, with regard to (2.21) and the fact that the functions $\psi_1, \psi_2, \dots,$ are orthonormal in $L^{(2)}_{2,\rho_2}$, provides us with

$$a_i + \sum_{k=1}^{n} a_k [K\varphi_k, \psi_i] = [f, \psi_i], \quad i = 1, 2, \dots, n. \tag{2.23}$$

To compose the above system, we have to calculate the integrals

$$[K\varphi_k, \psi_i] = -\frac{2}{\pi^2} \int_{-1}^{1}\int_{-1}^{1} K(x,t)(1-x^2)^{-\frac{1}{2}} \sin(k \arccos t) \cos(i \arccos x) dt\, dx,$$

$$[f, \psi_i] = -\Big(\frac{2}{\pi^2}\Big)^{\frac{1}{2}} \int_{-1}^{1} f(x)(1-x^2)^{-\frac{1}{2}} \cos(i \arccos x) dx, \quad k, i = 1, 2, \dots, n.$$

Theorem 2.2. *If there exists the inverse operator $(I+KS^{-1})^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto itself, then the algebraic system* (2.23) *for sufficiently large n has a unique solution $(a_1, a_2, \dots, a_n)$, and the sequence of approximate solutions $\{\varphi^{(n)}\}$ converges to the exact solution in the space L_{2,ρ_2}, i.e.,*

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} (\varphi^{(n)} - \varphi)^2 dx \to 0, \quad n \to \infty.$$

This theorem can be proved just in the similar way as Theorem 2.1.

If the operator $K = 0$, then the algebraic system (2.23) takes the form

$$a_i = [f, \psi_i], \quad i = 1, 2, \dots, n,$$

where

$$[f, \psi_i] = -\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \int_{-1}^{1} f(x)(1-x^2)^{-\frac{1}{2}} T_i(x) dx, \quad i = 1, 2, \dots, n.$$

3) $\varkappa = 0$, $\alpha = -\beta$, $|\alpha| = |\beta| = \frac{1}{2}$. Here we may have two cases (a) $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and (b) $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$. Let us introduce the space $L_{2,\rho_3}[-1,1]$ with the weight $\rho_3 = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$. Any function of the type $\varphi(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}\varphi_0(x)$, where $\varphi_0(x)$ is bounded and measurable, belongs to the space L_{2,ρ_3}.

Let

$$\int_{-1}^{1}\int_{-1}^{1} K^2(x,t) \Big(\frac{1-x}{1+x}\Big)^{\frac{1}{2}} \Big(\frac{1+t}{1-t}\Big)^{\frac{1}{2}} dt dx < +\infty.$$

Then the integral operator K is completely continuous in L_{2,ρ_3} and the operator S is bounded in L_{2,ρ_3} [4]. The equation

$$S\varphi + K\varphi = f, \quad \varphi, f \in L_{2,\rho_3} \tag{2.24}$$

is considered in the space L_{2,ρ_3}. The function $(1-x^2)^{-\frac{1}{2}} \overline{\in} L_{2,\rho_3}$, thus the equation $S\varphi = 0$ in the space L_{2,ρ_3} has only zero solution. The conjugate operator has the form

$$S^* = -\Big(\frac{1+t}{1-t}\Big)^{\frac{1}{2}} S \Big(\frac{1-x}{1+x}\Big)^{\frac{1}{2}}.$$

The equation $S^*\varphi = 0$ has only zero solution in L_{2,ρ_3}. Consider the system of functions [11]

$$\varphi_k \equiv c_k(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}} P_k^{(-\frac{1}{2},\frac{1}{2})}(x), \quad k = 0, 1, \dots,$$

where $P_k^{(-\frac{1}{2},\frac{1}{2})}(x)$ are Jacobi polynomials,

$$c_k \equiv (h_k^{(-\frac{1}{2},\frac{1}{2})})^{-\frac{1}{2}} = \Big[\frac{2\Gamma(k+\frac{1}{2})\Gamma(k+\frac{3}{2})}{(2k+1)(k!)^2}\Big]^{-\frac{1}{2}}, \quad k = 0, 1, \dots,$$

$$P_k^{(-\frac{1}{2},\frac{1}{2})}(x) = e_k \frac{\cos(\frac{2k+1}{2}\arccos x)}{\cos(\frac{1}{2}\arccos x)}, \quad k = 0, 1, \dots,$$

$$e_0 = 1, \quad e_k = \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}, \quad k = 1, 2, \dots.$$

Thus the system of functions $\varphi_0, \varphi_1, \dots$ is orthonormal and complete in the space L_{2,ρ_3}.

Take the Jacobi polynomials [11]:

$$P_k^{(\frac{1}{2},-\frac{1}{2})}(x) = e_k \frac{\sin(\frac{2k+1}{2}\arccos x)}{\sin(\frac{1}{2}\arccos x)}, \quad k = 0, 1, \dots,$$

and denote

$$\psi_k \equiv c_k P_k^{(\frac{1}{2},-\frac{1}{2})}(x), \quad k = 0, 1, \dots.$$

The system of functions $\psi_0, \psi_1, \dots$ is complete and orthonormal in the space L_{2,ρ_3},

$$(c_k^{-\frac{1}{2}} = h_k^{(-\frac{1}{2},\frac{1}{2})} = h_k^{(\frac{1}{2},-\frac{1}{2})}).$$

Let us show that

$$S\varphi_k = \psi_k, \quad k = 0, 1, \dots. \tag{2.25}$$

We have

$$S\varphi_k = c_k e_k \pi^{-1} \int_{-1}^{1} \frac{(1-t)^{-\frac{1}{2}}(1+t)^{\frac{1}{2}} \cos(\frac{2k+1}{2}\arccos t)}{(t-x)\cos(\frac{1}{2}\arccos t)}\, dt =$$

$$= c_k e_k \pi^{-1} \int_{0}^{\pi} \frac{(1-\cos\tau)^{-\frac{1}{2}}(1+\cos\tau)^{\frac{1}{2}} \cos\frac{2k+1}{2}\tau \sin\tau d\tau}{\cos\frac{\tau}{2}(\cos\tau - \cos\xi)} =$$

$$= c_k e_k \pi^{-1} \int_0^\pi \frac{(1-\cos\tau)^{-\frac12}(1+\cos\tau)^{\frac12}(1-\cos x)^{\frac12}\cos\frac{2k+1}{2}\tau d\tau}{\cos\frac{\tau}{2}(\cos\tau-\cos\xi)} =$$

$$= c_k e_k \pi^{-1} \int_0^\pi \frac{(1+\cos\tau)\cos\frac{2k+1}{2}\tau}{\cos\frac{\tau}{2}(\cos\tau-\cos\xi)}\,d\tau = c_k e_k \pi^{-1} \int_0^\pi \frac{2\cos\frac{2k+1}{2}\tau\cos\frac{\tau}{2}}{\cos\tau-\cos\xi}\,d\tau =$$

$$= c_k e_k \pi^{-1} \int_0^\pi \frac{\cos(k+1)\tau+\cos k\tau}{\cos\tau-\cos\xi}\,d\tau.$$

Taking into account the known relation

$$\pi^{-1} \int_0^\pi \frac{\cos k\tau}{\cos\tau-\cos\xi}\,d\tau = \frac{\sin k\xi}{\sin\xi}, \quad k = 0, 1, \dots,$$

which is obtained from (2.2), we have

$$S\varphi_k = c_k e_k \Big(\frac{\sin(k+1)\xi}{\sin\xi} + \frac{\sin k\xi}{\sin\xi}\Big) = c_k e_k \frac{2\sin\frac{2k+1}{2}\xi\cos\frac{\xi}{2}}{2\sin\frac{\xi}{2}\cos\frac{\xi}{2}} =$$

$$= c_k e_k \frac{\sin\frac{2k+1}{2}\xi}{\sin\frac{\xi}{2}} = c_k P_k^{(\frac12,-\frac12)}(x), \quad k = 0, 1, \dots, \quad x = \cos\xi.$$

Thus the operator S transforms an orthonormal complete system of functions $\varphi_0, \varphi_1, \dots$ of the space L_{2,ρ_3} onto another orthonormal complete system $\psi_0, \psi_1, \dots$ of the same space. Hence the operator S is unitary; $S(L_{2,\rho_3}) = L_{2,\rho_3}$, $S^{-1}(L_{2,\rho_3}) = L_{2,\rho_3}$.

An approximate solution of the equation (2.24) is sought in the form

$$\varphi^{(n)} = \sum_{k=0}^n a_k \varphi_k$$

and the algebraic system is composed by means of the following conditions:

$$[(S+K)\varphi^{(n)} - f, \psi_i] = 0, \quad i = 0, 1, \dots, n,$$

which, taking into account (2.25) and the fact that the functions $\psi_0, \psi_1, \dots$ are orthonormal, provides us with

$$a_i + \sum_{k=0}^n a_k [K\varphi_k, \psi_i] = [f, \psi_i], \quad i = 0, 1, \dots, n. \tag{2.26}$$

To compose this system, we have to calculate the integrals

$$[K\varphi_k, \psi_i] = \frac{(h_k^{(-\frac12,\frac12)} h_i^{(\frac12,-\frac12)})^{-\frac12} 1\cdot 3\cdots(2k-1)\cdot 1\cdot 3\cdots(2i-1)}{\pi(2\cdot 4\cdots 2k)(2\cdot 4\cdots 2i)} \times$$

$$\times \int_{-1}^1 \int_{-1}^1 K(x,t)\Big(\frac{1-x}{1+x}\Big)^{\frac12}\Big(\frac{1+t}{1-t}\Big)^{\frac12} \cos\Big(\frac{2k+1}{2}\arccos t\Big) \times$$

$$\times \sin\Big(\frac{2i+1}{2}\arccos x\Big)[\cos(\frac{1}{2}\arccos t)\sin(\frac{1}{2}\arccos x)]^{-1}\,dtdx,$$

$$[f,\psi_i]=\Big(h_i^{(\frac{1}{2},-\frac{1}{2})}\Big)^{-\frac{1}{2}}\frac{1\cdot 3\cdot(2i-1)}{2\cdot 4\cdots 2i}\times$$

$$\times\int\limits_{-1}^{1}\Big(\frac{1-x}{1+x}\Big)^{\frac{1}{2}}f(x)\sin\Big(\frac{2i+1}{2}\arccos x\Big)[\sin(\frac{1}{2}\arccos x)]^{-1}\,dx,$$

$k,i=0,1,\dots,n$ (for $k=0$, instead of $\frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}$ we take unity).

Theorem 2.3. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *mapping* L_{2,ρ_3} *onto itself, then the algebraic system* (2.26) *for sufficiently large n has a unique solution* $(a_0,a_1,\dots,a_n)$*, and the sequence of approximate solutions* $\{\varphi^{(n)}\}$ *converges to the exact solution* φ *in the space* $L_{2,\rho}$.

This theorem can be proved by repeating word by word the arguments of Theorem 2.1; the subspace is the linear span of the functions $\psi_0,\psi_1,\dots,\psi_n$.

Consider now the case $\alpha=\frac{1}{2}$ and $\beta=-\frac{1}{2}$. The reasoning is the same, so we give only the formulas needed for practical application. $\rho_3=(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}$ is the weight. The kernel $k(x,t)$ must satisfy the condition

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1}k^2(x,t)\Big(\frac{1+x}{1-x}\Big)^{\frac{1}{2}}\Big(\frac{1-t}{1+t}\Big)^{\frac{1}{2}}\,dtdx<+\infty.$$

We take the following systems of functions:

1) $\varphi_k\equiv c_k(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}P_k^{(\frac{1}{2},-\frac{1}{2})}(x)$, $k=0,1,\dots$; $c_k\equiv(h_k^{(\frac{1}{2},-\frac{1}{2})})^{-\frac{1}{2}}=(h_k^{(-\frac{1}{2},\frac{1}{2})})^{-\frac{1}{2}}=\Big[\frac{2\Gamma(k+\frac{1}{2})\Gamma(k+\frac{3}{2})}{(2k+1)(k!)^2}\Big]^{-\frac{1}{2}}$, $k=0,1,\dots$, $P_k^{(\frac{1}{2},-\frac{1}{2})}(x)=e_k\times\frac{\sin(\frac{2k+1}{2}\arccos x)}{\sin(\frac{1}{2}\arccos x)}$, $k=0,1,\dots$, $e_0=1$, $e_k=\frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}$, $k=1,2,\dots$.

2) $\psi_k=-c_kP_k^{(-\frac{1}{2},\frac{1}{2})}(x)$, $k=0,1,\dots$.

The formula

$$S\varphi_k=\psi_k,\quad k=0,1,\dots,$$

is valid. The weight $\rho_3=(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}$, and the equation

$$S\varphi+K\varphi=f,\quad \varphi,f\in L_{2,\rho_3}.$$

An approximate solution is sought in the form

$$\varphi^{(n)}=\sum_{k=0}^{n}a_k\varphi_k$$

and we obtain the algebraic system

$$a_i+\sum_{k=0}^{n}a_k[K\varphi_k,\psi_i]=[f,\psi_i],\quad i=0,1,\dots,n, \tag{2.27}$$

where

$$[K\varphi_k,\psi_i]=\frac{-(h_k^{(\frac12,-\frac12)}h_i^{(-\frac12,\frac12)})^{-\frac12}1\cdot3\cdots(2k-1)\cdot1\cdot3\cdots(2i-1)}{\pi(2\cdot4\cdots2k)(2\cdot4\cdots2i)}\times$$

$$\times\int_{-1}^{1}\int_{-1}^{1}K(x,t)\Big(\frac{1+x}{1-x}\Big)^{\frac12}\Big(\frac{1-t}{1+t}\Big)^{\frac12}\sin\Big(\frac{2k+1}{2}\arccos t\Big)\times$$

$$\times\cos\Big(\frac{2i+1}{2}\arccos x\Big)\Big[\sin\Big(\frac12\arccos t\Big)\cos\Big(\frac12\arccos x\Big)\Big]^{-1}dt\,dx,$$

$$[f,\psi_i]=-\Big(h_i^{(-\frac12,\frac12)}\Big)^{-\frac12}\frac{1\cdot3\cdot(2i-1)}{2\cdot4\cdots2i}\times$$

$$\times\int_{-1}^{1}\Big(\frac{1+x}{1-x}\Big)^{\frac12}f(x)\cos\Big(\frac{2i+1}{2}\arccos x\Big)[\cos(\frac12\arccos x)]^{-1}dx,$$

$k,i=0,1,\dots,n$ (for $k=0$ instead of $\frac{1\cdot3\cdots(2k-1)}{2\cdot4\cdots2k}$ we take unity).

3. Projective Method for the Singular Integral Equation of Second Kind

We take the equation

$$a\varphi(x)+\frac{b}{\pi}\int_{-1}^{1}\frac{\varphi(t)dt}{t-x}+\frac{1}{\pi}\int_{-1}^{1}K(x,t)\varphi(t)dt=f(x),$$

which in the notation of Section 1 is written as

$$(a+bS+K)\varphi=f. \tag{3.1}$$

For that equation we consider three values of the index:
1) $\varkappa=1$, $\varkappa=-(\alpha+\beta)$, $-1<\alpha,\beta<0$, $\beta=-1-\alpha$, $|\alpha|\neq\frac12$,
2) $\varkappa=-1$, $\varkappa=-(\alpha+\beta)$, $0<\alpha,\beta<1$, $\beta=1-\alpha$, $|\alpha|\neq\frac12$,
3) $\varkappa=0$, $\varkappa=-(\alpha+\beta)$, $0<|\alpha|,|\beta|<1$, $\beta=-\alpha$, $|\alpha|\neq\frac12$.

Here we present the needed formulas from [1], [6], [11] and [12]. The Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, $\alpha>-1$, $\beta>-1$, $k=0,1,\dots$, are defined by means of the condition

$$(1-x)^\alpha(1+x)^\beta P_k^{(\alpha,\beta)}(x)=\frac{(-1)^k}{2^kk!}\frac{d^k}{dx^k}[(1-x)^{k+\alpha}(1+x)^{k+\beta}],\quad -1\le x\le1.$$

The polynomials

$$\widehat{P}_k^{(\alpha,\beta)}(x)=(h_k^{(\alpha,\beta)})^{-\frac12}P_k^{(\alpha,\beta)}(x),\quad k=0,1,\dots,$$

where

$$h_k^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}$$

(for $k = 0$, we replace the product $(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)$ by $\Gamma(\alpha+\beta+2)$), $\widehat{P}_k^{(\alpha,\beta)}(x)$ are orthonormal with the weight $(1-x)^\alpha(1+x)^\beta$. For all values of the index $\varkappa$ the formula [12]

$$a(1-x)^\alpha(1+x)^\beta P_i^{(\alpha,\beta)}(x) + bS[(1-t)^\alpha(1+t)^\beta P_i^{(\alpha,\beta)}(t)] =$$
$$= -2^{-\varkappa}\pi^{-1}\Gamma(\alpha)\Gamma(1-\alpha)P_{i-\varkappa}^{(-\alpha,-\beta)}(x), \quad i = 0,1,\dots, -1 < x < 1, \quad (3.2)$$

is valid; for $\varkappa = 1$ and $i = 0$ we assume that $P_{-1}^{(-\alpha,-\beta)}(x) \equiv 0$.

We consider every value of the above-mentioned index separately.

1. $\varkappa = 1$, $\varkappa = -(\alpha+\beta)$ and α, β are defined from the conditions $a + b\operatorname{ctg}\alpha\pi = 0$, $-1 < \alpha < 0$, $\beta = -1-\alpha$. Introduce here a space with the weight $L_{2,\rho_1}[-1,1]$, where $\rho_1 = (1-x)^{-\alpha}(a+x)^{-\beta}$. A solution of the equation (3.1) of the type $\varphi(x) = (1-x)^\alpha(1+x)^\beta\varphi_0(x)$, $-1 < \alpha, \beta < 0$, where $\varphi_0(x)$ is a bounded measurable function, belongs to the space L_{2,ρ_1}.

Let the function $K(x,t)$ satisfy the condition

$$\int_{-1}^{1}\int_{-1}^{1} K^2(x,t)(1-x)^{-\alpha}(1+x)^{-\beta}(1-t)^\alpha(1+t)^\beta dt dx < +\infty.$$

Then the integral operator K is completely continuous in L_{2,ρ_1}. The singular integral operator S is bounded in L_{2,ρ_1} [4]. The homogeneous equation $(a+bS)\varphi = 0$ has the nonzero solution $(1-x)^\alpha(1+x)^\beta \in L_{2,\rho_1}$, and the conjugate operator has the form

$$(a+bS)^* = a - b(1-t)^\alpha(1+t)^\beta S(1-x)^{-\alpha}(1+x)^{-\beta}.$$

The conjugate homogeneous equation $(a+bS)^*\varphi = 0$ has in the space L_{2,ρ_1} only zero solution. For $\varkappa = 1$, to have a unique solution, we have to prescribe the additional condition

$$\int_{-1}^{1} \varphi(t)dt = P, \quad (3.3)$$

where P is a given number.

Introduce the new unknown function

$$\phi \equiv \varphi - \pi^{-1}P\sin(|\alpha|\pi)(1-x)^\alpha(1+x)^\beta.$$

Then

$$\int_{-1}^{1} \phi(t)dt = 0.$$

Indeed, from the conditions [11]

$$\int_{-1}^{1}(1-x)^\alpha(a+x)^\beta[\widehat{P}_0^{(\alpha,\beta)}(x)]^2dx = \int_{-1}^{1}(h_0^{(\alpha,\beta)})^{-1}(1-x)^\alpha(1+x)^\beta dx = 1,$$

we have

$$\int_{-1}^{1}(1-x)^{\alpha}(a+x)^{\beta}dx = h_0^{(\alpha,\beta)},$$

$$h_0^{(\alpha,\beta)} = \Gamma(1+\alpha)\Gamma^{1+\beta}(\alpha) = \Gamma(1-|\alpha|)\Gamma(|\alpha|) = \frac{\pi}{\sin(|\alpha|\pi)},$$

i.e.,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}dx = \frac{\pi}{\sin(|\alpha|\pi)}.$$

The equation (3.1) for the new unknown function can be written in the form

$$(a+bS+K)\phi = f_1, \tag{3.4}$$

where

$$f_1 \equiv f - \pi^{-1}P\sin(|\alpha|\pi)K[(1-t)^{\alpha}(1+t)^{\beta}],$$

with the condition

$$[\phi,(1-x)^{\alpha}(1+x)^{\beta}] = \int_{-1}^{1}\phi(t)dt = 0. \tag{3.5}$$

We decompose the space L_{2,ρ_1} into an orthogonal sum $L_{2,\rho_1} = L_{2,\rho_1}^{(1)} \oplus L_{2,\rho_1}^{(2)}$, where $L_{2,\rho_1}^{(1)}$ is the linear span of the function $\varphi_0 = (1-x)^{\alpha}(1+x)^{\beta}$. Then the solution ϕ of the equation (3.4) with the condition (3.5) belongs to $L_{2,\rho_1}^{(2)}$. Therefore the problem (3.4)–(3.5) is replaced by the equation

$$(a+bS+K)\phi = f_1, \quad \phi \in L_{2,\rho_1}^{(2)}, \quad f_1 \in L_{2,\rho_1}. \tag{3.6}$$

Take now two systems of functions

$$\varphi_k \equiv 2b^{-1}\sin(|\alpha|\pi)(h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x), \quad k=1,2,\dots,$$

and

$$\psi_k \equiv (h_{k-1}^{-\alpha,-\beta})^{-\frac{1}{2}}P_{k-1}^{(-\alpha,-\beta)}(x), \quad k=1,2,\dots.$$

The formula

$$(a+bS)\varphi_k = \psi_k, \quad k=1,2,\dots, \quad -1<x<1, \tag{3.7}$$

is valid.

Indeed, on the basis of (3.2) we have

$$(a+bS)\varphi_k = 2b^{-1}\sin(|\alpha|\pi)(h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}\times$$

$$\times[a(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x) + bS(1-t)^{\alpha}(1+t)^{\beta}P_k^{(\alpha,\beta)}(t)] =$$

$$= 2b^{-1}\sin(|\alpha|\pi)(h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}(-2^{-1}\pi^{-1}\Gamma(\alpha)\Gamma(1-\alpha)bP_{k-1}^{(-\alpha,-\beta)(x)}) =$$

$$= 2b^{-1}\sin(|\alpha|\pi)(h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}(-\frac{1}{2})\frac{\pi}{\pi\sin(|\alpha|\pi)}bP_{k-1}^{(-\alpha,-\beta)(x)} =$$

$$= h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}} p_{K-1}^{(-\alpha,-\beta)}(x), \quad -1 < x < 1.$$

The system of functions $\psi_1, \psi_2, \dots$ is orthonormal and complete in L_{2,ρ_1}, and the system of functions $\varphi_1, \varphi_2, \dots$ is orthonormal and complete in $L_{2,\rho_1}^{(2)}$ [11].

The inverse operator $(a + bS)^{-1}$ mapping L_{2,ρ_1} onto $L_{2,\rho_1}^{(2)}$ exists. We will require the inverse operator $(a + bS + K)^{-1}$ mapping L_{2,ρ_1} onto $L_{2,\rho_1}^{(2)}$ to exist as well. This is equivalent to the existence of the inverse operator $[I + K(a + bS)^{-1}]^{-1}$ mapping L_{2,ρ_1} onto itself.

An approximate solution of the equation (3.6) is sought in the form

$$\phi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k.$$

We compose an algebraic system using the condition

$$[(a + bS)\phi^{(n)} + K\phi^{(n)} - f_1, \psi_1] = 0, \quad i = 1, 2, \dots, n.$$

Taking into account (3.7) and the fact that the system $\psi_1, \psi_2, \dots,$ is orthonormal, we obtain

$$a_i + \sum_{k=1}^{n} a_k [K\varphi_k, \psi_i] = [f_1, \psi_i], \quad i = 1, 2, \dots, n. \tag{3.8}$$

To compose this algebraic system, we have to calculate the integrals

$$[K\varphi_k, \psi_i] = 2(nb)^{-1} \sin(|\alpha|\pi)(h_{k-1}^{(-\alpha_1-\beta)} h_{i-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}} \times$$

$$\times \int_{-1}^{1}\int_{-1}^{1} K(x,t)(1-t)^{\alpha}(1+t)^{\beta}(1-x)^{-\alpha}(1+x)^{-\beta} P_k^{(\alpha,\beta)}(t) P_{i-1}^{(-\alpha,-\beta)}(x)dt\,dx,$$

$$[f, \psi_i] = (h_{i-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}} \int_{-1}^{1} (1 - x)^{-\alpha}(1 + x)^{-\beta} f_1(x) P_{i-1}^{(-\alpha,-\beta)}(x)dx,$$

$$k, i = 1, 2, \dots, n.$$

Theorem 3.1. *If there exists the inverse operator* $[I+K(a+bS)^{-1}]$ *mapping* L_{2,ρ_1} *onto itself, then the algebraic system* (3.8) *for sufficiently large* n *has a unique solution* $(a_1, a_2, \dots, a_n)$*, and the sequence of approximate solutions* $\{\varphi^{(n)}\}$ *converges to the exact solution* φ *of the equation* (3.1) *in the space* L_{2,ρ_1}.

This theorem is proved analogously to Theorem 2.1.

If the operator $K \equiv 0$, i.e., we have the characteristic equation

$$(a + bS)\varphi = f \tag{3.9}$$

with the condition

$$\int_{-1}^{1} \varphi(t)dt = P,$$

then assuming

$$f_1 \equiv f, \quad \phi = \varphi - P\pi^{-1}\sin(|\alpha|\pi)(1-x)^{\alpha}(1+x)^{\beta}$$

we arrive at the equation

$$(a+bS)\phi = f, \quad \phi \in L^{(2)}_{2,\rho_1}, \quad f \in L_{2,\rho_1}. \tag{3.10}$$

The algebraic system (3.8) takes the form

$$a_i = [f, \psi_i], \quad i = 1, 2, \dots, n, \tag{3.11}$$

and the matter is now reduced to the calculation of the Fourier coefficients

$$[f, \psi_i] = (h^{(-\alpha,-\beta)}_{i-1})^{-\frac{1}{2}} \int_{-1}^{1} (1-x)^{-\alpha}(1+x)^{-\beta} f(x) P^{(-\alpha,-\beta)}_{i-1}(x)dx,$$
$$i = 1, 2, \dots, n.$$

2) $\varkappa = -1$, $0 < \alpha < 1$, $\beta = 1-\alpha$, $\alpha \neq \beta$. We introduce the weight space $L_{2,\rho_2}[-1,1]$, where $\rho_2 = (1-x)^{-\alpha}(1+x)^{-\beta}$. The solution of the equation (3.1) of the type $\varphi = (1-x)^{\alpha}(1+x)^{\beta}\varphi_0(x)$, $0 < \alpha,\ \beta < 1$, where $\varphi_0(x)$ is a bounded measurable function, belongs to the space L_{2,p_2}.

Let the function $K(x,t)$ satisfy the condition

$$\int_{-1}^{1}\int_{-1}^{1} K^2(x,t)(1-x)^{-\alpha}(1+x)^{-\beta}(1-t)^{\alpha}(1+t)^{\beta} dt dx < +\infty.$$

Then the integral operator K is completely continuous in L_{2,ρ_2}, and the singular integral operator S is bounded in L_{2,ρ_2} [4]. The homogeneous equation $(a+bS)\varphi = 0$ in the space L_{2,ρ_2} has only zero solution, and the conjugate operator has the form

$$(a+bS)^* = a - b(1-t)^{\alpha}(1+t)^{\beta} S(1-x)^{-\alpha}(1+x)^{-\beta}.$$

The equation $(a-bS)\varphi = 0$ has the nonzero solution $\varphi = (1-x)^{-\alpha}(1+x)^{-\beta}$, therefore the conjugate equation $(a+bS)^*\varphi = 0$ has the non-trivial solution $1 \in L_{2,\rho_2}$.

If the equation (3.1) has a solution, then [1]

$$[K\varphi - f, 1] = 0. \tag{3.12}$$

The condition (3.12) is fulfilled if $[f,1] = 0$ and $K(L_{2,\rho_2}) \perp 1$; the latter condition is fulfilled if

$$\int_{-1}^{1} K(x,t)(1-x)^{-\alpha}(1+x)^{-\beta}\, dx = 0.$$

Assume that these conditions are fulfilled. We decompose the space L_{2,ρ_2} into the orthogonal sum $L_{2,\rho_2} = L^{(1)}_{2,\rho_2} \oplus L^{(2)}_{2,\rho_2}$, where $L^{(1)}_{2,\rho_2}$ is the subspace corresponding to $\varphi = 1$. Then we have the equation

$$(a+bS+K)\varphi = f, \quad \varphi \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}. \tag{3.13}$$

The bounded inverse operator $(a+bS)^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto L_{2,ρ_2} exists [4]. The inverse operator $[I+K(a+bS)^{-1}]^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ into itself is required to exist.

Take two systems of functions:

1) $\varphi_{k+1} \equiv (2b)^{-1}\sin(\alpha\pi)(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{\beta}P^{(\alpha,\beta)}_k(x)$, $k = 0,1,\dots,$ and

2) $\psi_{k+1} \equiv -(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}P^{(-\alpha,-\beta)}_{k+1}(x)$, $k = 0,1,\dots,$ $(\psi_0 \equiv (\pi^{-1}\times\sin(\alpha\pi))^{\frac{1}{2}})$.

The system $\varphi_1,\varphi_2,\dots$ is orthogonal and complete in L_{2,ρ_2}, and the system $\psi_1,\psi_2,\dots$ is orthogonal and complete in $L^{(2)}_{2,\rho_2}$.

The formula

$$(a+bS)\varphi_k = \psi_k, \quad k=1,2,\dots \tag{3.14}$$

is valid.

Indeed, on the basis of (3.2) we have

$$\begin{aligned}(a+bS)\varphi_{k+1} &= (2b)^{-1}\sin(\alpha\pi)(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}[a(1-x)^{\alpha}(1+x)^{\beta}P^{(\alpha,\beta)}_k(x)+\\ &\quad +bS(1-t)^{\alpha}(1+t)^{\beta}P^{(\alpha,\beta)}_k(t)] =\\ &= (2b)^{-1}\sin(\alpha\pi)(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}(-2\pi^{-1}\Gamma(\alpha)\Gamma(1-\alpha)bP^{(-\alpha,-\beta)}_{k+1})(x) =\\ &= (2b)^{-1}\sin(\alpha\pi)(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}\Big(-\frac{2b}{\sin(\alpha\pi)}\Big)P^{(-\alpha,-\beta)}_{k+1}(x) =\\ &= -(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}P^{(-\alpha,-\beta)}_{k+1}(x) = \psi_{k+1}, \quad k=0,1,\dots.\end{aligned}$$

An approximate solution of the equation (3.13) is sought in the form

$$\varphi(n) = \sum_{k=1}^{n} a_k\varphi_k.$$

We compose an algebraic system by using the conditions

$$[(a+bS+K)\varphi^{(n)} - f, \psi_i] = 0, \quad i=1,2,\dots,n,$$

which, with regard to (3.14) and the fact that the functions $\psi_1,\psi_2,\dots$ are orthonormal, provides us with

$$a_i + \sum_{k=1}^{n} a_k[K\varphi_k,\psi_i] = [f,\psi_i], \quad i=1,2,\dots,n. \tag{3.15}$$

To compose this algebraic system we have to calculate the integrals

$$[K\varphi_k,\psi_i] = -(2\pi b)^{-1}\sin(\alpha\pi)[h^{(-\alpha,-\beta)}_k h^{(-\alpha,-\beta)}_i]^{-\frac{1}{2}}\times$$

$$\times\int_{-1}^{1}\int_{-1}^{1} K(x,t)(1-t)^{\alpha}(1+t)^{\beta}(1-x)^{-\alpha}(1+x)^{-\beta}P^{(\alpha,\beta)}_{k-1}(t)P^{(\alpha,\beta)}_i(x)dtdx,$$

$$[f,\psi_i] = -(h^{(-\alpha,-\beta)}_i)^{-\frac{1}{2}}\times$$

$$\times \int_{-1}^{1} (1-x)^{-\alpha}(1+x)^{-\beta} f(x) P_i^{(-\alpha,-\beta)}(x)dx, \quad i,k = 1,2,\dots,n.$$

Theorem 3.2. *If there exists the inverse operator* $[I + K(a+bS)^{-1}]^{-1}$ *mapping* $L^{(2)}_{2,\rho_2}$ *onto itself, then the algebraic system* (3.15) *for sufficiently large* n *has a unique solution* $(a_1, a_2, \dots, a_n)$, *and the sequence of approximate solutions* $\{\varphi^{(n)}\}$ *converges to the exact solution* φ *in the space* L_{2,ρ_2}.

This theorem is proved just in the same way as Theorem 2.1.

If $K \equiv 0$, i.e., we have the equation $(a+bS)\varphi = f$, then

$$a_i = [f, \psi_i], \quad i = 1,2,\dots,n,$$

where

$$[f,\psi_i] = -(h_i^{(-\alpha,-\beta)})^{-\frac{1}{2}} \int_{-1}^{1} (1-x)^{-\alpha}(1+x)^{-\beta} f(x) P_i^{(-\alpha,-\beta)}(x)dx,$$
$$i = 1,2,\dots,n.$$

3) $\varkappa = 0$, $\varkappa = -(\alpha+\beta)$, $\beta = -\alpha$, $|\alpha| \neq \frac{1}{2}$, $0 < |\alpha| < 1$. Introduce a space $L_{2,\rho_3}[-1,1]$ with the weight $\rho_3 = (1-x)^{-\alpha}(1+x)^{\alpha}$. The function of the type $\varphi(x) = (1-x)^{\alpha}(1+x)^{-\alpha}\varphi_0(x)$, where $\varphi_0(x)$ is a bounded measurable function, belongs to the space L_{2,ρ_3}.

Let the function $K(x,t)$ satisfy the condition

$$\int_{-1}^{1}\int_{-1}^{1} K^2(x,t)(1-t)^{\alpha}(1+t)^{-\alpha}(1-x)^{-\alpha}(1+x)^{\alpha} dt dx < +\infty.$$

Then the integral operator K is completely continuous in L_{2,ρ_3} and the singular integral operator S is bounded in L_{2,ρ_3} [4]. The homogeneous equation $(a+bS)\varphi = 0$ in the space L_{2,ρ_3} has only zero solution. The conjugate operator has the form

$$(a+bS)^* = a - b\Big(\frac{1+t}{1-t}\Big)^{-\alpha} S \Big(\frac{1+x}{1-x}\Big)^{\alpha}.$$

The conjugate homogeneous equation $(a+bS)^*\varphi = 0$ in the space L_{2,ρ_3} has only zero solution, and the inverse operator $(a+bS)^{-1}$ mapping L_{2,ρ_3} onto itself exists.

If $a^2 + b^2 = 1$, then the systems of functions $\{\varphi_k\}$ and $\{\psi_k\}$ employed in the paper are orthonormal [13].

The operator $(a+bS)$ is unitary.

If the index is zero, we have the equation

$$(a+bS+K)\varphi = f, \quad \varphi, f \in L_{2,\rho_3}. \tag{3.16}$$

Let us take two systems of functions:

1) $\varphi_k \equiv b^{-1}\sin(|\alpha|\pi)(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{-\alpha}P_k^{(\alpha,-\alpha)}(x)$, $k = 0,1,\dots,$ and

2) $\psi_k \equiv -(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}} P_k^{(-\alpha,\alpha)}(x)$, $k = 0, 1, \dots$.

The formula

$$(a + bS)\varphi_k = \psi_k, \quad k = 0, 1, \dots, \tag{3.17}$$

is valid.

Indeed, on the basis of (3.2) we have

$$(a + bS)\varphi_k = b^{-1}\sin(|\alpha|\pi)(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}}[a(1-x)^{\alpha}(1+x)^{-\alpha}P_k^{(\alpha,-\alpha)}(x)+$$
$$+bS(1-t)^{\alpha}(1+t)^{-\alpha}P_k^{(\alpha,-\alpha)}(t)] =$$
$$= b^{-1}\sin(|\alpha|\pi)(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}}\Big(-\frac{\pi b}{\sin(|\alpha|\pi)}\Big)P_k^{(-\alpha,\alpha)}(x) =$$
$$= -(h_k^{(-\alpha,\alpha)})^{-1}P_k^{(-\alpha,\alpha)}(x), \quad k = 0, 1, \dots .$$

An approximate solution of the equation (3.15) is sought in the form

$$\varphi^{(n)} = \sum_{k=0}^{n} a_k\varphi_k.$$

We compose an algebraic system by using the conditions

$$[(a + bS + K)\varphi^{(n)} - f, \psi_i] = 0, \quad i = 0, 1, \dots, n,$$

which, with regard to (3.17) and the fact that the system $\psi_0, \psi_1, \dots$ is orthonormal, provides us with

$$a_i + \sum_{k=0}^{n} a_k[K\varphi_k, \psi_i] = [f, \psi_i], \quad i = 0, 1, \dots, n. \tag{3.18}$$

To compose this algebraic system we have to calculate the integrals

$$[K\varphi_k, \psi_i] = -b\sin(|\alpha|\pi)[h_k^{(-\alpha,\alpha)}h_i^{(-\alpha,\alpha)}]^{-\frac{1}{2}}\times$$
$$\times\int_{-1}^{1}\int_{-1}^{1} K(x,t)\frac{(1-x)^{-\alpha}(1+x)^{\alpha}}{(1-t)^{-\alpha}(1+t)^{\alpha}}P_k^{(\alpha,-\alpha)}(t)P_i^{(-\alpha,\alpha)}(x)dtdx,$$
$$[f, \psi_i] = -(h_i^{(-\alpha,\alpha)})^{-\frac{1}{2}}\times$$
$$\times\int_{-1}^{1}(1-x)^{-\alpha}(1+x)^{\alpha}f(x)P_i^{(-\alpha,\alpha)}(x)dx, \quad k, i = 0, 1, \dots, n.$$

Theorem 3.3. *If there exists the inverse operator $[I + K(a + bS)^{-1}]^{-1}$ mapping L_{2,ρ_3} onto itself, then the algebraic system* (3.18) *for sufficiently large n has a unique solution $(a_0, a_1, \dots, n)$, and the sequence of approximate solutions $\{\varphi^{(n)}\}$ converges to the exact solution φ in the metric of the space L_{2,ρ_3}.*

If the operator $K \equiv 0$, i.e., we have the characteristic singular equation

$$(a + bS)\varphi = f,$$

then

$$a_i = [f, \psi_i], \quad i = 0, 1, \dots, n.$$

4. Collocation Method for the Singular Integral Equation of the First Kind

We consider the equation

$$S\varphi + K\varphi = f. \tag{4.1}$$

In this section the use will be made of the formulas and reasoning of Section 2. The kernels $K(x,t)$ and $f(x)$ are required to be continuous in a closed domain. Each value of the index will be considered separately.

1) $\varkappa = 1$. Taking into account our reasoning in Section 2, we have

$$\begin{gathered} (S+K)\phi = f_1, \quad \phi \in L^{(2)}_{2,\rho_1}, \quad f_1 \in L_{2,\rho_1}, \\ f_1 \equiv f - P\pi^{-1}K(1-t^2)^{-\frac{1}{2}}, \quad \phi \equiv \varphi - P\pi^{-1}(1-t^2)^{-\frac{1}{2}}. \end{gathered} \tag{4.2}$$

An approximate solution in the collocation method is sought in the form

$$\phi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k,$$

where

$$\varphi_k = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} T_k(x), \quad k = 1, 2 \dots .$$

It is required that the difference $(S+K)\phi^{(n)} - f_1$ at the discrete points $x_1, x_2, \dots, x_n$ be zero, i.e.,

$$((S+K)\phi^{(n)} - f_1)(x_j) = 0, \quad j = 1, 2, \dots, n.$$

This, with regard for (2.3), yields

$$\sum_{k=1}^{n} a_k \psi_k(x_j) + \sum_{k=1}^{n} a_k (K\varphi_k)(x_j) = f_1(x_j), \quad j = 1, 2, \dots, n, \tag{4.3}$$

where

$$\psi_k \equiv \left(\frac{2}{\pi}\right)^{\frac{1}{2}} U_{k-1}(x), \quad k = 1, 2, \dots .$$

Theorem 4.1. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *mapping* L_{2,ρ_1} *onto itself, and as collocation nodes we take the roots of the Chebyshev polynomials* $U_n(x)$,

$$x_j^{(n)} = \cos\frac{\pi j}{n+1}, \quad j = 1, 2, \dots, n,$$

then the algebraic system (4.3) *for sufficiently large* n *has a unique solution* $(a_1, a_2, \dots, a_n)$, *and the collocation process converges in the space* L_{2,ρ_1}.

Proof. Introduce the notation $w \equiv S\phi$ and $w^{(n)} \equiv \phi^{(n)}$. Then we write the equations (4.2) and (4.3), respectively, as follows:

$$w + KS^{-1}w = f_1, \quad w, f_1 \in C[-1,1], \tag{4.4}$$

and

$$w^{(n)}(x_j) + (KS^{-1}w^{(n)})(x_j)f_1(x_j), \quad j = 1,2,\dots,n. \tag{4.5}$$

Take the space of continuous functions $C[-1,1]$ and let P_m be the projector defined by the Lagrange interpolation polynomial of the m-th degree; $P_m v \equiv L_m(v)$, $v \in C$. Then, taking into account that the Lagrange interpolation polynomial is defined uniquely by the function values at nodes, we can write the algebraic system (4.5) in the form

$$w^{(n)} + P_{n-1}KS^{-1}w^{(n)} = P_{n-1}f_1, \quad w^{(n)} \in \overline{L}^{(n)}_{2,\rho_1}, \tag{4.6}$$

where $\overline{L}^{(n)}_{2,\rho_1}$ is the linear span of the functions $\psi_1, \psi_2, \dots, \psi_n$ (ψ_k is the polynomial of degree $k-1$).

If as interpolation nodes we take the roots of the Chebyshev polynomial $U_n(x)$, then by the Erdös–Turan theorem ([14], [11], [15]) the Lagrange interpolation polynomial for any continuous function converges in the space L_{2,ρ_1}, i.e.,

$$\|P_n v - Ev\|_{L_{2,\rho_1}} \to 0, \quad n \to \infty, \quad \forall v \in C[-1,1], \tag{4.7}$$

where E is the operator of the embedding $C[-1,1]$ in $L_{2,\rho_1}[-1,1]$, i.e., $P_n \to E$ strongly.

The Banach space $C[-1,1]$ is embedded continuously in $L_{2,\rho_1}[-1,1]$, i.e.,

$$\|v\|_{L_{2,\rho_1}} = \Big(\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} v^2 dx\Big)^{\frac{1}{2}} \le \Big(\int_{-1}^{1} v^2 dx\Big)^{\frac{1}{2}} \le$$
$$\le 2^{\frac{1}{2}} \|v\|_c, \quad \forall v \in C[-1,1]. \tag{4.8}$$

The operator KS^{-1} from the space $L_{2\rho_1}[-1,1]$ to $C[-1,1]$ is completely continuous. Indeed, S^{-1} transforms a bounded set from L_{2,ρ_1} into a bounded set in $L^{(2)}_{2,\rho_1}$, the kernel $K(x,t)$ is continuous. Thus the operator K transforms a set bounded in $L^{(2)}_{2,\rho_1}$ to a uniformly bounded and equicontinuous set of functions on $[-1,1]$.

The projector P_n is bounded as the operator from $C[-1,1]$ to $L_{2,\rho_1}[-1,1]$,

$$\|P_n v\|_{L_{2,\rho_1}} \le 2^{\frac{1}{2}} \|P_n v\|_c \le 2^{\frac{1}{2}} C(n) \|v\|_c,$$

i.e.,

$$\|P_n\|_{c \to L_{2,\rho_1}} \le 2^{\frac{1}{2}} C(n), \quad C(n) \sim \ln n. \tag{4.9}$$

By the Banach–Steinhaus theorem, they are uniformly bounded.

As (4.7), (4.8) and (4.9) show, all the conditions of Theorem 15.5 from [9] are fulfilled, and we can conclude that

$$\|P^{(n-1)}KS^{-1}\|_{L_{2,\rho_1}} \to 0 \tag{4.10}$$

as $n \to \infty$, $P^{(n-1)} \equiv I - P_{n-1}$.

Next, by Theorem 15.5 from [9], the equation (4.6) for sufficiently large n has a unique solution $w^{(n)}$ (hence the algebraic system (4.5) has a unique solution), and the estimate

$$\|w^{(n)} - w\| \le \frac{\|(I + KS^{-1})^{-1}\|}{1-q}\|P^{(n-1)}w\| \tag{4.11}$$

is valid for

$$\|P^{(n-1)}KS^{-1}\| \cdot \|(I + KS^{-1})^{-1}\| \le q < 1.$$

As (4.7) shows, the norm $\|P^{(n-1)}w\| \to 0$ as $n \to \infty$. Further, taking into account (4.11),

$$\|\phi^{(n)} - \phi\|_{L^{(2)}_{2,\rho_1}} = \|S^{-1}S(\phi^{(n)} - \phi)\| = \|S^{-1}(w^{(n)} - w)\| \le$$
$$\le \|S^{-1}\|_{L_{2,\rho_1} \to L^{(2)}_{2,\rho_1}} \cdot \|w^{(n)} - w\|_{L_{2,\rho_1}} \to 0, \quad n \to \infty. \tag{4.12}$$

An approximate solution of the initial problem is given by the equality

$$\varphi^{(n)} = \phi^{(n)} + P\pi^{-1}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}.$$

Therefore $\varphi^{(n)} - \varphi = \phi^{(n)} - \phi$, and on the basis of (4.12) we find that

$$\|\varphi^{(n)} - \varphi\|^2 = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}}(\varphi^{(n)} - \varphi)^2 dx \to 0, \quad n \to \infty, \tag{4.13}$$

which completes the proof of our theorem. □

If the operator $K \equiv 0$, i.e., we have the equation $S\varphi = f$ under the additional condition $\int_{-1}^{1} \varphi(t)dt = P$ (index $\varkappa = 1$), then we obtain the algebraic system

$$\sum_{k=1}^{n} a_k \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} U_{k-1}(x_j) = f_1(x_j), \quad j = 1, 2, \dots, n. \tag{4.14}$$

Theorem 4.1 results in the following

Corollary. *For any $n = 1, 2, \dots,$ the algebraic system* (4.14) *has the unique solution $(a_1, a_2, \dots, a_n)$, and the process converges in L_{2,ρ_1}.*

2) $\varkappa = -1$. On the basis of our reasoning in Section 2, we have the equation

$$S\varphi + K\varphi = f, \quad \varphi \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}, \quad \rho_2 = (1-x^2)^{-\frac{1}{2}}, \tag{4.15}$$

under the restrictions $K(L_{2,\rho_2}) \perp 1$, $f \perp 1$.

In Section 2 we had the functions

$$\varphi_{k+1} \equiv \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} U_k(x), \quad k = 0, 1, \dots,$$
$$\psi_{k+1} \equiv -\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} T_{k+1}(x), \quad k = 0, 1, \dots,$$

Take the subspace $C_0[-1,1]$, $v \in C_0[-1,1]$ meaning that $[v,1]=0$, $C_0[-1,1] \subset C[-1,1]$. Let $L_n(v) = b_0 + b_1 x + \cdots + b_n x^n$ be the Lagrange interpolation polynomial and $C^{(n)}[-1,1] \subset C[-1,1]$ be the linear manifold of polynomials $\psi_0, \psi_1, \dots, \psi_n$ ($\psi_0 = \widetilde{T}_0$). The Lagrange polynomial $L_n(v)$ can be represented uniquely in the form

$$b_0 + b_1 x + \cdots + b_n x^n = a_0\psi_0 + a_1\psi_1 + \cdots + a_n\psi_n.$$

The algebraic system

$$a_0 + a_1\psi_1(x_j) + \cdots + a_n\psi_n(x_j) = v_j, \quad j = 0,1,\dots,n,$$

has the unique solution $a_0^{(n)}, \dots, a_n^{(n)}$.

We define the projector as follows:

$$\overline{P}_n(v) = L_n(v) - a_0^{(n)}\psi_0. \tag{4.16}$$

Then $\overline{P}_n^2 = \overline{P}_n$, $\overline{P}_n(v) \in C_0[-1,1]$ for $\forall v \in C_0[-1,1]$.

If as interpolation nodes we take the roots of the Chebyshev polynomial of the first kind $T_{n+1}(x)$, then by the Erdös-Turan theorem [11],

$$\|v - L_n(v)\|_{2,\rho_2} \to 0, \quad n \to \infty, \quad \forall v \in C[-1,1]. \tag{4.17}$$

For $v \in C_0(-1,1]$ we have

$$\|v - L_n(v)\|^2_{L_{2,\rho_2}} = (a_0^{(n)})^2 + \sum_{k=1}^{\infty}[v - L_n(v), \psi_k]^2,$$

which, with regard for (4.17), yields

$$a_0^{(n)} \to 0, \quad n \to \infty, \quad \forall v \in C_0[-1,1]. \tag{4.18}$$

An approximate solution of the equation (4.15) is sought in the form

$$\varphi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k.$$

We compose the collocation system from the condition

$$\overline{P}_n(S\varphi^{(n)} + K\varphi^{(n)} - f) = 0. \tag{4.19}$$

Taking into account the relation $S\varphi_k = \psi_k$, $k = 1,2,\dots$, and the type of functions φ_k, ψ_k, $k = 1,2,\dots$, the above condition provides us with the algebraic system

$$a_0(-\widehat{T}_0) + \sum_{k=1}^{n} a_k(-\widehat{T}_k(x_j)) + \sum_{k=1}^{n} a_k(K\varphi_k)(x_j) = f(x_j), \tag{4.20}$$
$$j = 0,1,\dots,n.$$

Theorem 4.2. *If there exists the inverse operator $(I + KS^{-1})^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto itself, and as interpolation nodes we take the roots of the Chebyshev polynomial $T_{n+1}(x)$*

$$x_j = \cos\frac{(2j+1)}{2(n+1)}, \quad j = 0,1,\dots,n,$$

then the algebraic system (4.20) *for sufficiently large* n *has the unique solution* $(a_0, a_1, \dots, n)$*, and the collocation process converges in the space* L_{2,ρ_2}*.*

Proof. We rewrite the initial equation (4.15) in the form

$$(I + KS^{-1})w = f, \quad w \equiv S\varphi, f, w \in L^{(2)}_{2,\rho_2}, \tag{4.21}$$

and the approximation (4.19) in the form

$$w^{(n)} + \overline{P}_n KS^{-1} w^{(n)} = \overline{P}_n f, \quad w^{(n)} \equiv S\varphi^{(n)}. \tag{4.22}$$

Thus we have:

a) $$\|v - \overline{P}_n(v)\|^2_{L^{(2)}_{2,\rho_2}} = \|v - L_n(v) + a_0^{(n)}\psi_0\|^{(2)}_{L_{2,\rho_2}} \le$$

$$\le \|v - L_n(v)\|^2_{L_{2,\rho_2}} + (a_0^{(n)})^2 \to 0, \quad \forall v \in C_0[-1,1], \quad n \to \infty, \tag{4.23}$$

on the basis of (4.17) and (4.18) we have

$$\|\overline{P}_n v\|^2_{L^{(2)}_{2,\rho_2}} \le \|L_n(v)\|^2_{L_{2,\rho_2}} \le, (B_0 \ln n)^2 \|v\|^2_{C_0},$$

i.e.,

$$\|\overline{P}_n\|_{C_0 \to L^{(2)}_{2,\rho_2}} \le B_0 \ln n;$$

b) the operator KS^{-1} is completely continuous from $L^{(2)}_{2,\rho_2}$ in $C_0[-1,1]$;

c) $$\begin{aligned} &\|v\|^2_{L^{(2)}_{2,\rho_2}} = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} v^2 dx \le \pi \|v\|^2_{C_0}, \\ &\|v\|_{L^{(2)}_{2,\rho_2}} \le \pi^{\frac{1}{2}} \|v\|_{C_0}. \end{aligned} \tag{4.24}$$

From a), b), c) it follows by [9] that

$$\|\overline{P}^{(n)} KS^{-1}\| \to 0, \quad n \to \infty, \quad \overline{P}^{(n)} \equiv I - \overline{P}_n.$$

Next, reasoning as in the proof of Theorem 4.1 we will see that the algebraic system (4.20) for sufficiently large n has a unique solution, and the process converges, i.e.,

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} (\varphi^{(n)} - \varphi)^2 dx \to 0 \quad \text{as} \quad n \to \infty. \tag{4.25}$$

The theorem is proved. □

If the operator $K \equiv 0$, then under the equation $S\varphi = f$ for the condition $[f, 1] = 0$ we obtain the algebraic system

$$a_0(-\widehat{T}_0) + \sum_{k=1}^{n} a_k(-\widehat{T}_k(x_j)) = f(x_j), \quad j = 0, 1, \dots, n, \tag{4.26}$$

where x_j, $j = 0, 1, \dots, n$ are the roots of the Chebyshev polynomial $T_{n+1}(x)$. From the above proven Theorem 4.2 we have

Corollary. *For any $n = 1, 2, \dots,$ the algebraic system* (4.26) *has the unique solution $(a_0, a_1, \dots, a_n)$, and the process converges in L_{2,ρ_2}.*

3) $\varkappa = 0$. Arguing as in Section 2, we have

$$S\varphi + K\varphi = f, \quad f, \varphi \in L_{2,\rho_3}. \tag{4.27}$$

Consider first the case $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$; the weight $\rho_3 = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$,

$$\varphi_k \equiv (h_k^{(-\frac{1}{2},\frac{1}{2})})^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}} P_k(x)^{(-\frac{1}{2},\frac{1}{2})}, \quad k = 0, 1, \dots,$$

where

$$P_k^{(-\frac{1}{2},\frac{1}{2})}(x) = e_k \frac{\cos(\frac{2k+1}{2}\arccos x)}{\cos(\frac{1}{2}\arccos x)}, \quad k = 0, 1, \dots,$$

$$e_0 = 1, \quad e_k = \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots 2k}, \quad k = 1, 2, \dots,$$

and

$$\psi_k \equiv \left(h_k^{(-\frac{1}{2},\frac{1}{2})}\right)^{-\frac{1}{2}} P_k^{(\frac{1}{2},-\frac{1}{2})}(x), \quad k = 0, 1, \dots .$$

where

$$P_k^{(\frac{1}{2},-\frac{1}{2})}(x) = e_k \frac{\sin(\frac{2k+1}{2}\arccos x)}{\sin(\frac{1}{2}\arccos x)}, \quad k = 0, 1, \dots .$$

An approximate solution is sought in the form

$$\varphi^{(n)} = \sum_{k=0}^{n} a_k \varphi_k.$$

The coefficients $(a_0, a_1, \dots, a_n)$ in the collocation method are defined from the following conditions:

$$(s\varphi^{(n)} + K\varphi^{(n)} - f)(x_j) = 0, \quad j = 0, 1, \dots, n,$$

which, with regard for the equality $S\varphi_k = \psi_k$, $k = 0, 1, \dots,$ provides us with the algebraic system

$$\sum_{k=0}^{n} a_k \psi_k(x_j) + \sum_{k=0}^{n} a_k (K\varphi_k)(x_j), \quad j = 0, 1, \dots, n. \tag{4.28}$$

Theorem 4.3. *If there exists the inverse operator $(I + KS^{-1})^{-1}$ mapping L_{2,ρ_3} onto itself and as interpolation nodes we take the roots of the Jacobi polynomial $P_{n+1}^{(\frac{1}{2},-\frac{1}{2})}(x)$,*

$$x_j = \cos\frac{2(j+1)\pi}{2n+3}, \quad j = 0, 1, \dots, n,$$

then the algebraic system (4.28) *for sufficiently large n has a unique solution, and the collocation process converges in the space L_{2,ρ_3} ($\rho_3 = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$).*

Proof. We denote $w \equiv S\varphi$, $w^{(n)} \equiv S\varphi^{(n)}$. Then the equations (4.27) and (4.28) can be rewritten as follows:

$$w + KS^{-1}w = f, \quad f, w \in L_{2,\rho_3}, \tag{4.29}$$

$$w^{(n)}(x_j) + (KS^{-1}w^{(n)})(x_j) = f(x_j), \quad j = 0, 1, \dots, n. \tag{4.30}$$

Let P_n be the projector defined by the Lagrange interpolation polynomial of the n-th degree; $P_n v \equiv L_n(v)$, $v \in C[-1,1]$. Then we can write the algebraic system (4.30) in the form

$$w^{(n)} + P_n KS^{-1}w^{(n)} = P_n f, \quad w^{(n)} \in \overline{L}^{(n)}_{2,\rho_3}, \tag{4.31}$$

where $\overline{L}^{(n)}_{2,\rho_3}$ is the linear span of the functions $\psi_0, \psi_1, \dots, \psi_n$ (ψ_k is a polynomial of the k-th degree).

If in the Lagrange polynomial we take as interpolation nodes the roots of the Jacobi polynomial $P_{n+1}^{(\frac{1}{2};-\frac{1}{2})}(x)$, then by the Erdös–Turan theorem [11], the Lagrange interpolation polynomial for any continuous function converges in the space L_{2,ρ_3}, i.e.,

$$\|P_n v - Ev\|_{L_{2,\rho_3}} \to 0, \quad n \to \infty, \quad \forall v \in C[-1,1]; \tag{4.32}$$

$P_n \to E$ strongly (E is the operator of embedding $C[-1,1]$ in L_{2,ρ_3}.

The Banach space $C[-1,1]$ is embedded continuously in $L_{2,\rho_3}[-1,1]$, since

$$\|v\|_{L_{2,\rho_3}} = \left(\int_{-1}^{1} (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} v^2 dx \right)^{\frac{1}{2}} \le 2\|v\|_c. \tag{4.33}$$

The operator KS^{-1} from the space L_{2,ρ_3} in $C[-1,1]$ is completely continuous; S^{-1} is bounded in L_{2,ρ_3}.

The projector P_n is bounded as an operator from C to L_{2,ρ_3},

$$\|P_n v\|_{L_{2,\rho_3}} \le 2\|P_n v\|_c \le 2C_1(n)\|v\|_c,$$

i.e.,

$$\|P_n\|_{c \to L_{2,\rho_3}} \le 2C_1(n) \tag{4.34}$$

(by the Banach–Steinhaus theorem, they are uniformly bounded). This yields

$$\|P^{(n)}KS^{-1}\|_{L_{2,\rho_3}} \to 0 \tag{4.35}$$

as $n \to \infty$, $P^{(n)} \equiv I - P_n$.

From (4.35) we conclude that for sufficiently large n the system (4.28) has a unique solution, and the process

$$\int_{-1}^{1} (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} (\varphi^{(n)} - \varphi)^2 dx \to 0 \tag{4.36}$$

converges as $n \to \infty$, which completes the proof of the theorem. □

If the operator $K \equiv 0$, i.e., we have the equation $S\varphi = f$, then we obtain the algebraic system

$$\sum_{k=0}^{n} a_k \psi_k(x_j) = f(x_j), \quad j = 0, 1, \dots, n. \tag{4.37}$$

Theorem 4.3 results in the corollary: the algebraic system (4.37) for any $n = 1, 2, \dots,$ has the unique solution $(a_0, a_1, \dots, a_n)$, and the collocation process converges in L_{2,ρ_3}.

If $\alpha = \frac{1}{2}$, and $\beta = -\frac{1}{2}$, then in the weight space L_{2,ρ_3}, where $\rho_3 = (1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}$, we again take

$$\varphi_k \equiv \Big(h_k^{(\frac{1}{2},-\frac{1}{2})}\Big)^{-\frac{1}{2}} (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} P_k^{(\frac{1}{2},-\frac{1}{2})}(x), \quad k = 0, 1, \dots,$$

$$\psi_k \equiv -\Big(h_k^{(\frac{1}{2},-\frac{1}{2})}\Big)^{-\frac{1}{2}} P_k^{(\frac{1}{2},-\frac{1}{2})}(x), \quad k = 0, 1, \dots.$$

The collocation method provides us with the algebraic system

$$\sum_{k=0}^{n} a_k \psi_k(x_j) + \sum_{k=0}^{n} a_k (K\varphi_k)(x_j) = f(x_j), \quad j = 0, 1, \dots, n, \tag{4.38}$$

in which as collocation nodes we take the roots of the Jacobi polynomial $P_{n+1}^{(-\frac{1}{2},\frac{1}{2})}(x)$,

$$x_j = \cos\frac{(2j+1)\pi}{2n+3}, \quad j = 0, 1, \dots, n.$$

Repeating here our reasoning, we can get an analogous result.

5. Collocation Method for the Singular Integral Equation of the Second Kind

Consider the equation of the second kind

$$(a + bS + K)\varphi = f, \quad f \in C[-1, 1]. \tag{5.1}$$

In this section the use will be made of the formulas and arguments of Section 3. As in Section 4, the function $K(x,t)$ is required to be continuous on the square $[-1,1] \times [-1,1]$. Every value of the index $\varkappa = 1, -1, 0$ will be considered separately.

1) $\varkappa = 1$. In this case we have the equation

$$(a + bS + K)\phi = f_1, \quad \phi \in L_{2,\rho_1}^{(2)}, \quad f_1 \in L_{2,\rho_1}, \tag{5.2}$$

where

$$\phi \equiv \varphi - P\pi^{-1}\sin(|\alpha|\pi)(1-x)^{\alpha}(1+x)^{\beta}, \quad \int_{-1}^{1} \Phi(t)dt = 0,$$

$$f_1 \equiv f - P\pi^{-1}\sin(|\alpha|\pi)K[(1-t)^{\alpha}(1+)^{\beta}], \quad -1 < \alpha, \ \ \beta < 0.$$

We use the functions

$$\varphi_k \equiv 2b^{-1}\sin(|\alpha|\pi)(h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x), \quad k=0,1,\dots,$$

and

$$\psi_k \equiv (h_{k-1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}P_{k-1}^{(-\alpha,-\beta)}(x), \quad k=1,2,\dots.$$

An approximate solution is sought in the form

$$\phi^{(n)} = \sum_{k=1}^{n} a_k\varphi_k.$$

It is required that the difference $(a+bS+K)\phi^{(n)} - f_1$ at the discrete points $x_1, x_2, \dots, x_n$ be equal to zero,

$$((a+bS+K)\phi^{(n)} - f_1)(x_j) = 0, \quad j=1,2,\dots,n,$$

which, on the basis of (3.7), provides us with the algebraic system

$$\sum_{k=1}^{n} a_k\psi_k(x_j) + \sum_{k=1}^{n} a_k(K\varphi_k)(x_j) = f_1(x_j), \quad j=1,2,\dots,n. \tag{5.3}$$

Theorem 5.1. *If there exists the inverse operator* $[I+K(a+bS)^{-1}]^{-1}$ *mapping* L_{2,ρ_1} *onto itself and as collocation nodes we take the roots of the Jacobi polynomial* $P_n^{(-\alpha,-\beta)}(x)$*, then the algebraic system* (5.3) *for sufficiently large* n *has a unique solution, and the process converges in the space* L_{2,ρ_1}*.*

Proof. Denote $w \equiv (a+bS)\phi$ and $w^{(n)} \equiv (a+bS)\phi^{(n)}$. Then the equations (5.2) and (5.3) can, respectively, be rewritten as follows:

$$w + K(a+bS)^{-1}w = f_1, \quad w, f_1 \in C[-1,1], \tag{5.4}$$

and

$$w^{(n)}(x_j) + (K(a+bS)^{-1}w^{(n)})(x_j) = f_1(x_j), \quad j=1,2,\dots,n. \tag{5.5}$$

Let P_m be the projector defined by the Lagrange interpolation polynomial of the m-th degree; $P_m v = L_m(v)$, $v \in C[-1,1]$. Then we write the algebraic system (5.5) in the form

$$w^{(n)} + P_{n-1}K(a+bS)^{-1}w^{(n)} = P_{n-1}f_1, \quad w^{(n)} \in L_{2,\rho_1}^{(n)}, \tag{5.6}$$

where $L_{2,\rho_1}^{(n)}$ is the linear span of functions $\psi_1, \psi_2, \dots, \psi_n$ (ψ_k is the polynomial of $(k-1)$-th degree).

If as interpolation nodes we take the roots of the Jacobi polynomial $P_n^{(-\alpha-\beta)}(x)$, then by the Erdös–Turan theorem [11], the Lagrange interpolation polynomial for any continuous function converges in the space L_{2,ρ_1}, i.e., $\|P_n v - Ev\| \to 0$ as $n \to \infty$, $\forall v \in C[-1,1]$, E is the operator of embedding of $C[-1,1]$ into $L_{2,\rho_1}[-1,1]$. The Banach space $C[-1,1]$ is embedded continuously in $L_{2,\rho_1}[-1,1]$. Every norm $\|P_m\|_{c\to L_{2,\rho_1}}$ is bounded and, moreover, the operator $K(a+bS)^{-1}$ from the space L_{2,ρ_1} in C is completely continuous. Then

$$\|P^{(n-1)}K(a+bS)^{-1}\|_{L_{2,\rho_1}} \to 0$$

as $n \to \infty$ ($P^{(n)} \equiv I - P_n$).

Therefore the equation (5.6) for sufficiently large n has the unique solution $w^{(n)}$ (then the algebraic system (5.5) also has a unique solution), and the estimate

$$\|w^{(n)} - w\| \le \frac{\|[I + K(a+bS)^{-1}]^{-1}\|}{1-q}\|P^{(n-1)}w\|$$

is valid for

$$\|P^{(n-1)}K(a+bS)^{-1}\| \, \|[I + K(a+bS)^{-1}]^{-1}\| \le q < 1.$$

The norm $\|P^{(n-1)}w\| \to 0$ as $n \to \infty$, since $w \in C[-1,1]$.

Further,

$$\|\phi^{(n)} - \phi\|_{L^{(2)}_{2,\rho_1}} = \|(a+bS)^{-1}(a+bS)(\phi^{(n)} - \phi)\| = \|(a+bS)^{-1}w^{(n)} - w\| \le$$
$$\le \|(a+bS)^{-1}\|_{L_{2,\rho_1} \to L^{(2)}_{2,\rho_1}} \|w^{(n)} - w\|_{L_{2,\rho_1}}.$$

An approximate solution of the initial problem has the form

$$\varphi^{(n)} = \phi^{(n)} + P\pi^{-1}\sin(|\alpha|\pi)(1-x)^{\alpha}(1+x)^{\beta}.$$

Owing to $\varphi^{(n)} - \varphi = \phi^{(n)} - \phi$, we finally get

$$\|\varphi^{(n)} - \varphi\| = \left(\int_{-1}^{1}(1-x)^{-\alpha}(1+x)^{-\beta}(\varphi^{(n)} - \varphi)^2 dx\right)^{\frac{1}{2}} \to 0 \tag{5.7}$$

as $n \to \infty$. □

If $K \equiv 0$, i.e., we have the characteristic singular integral equation

$$(a+bS)\varphi = f,$$

with the additional condition $\int_{-1}^{1}\varphi(x)dx = P$, from (5.3) we obtain the algebraic system

$$\sum_{k=1}^{n} a_k\psi_k(x_j) = f(x_j), \quad j = 1,2,\ldots,n. \tag{5.8}$$

Theorem 5.1 results in the following

Corollary. *The algebraic system* (5.8) *for any* $n = 1,2,\ldots$ *has the unique solution* $(a_1, a_2, \ldots, a_n)$*, and the process converges in the space* L_{2,ρ_1}*.*

2) $\varkappa = -1$. In this case, on the basis of our reasoning in Section 3, we have the equation

$$(a+bS+K)\varphi = f, \quad \varphi \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}, \tag{5.9}$$

under the restriction $K(L_{2,\rho_2}) \perp 1$, where $\rho_2 = (1-x)^{-\alpha}(1+x)^{-\beta}$, $0<\alpha$, $\beta<1$ is the weight. We have the functions:

1) $\varphi_{k+1} \equiv (2b)^{-1} \sin(|\alpha|\pi)(h_{k+1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)$, $k = 0,1,\dots,$ and

2) $\psi_{k+1} = -(h_{k+1}^{(-\alpha,-\beta)})^{-\frac{1}{2}}P_{k+1}^{(-\alpha,-\beta)}(x)$, $k=0,1,\dots,$ $\alpha+\beta=1$.

The systems $\psi_1,\psi_2,\dots$ and $\varphi_1,\varphi_2,\dots$ are the bases in L_{2,ρ_2} and $L_{2,\rho_2}^{(2)}$, respectively.

Just is in point 2 of Section 4, we take the subspace $C_0[-1,1] \subset C[-1,1]$; $C_0[-1,1] \subset L_{2,\rho_2}^{(2)}$, $v \in C_0[-1,1]$ if $[v,1]=0$. Let $L_n(v)$ be the Lagrange interpolation polynomial for the function v, $C^{(n)} \subset C[-1,1]$ be the linear span of the polynomials $\psi_0,\psi_1,\dots,\psi_n$. The Lagrange polynomial $L_n(v) \in C^{(n)}$ is representable uniquely in the form $L_n(v) = a_0\psi_0 + a_1\psi_1 + \dots + a_n\psi_n$ for any $v \in C[-1,1]$.

The algebraic system

$$a_0\psi_0 + a_1\psi_1(x_j) + \dots + a_n\psi_n(x_j) = v_j, \quad j=0,1,\dots,n,$$

has the unique solution $a_0^{(n)}, a_1^{(n)}, \dots, a_n^{(n)}$.

We take the projector

$$\overline{P}_n v = L_n(v) - a_0^{(n)}\psi_0 \quad (\psi_0 = (\pi^{-1}\sin\alpha\pi)^{\frac{1}{2}}),$$
$$\overline{P}_n^2 = \overline{P}_n, \quad \overline{P}_n v \in C_0[-1,1], \quad \forall v \in C_0[-1,1].$$

If as interpolation nodes we take the roots of the Jacobi polynomial $P_{n+1}^{-\alpha,-\beta}(x)$, then by the Erdös–Turan theorem [11],

$$\|v - L_n(v)\|_{L_{2,\rho_2}} \to 0, \quad n \to \infty, \quad \forall v \in C[-1,1].$$

For $v \in C_0[-1,1]$ we have

$$\|v - L_n(v)\|_{L_{2,\rho_2}}^2 = (a_0^{(n)})^2 + \sum_{k=1}^{\infty}[v - L_n(v,\psi_k)]^2.$$

Therefore

$$a_0^{(n)} \to 0, \quad n \to \infty, \quad \forall v \in C_0[-1,1].$$

An approximate solution of the equation (5.9) is sought in the form

$$\varphi^{(n)} = \sum_{k=1}^{n} a_k\varphi_k.$$

Using the condition

$$\overline{P}_n[(a+bS)\varphi^{(n)} + K\varphi^{(n)} - f] = 0,$$

we compose the collocation system which, with regard for the equality $(a+bS)\varphi_k = \psi_k$, $k=1,2,\dots,$ provides us with the algebraic system

$$a_0\psi_0 + \sum_{k=1}^{n} a_k\psi_k(x_j) + \sum_{k=1}^{n} a_k(K\varphi_k)(x_j) = f(x_j), \quad j=0,1,\dots,n. \quad (5.10)$$

Theorem 5.2. *If there exists the inverse operator* $[I + K(a + bS)^{-1}]^{-1}$ *mapping* $L^{(2)}_{2,\rho_2}$ *onto itself and as interpolation nodes we take the roots of the Jacobi polynomial* $P^{(-\alpha,-\beta)}_{n+1}(x)$*, then the algebraic system* (5.10) *for sufficiently large* n *has a unique solution, and the process converges in the space* L_{2,ρ_2}.

Proof. We rewrite the equation (5.9) in the form

$$[I + K(a + bS)^{-1}]w = f, \quad w \equiv (a + bS)\varphi, \quad w, f \in L^{(2)}_{2,\rho_2},$$

and the approximate equation (5.10) in the form

$$w^{(n)} + \overline{P}_n K(a + bS)^{-1} w^{(n)} = \overline{P}_n f, \quad w^{(n)} \equiv (a + bS)\varphi^{(n)},$$

$w^{(n)} \in \overline{L}^{(n)}_{2,\rho}$, where $\overline{L}^{(n)}_{2,\rho_2}$ is the linear span of the functions $\psi_1, \psi_2, \ldots, \psi_n$.

We have

a) $\|v - \overline{P}_n v\|_{L^{(2)}_{2,\rho_2}} \to 0$, $n \to \infty$,

b) $\|\overline{P}_n v\|_{L^{(2)}_{2,\rho_2}} \leq B_0 C(n) \|v\|_{C_0}$, that is,

$$\|\overline{P}_n\|_{C_0 \to L^{(2)}_{2,\rho_2}} \leq B_0 c(n), \quad c(n) \sim \ln n,$$

where $c(n)$ is a number dependent of n;

c) the operator $K(a + bS)^{-1}$ is completely continuous from $L^{(2)}_{2,\rho_2}$ to C_0;

d) $\|v\|_{L^{(2)}_{2,\rho_2}} \leq c\|v\|_{C_0}$.

Arguing as in point 2 of Section 4, we find that the algebraic system (5.10) for sufficiently large n has a unique solution, and the process converges in the space L_{2,ρ_2}, i.e.,

$$\int_{-1}^{1} (1 - x)^{-\alpha}(1 + x)^{-\beta}(\varphi^{(n)} - \varphi)^2 dx \to 0 \tag{5.11}$$

as $n \to \infty$ $(0 < \alpha,\ \beta < 1)$

If the operator $K \equiv 0$, i.e., we have the equation

$$(a + bS)\varphi = f,$$

with the condition $[f, 1] = 0$, then the algebraic system (5.10) takes the form

$$a_0\psi_0 + \sum_{k=1}^{n} a_k \psi_k(x_j) = f(x_j), \quad j = 0, 1, \ldots, n, \tag{5.12}$$

where $x_0, x_1, \ldots, x_n$ are, as before, the roots of the Jacobi polynomial $P^{(-\alpha,-\beta)}_{n+1}(x)$. □

From Theorem 5.2 we get

Corollary. *The algebraic system* (5.12) *for any* $n = 1, 2, \ldots$ *has the unique solution* $(a_0, a_1, \ldots, a_n)$*, and the process converges in the space* L_{2,ρ_2}.

3) $\varkappa = 0$, $\varkappa = -(\alpha+\beta)$, $\beta = -\alpha$, $|\alpha| \neq \frac{1}{2}$, $0 < |\alpha| < 1$. As in Section 3, the equation will be considered in the weight space $L_{2,\rho_3}[-1,1]$, where the weight $\rho_3 = (1-x)^{-\alpha}(1+x)^{\alpha}$. We have the equation

$$(a + bS + K)\varphi = f, \quad \varphi, f \in L_{2,\rho_3}. \tag{5.13}$$

The kernel $K(x,t)$ and the right-hand side $f(x)$ are required to be continuous in a closed domain. Then the solution w of the equation $w + K(a + bS)^{-1}w = f$ likewise belongs to $C[-1,1]$.

The inverse operator $(a+bS^{-1})$ mapping the space L_{2,ρ_3} onto itself exists. It will be required that the operator $(a+bS+K)^{-1}$, transforming the space L_{2,ρ_3} onto itself, exist too.

To construct an approximate solution of the equation (5.13), just as in the projective method, we use the following two systems:

1) $\varphi_k \equiv b^{-1} \sin|\alpha|\pi)(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{-\alpha}P_k(x)^{(\alpha,-\alpha)}$

and

2) $\psi_k \equiv -(h_k^{(-\alpha,\alpha)})^{-\frac{1}{2}}P_k^{(-\alpha,\alpha)}(x)$, $k = 0, 1, \dots$.

The formula

$$(a + bS)\varphi_k = \psi_k, \quad k = 0, 1, \dots,$$

is valid. Each of these systems is an orthonormal basis in the space L_{2,ρ_3}.

An approximate solution of the equation (5.13) is sought in the form

$$\varphi^{(n)} = \sum_{k=0}^{n} a_k \varphi_k.$$

The method of collocation

$$(a + bS)\varphi^{(n)} + K\varphi^{(n)} - f)(x_j) = 0, \quad j = 0, 1, \dots, n,$$

with regard for $(a + bS)\varphi_k = \psi_k$ provides us with the algebraic system

$$\sum_{k=0}^{n} a_k \psi_k(x_j) + \sum_{k=0}^{n} a_k (K\varphi_k)(x_j) = f(x_j), \quad j = 0, 1, \dots, n. \tag{5.14}$$

Theorem 5.3. *If there exists the inverse operator* $[I + K(a + bS)^{-1}]^{-1}$ *mapping* L_{2,ρ_3} *onto itself and as collocation nodes we take the roots of the Jacobi polynomial* $P_{n+1}^{(-\alpha,\alpha)}(x)$, *then the algebraic system* (5.14) *for sufficiently large* n *has a unique solution, and the collocation process converges in the space* L_{2,ρ_3}.

Proof. Denote $w \equiv (a + bS)\varphi$ and $w^{(n)} \equiv (a + bS)\varphi^{(n)}$. Then the equations (5.13) and (5.14) can be rewritten as

$$w + K(a + bS)^{-1}w = f, \quad w, f \in L_{2,\rho_3}, \tag{5.15}$$

$$w^{(n)}(x_j) + (K(a + bS)^{-1}w^{(n)})(x_j) = f(x_j), \quad j = 0, 1, \dots, n. \tag{5.16}$$

Let P_n be the projector defined by means of the Lagrange interpolation polynomial $P_n v = L_n(v)$, $v \in C[-1,1]$. Then the algebraic system (5.16) can be written in the form

$$w^{(n)} + P_n K(a + bS)^{-1}w^{(n)} = P_n f, \quad w^{(n)} \in \overline{L}_{2,\rho_3}^{(n)}, \tag{5.17}$$

where $\overline{L}^{(n)}_{2,\rho_3}$ is the linear span of the functions $\psi_0, \psi_1, \dots, \psi_n$ (ψ_k is the polynomial of k-th degree).

If as interpolation nodes we take the roots of the Jacobi polynomial $P^{(-\alpha,\alpha)}_{n+1}(x)$, then

a) $\|v - P_n v\|_{L_{2,\rho_3}} \to 0$, $n \to \infty$, $\forall v \in C[-1,1]$. Moreover,

b) $\|P_n\|_{c \to L_{2,\rho_3}} \le B_0 C(n)$;

c) the operator $K(a+bS)^{-1}$ is completely continuous from the space L_{2,ρ_3} to $C[-1,1]$;

d) $\|v\|_{L_{2,\rho_3}} \le C\|v\|_c$.

Then the same arguments as in Theorem 5.1 allow us to show that the algebraic system (5.14) for sufficiently large n has a unique solution, and the process converges:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\alpha}(\varphi^{(n)} - \varphi)^2 dx \to 0 \tag{5.18}$$

as $n \to \infty$. □

If the operator $K \equiv 0$, i.e., we have the equation $(a+bS) = f$, then we obtain the algebraic system

$$\sum_{k=0}^{n} a_k \psi_k(x_j) = f(x_j), \quad j = 0, 1, \dots, n, \tag{5.19}$$

where $x_0, x_1, \dots, x_n$ are again the roots of the Jacobi polynomial $P^{(-\alpha,\alpha)}_{n+1}(x)$. From Theorem 5.3 it follows

Corollary. *The algebraic system* (5.19) *for any* $n = 1, 2, \dots$ *has the unique solution* $(a_0, a_1, \dots, a_n)$*, and the process converges in the space* L_{2,ρ_3}.

In Sections 2–5 the author has presented the results obtained by him in [16] and [17]. The collocation schemes for the equation (1.1) have been given by many authors, for example, by F. Erdogan and G.D. Gupta ([18], [19]). The author's schemes differ considerably from the already known schemes, especially for the index $\varkappa = 1, -1$. Moreover, in his above-mentioned works the author gives the proof for the convergence of approximate schemes.

6. Some Remarks on the Projective and Collocation Schemes

In this section (a) we prove that the process of replacement of the initial problem for the index $\varkappa = 1$ by one functional equation is stable; (b) for the index $\varkappa = -1$, we remove the restriction imposed on the kernel $K(x,t)$ in Sections 2–5; (c) we prove that the projective schemes are stable. Here the results obtained by the author in [20] are presented.

1) In case $\varkappa = 1$, we have

$$(a + bS + K)\varphi = f, \tag{6.1}$$

with the additional condition

$$\int_{-1}^{1} \varphi(t)dt = P, \tag{6.2}$$

where P is a given number.

In Sections 2–5 this problem has been replaced by one functional equation

$$(a + bS + K)\phi = f_1, \quad \phi \in L^{(2)}_{2,\rho_1}, \quad f_1 \in L_{2,\rho_1}, \tag{6.3}$$

where $\phi \equiv \varphi - P\pi^{-1}\sin(|\alpha|\pi)(1-x)^{\alpha}(1+x)^{\beta}$; $-1 < \alpha, \beta < 0$, $\varkappa = -(\alpha+\beta)$

$$f_1 \equiv f - P\pi^{-1}\sin(|\alpha|\pi)K(1-t)^{\alpha}(1+t)^{\beta},$$

$L_{2,\rho_1} = L^{(1)}_{2,\rho_1} \oplus L^{(2)}_{2,\rho_1}$, $L^{(1)}_{2,\rho_1}$ is the linear span of one function, $\varphi_0 = (1-x)^{\alpha}(1+x)^{\beta}$, $L^{(2)}_{2,\rho_1}$ is its orthogonal complement. When defining the function f_1, we have to calculate the integral $K(1-t)^{\alpha}(1+t)^{\beta}$ which, of course, not always can be taken exactly. Let there be an error $\delta(x) = f_1 - \widetilde{f}_1$. Then instead of (6.3) we have the perturbed equation

$$(a + bS + K)\widetilde{\phi} = \widetilde{f}_1, \quad \widetilde{\phi} \in L^{(2)}_{2,\rho_1}, \quad \widetilde{f}_1 \in L_{2,\rho_1}, \tag{6.4}$$

which will be solved approximately. Approximate schemes provide us with approximations $\widetilde{\phi}^{(n)}$, $n = 1, 2, \dots$; the sequence $\{\widetilde{\phi}^{(n)}\}$ converges to $\widetilde{\phi}$ in the metric of the space $L^{(2)}_{2,\rho_1}$. In our schemes we assume that there exists the inverse operator $[I + K(a + bS)^{-1}]^{-1}$ transforming L_{2,ρ_1} onto itself.

Take the difference of (6.3) and (6.4),

$$(a + bS + K)(\phi - \widetilde{\phi}) = \delta(x),$$

hence $\phi - \widetilde{\phi} = (a + bS + K)^{-1}\delta(x)$, or

$$\phi - \widetilde{\phi} = (a + bS)^{-1}[I + K(a + bS)^{-1}]^{-1}\delta(x), \tag{6.5}$$

i.e.,

$$\|\phi - \widetilde{\phi}\|_{L^{(2)}_{2,\rho_1}} \le \|(a + bS)^{-1}\|_{L_{2,\rho_1} \to L^{(2)}_{2,\rho_1}} \times \\ \times \|[I + K(a + bS)^{-1}]^{-1}\|_{L_{2,\rho_1}} \|\delta(x)\|_{L_{2,\rho_1}}. \tag{6.6}$$

The estimate (6.6) shows that the process of approximate substitution of the problem (6.1)–(6.2) by one functional equation (6.3) is stable.

2) $\varkappa = -1$, $\varkappa = -(\alpha + \beta)$, $0 < \alpha, \beta < 1$. We consider the equation $(a+bS+K)\varphi = f$ in the weight space $L_{2,\rho_2}[-1, 1]$, $\rho_2 = (1-x)^{-\alpha}(1+x)^{-\beta}$. If this equation has the solution $\varphi \in L_{2,\rho_2}$, then

$$[K\varphi - f, 1] = \int_{-1}^{1} (K\varphi - f)(1-x)^{-\alpha}(1+x)^{-\beta}\, dx = 0. \tag{6.7}$$

In Sections 2–5 the operator K is subject to the restriction

$$K(L_{2,\rho_2}) \perp 1, \tag{6.8}$$

and the problem is transformed to the functional equation

$$(a + bS + K)\varphi = f, \quad \varphi \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}. \tag{6.9}$$

We now remove the restriction (6.8) and take in the space L_{2,ρ_2} two systems of functions

$$\varphi_{k+1} \equiv (2b)^{-1} \sin(\alpha\pi)(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}}(1-x)^{\alpha}(1+x)^{\beta} P^{(\alpha,\beta)}_{k}(x), \quad k = 0, 1, \dots,$$

and

$$\psi_0 \equiv -\Big(\frac{\sin(\alpha\pi)}{\pi}\Big)^{-\frac{1}{2}}, \quad \psi_{k+1} \equiv -(h^{(-\alpha,-\beta)}_{k+1})^{-\frac{1}{2}} P^{(-\alpha,-\beta)}_{k+1}(x), \quad k = 0, 1, \dots .$$

These systems are orthonormal and complete in L_{2,ρ_2}. The system $\psi_1, \psi_2, \dots$ is orthonormal and complete in $L^{(2)}_{2,\rho_2}$.

Introduce the following notation:

1) $\widetilde{K}(t) \equiv [K(x,t), \psi_0]_x = \int_{-1}^{1} \rho_2(x) K(x,t)\psi_0 dx$;

2) $K_0(x,t) \equiv K(x,t) - \widetilde{K}(t)\psi_0$.

Then

$$[K_0(x,t), \psi_0]_x = [K(x,t), \psi_0]_x - \widetilde{K}(t) = 0.$$

The operator

$$K_0\varphi \equiv \pi^{-1} \int_{-1}^{1} K_0(x,t)\varphi(t)dt$$

satisfies the condition (6.8), i.e., $K_0(L_{2,\rho_2}) \perp 1$. In the above notation, the initial equation can now be written as

$$(a + bS + K_0)\varphi + \pi^{-1} \int_{-1}^{1} \widetilde{K}(t)\psi_0\varphi(t)dt = f, \tag{6.10}$$

and the condition (6.7) as

$$\Big[\pi^{-1} \int_{-1}^{1} \widetilde{K}(t)\psi_0\varphi(t)dt - f, \psi_0\Big] = 0, \tag{6.11}$$

owing to $[K_0\varphi, \psi_0]_x = 0$, $\forall\varphi \in L_{2,\rho_2}$.

The condition (6.11) yields

$$\pi^{-1} \int_{-1}^{1} \widetilde{K}(t)\varphi(t)dt\psi_0 = [f, \psi_0]\psi_0.$$

Therefore (6.10) allows us to get the functional equation

$$(a + bS + K_0)\varphi = f_1, \quad \varphi \in L_{2,\rho_2}, \quad f_1 \in L^{(2)}_{2,\rho_2}, \tag{6.12}$$

where $f_1 \equiv f - [f, \psi_0]\psi_0$. Obviously, $[f_1, \psi_0] = 0$.

Thus the matter is reduced to the case we have considered in Sections 2–5.

Under the condition of existence of the inverse operator $[I + K_0(a + bS)^{-1}]^{-1}$ mapping $L^{(2)}_{2,\rho_2}$ onto itself, let us show that the problem (6.1)–(6.7) and the functional equation (6.12) are equivalent.

Let φ be a solution of the initial problem (6.1)–(6.7). The problem (6.1)–6.7) is equivalent to the problem (6.10)–(6.11), hence

$$(a + bS + K_0)\widetilde{\varphi} = f_1, \quad \widetilde{\varphi} \in L_{2,\rho_2}, \quad f_1 \in L^{(2)}_{2,\rho_2}. \tag{6.13}$$

Taking the difference of formulas (6.12) and (6.13), we obtain

$$(a + bS + K_0)(\varphi - \widetilde{\varphi}) = 0,$$

i.e.,

$$\varphi - \widetilde{\varphi} = 0.$$

In both projective and collocation approximate schemes, we mean that the additional condition (6.8) is fulfilled. Let us now show that for practical realizations there is no need in calculating $\widetilde{K}(t)$ and $[f, \psi_0]$.

In the projective scheme, an approximate solution is sought in the form

$$\varphi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k,$$

and, just as in Section 2, we compose the algebraic system

$$a_i + \sum_{k=1}^{n} a_k \pi^{-1} \Big[[K(x,t), \varphi_k]_t, \psi_i\Big]_x = [f, \psi_i], \quad i = 1, 2, \dots, n. \tag{6.14}$$

Taking into account that

$$K(x,t) = \sum_{i=0}^{\infty} [K(x,t), \psi_i]_x \psi_i(x), \quad f(x) = \sum_{i=0}^{\infty} [f, \psi_i] \psi_i(x),$$

we obtain

$$[f, \psi_i] = [f_1, \psi_i], \quad i = 1, 2, \dots.$$

Next,

$$\Big[[K(x,t), \varphi_k]_t, \psi_i\Big]_x = \Big[[K(x,t), \psi_i]_x, \varphi_k\Big]_t = \Big[[K_0(x,t), \psi_i]_x, \varphi_k\Big]_t = $$
$$= \Big[[K_0(x,t), \varphi_k]_t, \psi_i\Big]_x.$$

Thus if in the system (6.14) we replace $K(x,t)$ by $K_0(x,t)$ and f by f_1, everything remains unchanged.

In the collocation scheme, an approximate solution is again sought in the form

$$\varphi^{(n)} = \sum_{k=1}^{n} a_k \varphi_k$$

and the algebraic system is composed from the condition

$$\overline{P}_n[(a + bS + K)\varphi^{(n)} - f_1] = 0. \tag{6.15}$$

For any continuous function, the projector $\overline{P}$ is defined as follows:

$$\overline{P}_n v = L_n(v) - a_0\psi_0,$$

where $L_n(v)$ is the Lagrange interpolation polynomial, and a_0 is the first coefficient of the decomposition $L_n(x) = a_0\psi_0 + a_1\psi_1 + \cdots + a_n\psi_n$. When v is unknown, the coefficient a_0 is undefined and the equation (6.15) can be written in the form

$$L_n[(a + bS + K_0)\varphi^{(n)} - f_1] - a_0\psi_0 = 0. \tag{6.16}$$

The Lagrange polynomial is defined uniquely by the values of the function at the discrete points, therefore (6.16) is equivalent to the algebraic system

$$-a_0\psi_0 + \sum_{k=1}^{n} a_k\psi_k(x_i) + \sum_{k=1}^{n} a_k(K_0\varphi_k)(x_i) = f_1(x_i), \tag{6.17}$$

$i = 0, 1, \dots, n$, x_i are the roots of the Jacobi polynomial $P_{n+1}^{(-\alpha,-\beta)}(x)$.

In Section 5 we have proved the existence of the unique solution (a_0, $a_1, \dots, a_n$) of the system (6.17) for $n > n_0$ and the convergence of the process.

Let now the projector $\overline{P}_n$ be used directly (not passing to $K_0(x,t)$ and f_1)

$$\overline{P}_n[(a + bS + K)\varphi^{(n)} - f] = 0, \tag{6.18}$$

i.e.,

$$\overline{P}_n\Bigg[(a + bS + K_0)\varphi^{(n)} + \pi^{-1}\int_{-1}^{1}\widetilde{K}(t)\varphi^{(n)}(t)\psi_0 dt - (f_1 + [f_1,\psi_0]\psi_0)\Bigg] = 0.$$

We require that

$$\pi^{-1}\int_{-1}^{1}\widetilde{K}(t)\varphi^{(n)}(t)dt\psi_0 = [f,\psi_0]\psi_0,$$

i.e.,

$$\pi^{-1}\int_{-1}^{1}\widetilde{K}(t)\varphi^{(n)}(t)dt = [f,\psi_0].$$

Then (6.18) transforms to (6.17).

The algebraic system shows that there is no need in calculating $\widetilde{K}(t)$ and $[f,\psi_0]$. We rewrite the formula (6.18) in the form

$$L_n[(a + bS + K)\varphi^{(n)} - f] - \widetilde{a}_0\psi_0 = 0, \tag{6.19}$$

i.e.,

$$-\widetilde{a}_0\psi_0 + \sum_{k=1}^{n}\widetilde{a}_k\psi_k(x_i) + \sum_{k=1}^{n}\widetilde{a}_k(K\varphi_k)(x_i) = f(x_i), \quad i = 0, 1 \dots, n. \tag{6.20}$$

Thus we have

$$(K\varphi_k)(x_i) = \left(\pi^{-1}\int_{-1}^{1} K(x_i,t)\varphi_k(t)dt\right)(x_i) =$$

$$= \left(\pi^{-1}\int_{-1}^{1} K_0(x_i,t)\varphi_k(t)dt + \pi^{-1}\int_{-1}^{1} \widetilde{K}(t)\psi_0\varphi_k(t)dt\right) =$$

$$= \left(\pi^{-1}\int_{-1}^{1} K_0(x_i,t)\varphi_k(t)dt\right)(x_i) + C_k = (K_0\varphi_k)(x_i) + C_k,$$

$$f(x_i) = f_1(x_i) + ([f,\psi_0]\psi_0) = f_1(x_i) + C,$$

where C_k and C are constants independent of i. Therefore the algebraic system (6.20) can be written as

$$-\widetilde{a}_0\psi_0 + \sum_{k=1}^{n}\widetilde{a}_k\psi_k(x_i) + \sum_{k=1}^{n}\widetilde{a}_k[(K_0\varphi_k)(x_i) + C_k] =$$
$$= f_1(x_i) + C, \quad i = 0,1,\dots,n. \tag{6.21}$$

The first column of the determinant of the system (6.21) is $(-\psi_0, -\psi_0,\dots,-\psi_0)^T$ (ψ_0 is a constant), hence the algebraic systems (6.17) and (6.21) have the same solutions $(a_1,a_2,\dots,a_n)$; the determinants of the algebraic systems (6.17) and (6.21) are equal and the numerators in Kramer formulas for determination of $(a_1,a_2,\dots,a_n)$ are also equal.

Thus for practical calculations there is no need finding $\widetilde{K}(t)$ and $[f,\psi_0]$ which are important only in proving the existence of a solution of the algebraic system and for the convergence of the process.

3) Consider now the questions connected with the stability of approximate schemes. Projective methods considered in Sections 2 and 3 are stable. The stability of the projective scheme will be shown for the equation of the first kind $(S+K)\varphi = f$ for the index $\varkappa = 1$. The stability of the projective scheme for another values of the index and for the equation of the second kind can be established analogously.

Let the matrix $R_n \equiv (([K\varphi_k,\psi_i]))_{k,i=1}^{(n)}$ and the vector $f_1^{(n)} = (f_1,f_2,\dots,f_n)$ be calculated with the errors $\Gamma_n \equiv ((\gamma_{k,i}))_{k,i=1}^{(n)}$ and $\delta^{(n)} \equiv (\delta_1,\delta_2,\dots,\delta_n)$, respectively. Then instead of

$$(I_n + R_n)a^{(n)} = f_1^{(n)} \tag{6.22}$$

we have the algebraic system

$$[I_n + (R_n + \Gamma_n)]\widetilde{a}^{(n)} = f_1^{(n)} + \delta^{(n)}, \tag{6.23}$$

and instead of

$$w^{(n)} + P_nKS^{-1}w^{(n)} = P_nf_1, \quad w^{(n)} \equiv S\phi^{(n)}, \tag{6.24}$$

we have the perturbed equation

$$\widetilde{w}^{(n)} + [P_n K S^{-1} + \Delta(P_n K S^{-1})]\widetilde{w}^{(n)} = P_n f_1 + \Delta(P_n f_1), \tag{6.25}$$

$\widetilde{w}^{(n)} \equiv S\widetilde{\phi}^{(n)}$.

Introduce the operator

$$\tau_n\bigg(\sum_{k=1}^{n} \alpha_k \psi_k\bigg) = \alpha^{(n)},$$

where the vector $\alpha^{(n)} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, the domain of definition of the operator τ_n, $D(\tau_n) = \overline{L}^{(n)}_{2,\rho_1}$ is the linear span of the functions $\psi_1, \psi_2, \dots, \psi_n$, the domain of values $R(\tau_n) = l_2^{(n)}$, where $l_2^{(n)}$ is the n-dimensional vector space. The inverse operator τ_n^{-1} maps $l_2^{(n)}$ onto $\overline{L}^{(n)}_{1,\rho_1}$. The norms $\|\tau_n\| = \|\tau_n^{-1}\| = 1$, since the system $\psi_1, \psi_2, \dots$ is orthonormal in L_{2,ρ_1}. Using the operators τ_n and τ_n^{-1}, we can easily realize the passage from the equation (6.23) to (6.25), and vice versa. For example, proceeding from the equation (6.25), we have

$$\begin{aligned}\tau_n \widetilde{w}^{(n)} + \tau_n[P_n K S^{-1} + \Delta(P_n K S^{-1})]\tau_n^{-1}\tau_n \widetilde{w}^{(n)} = \\ = \tau_n P_n f_1 + \tau_n(\Delta P_n f_1).\end{aligned}$$

The norm $\|P^{(n)} K S^{-1}\| \to 0$ as $n \to \infty$ ($P^n = I - P_n$), and therefore for $0 < q < 1$ there exists n_0 such that for $n \geq n_0$

$$\|(I + P_n K S^{-1})^{-1}\| \leq \frac{\|(I + K S^{-1})^{-1}\|}{1 - q}.$$

If

$$\frac{\|(I + K S^{-1})^{-1}\|}{1 - q}\, \|\Delta_n(P_n K S^{-1})\| < 1,$$

then the perturbed equation (6.25) has a unique solution $\widetilde{w}(n)$, and

$$\|\widetilde{w}^{(n)} - w^{(n)}\| \leq C_1\|\Delta(P_n K S^{-1})\| + C_2\|\Delta(P_n f_1)\|. \tag{6.26}$$

Then the algebraic system (6.23) for $n \geq n_0$ has likewise a unique solution.

Further, we have

$$\begin{aligned}\|\Delta(P_n K S^{-1})\|_{L_{2,\rho_1}} &= \|\tau_n^{-1}\Gamma_n \tau_n\| \leq \bigg(\sum_{i,k=1}^{n} \gamma_{i,k}^2\bigg)^{\frac{1}{2}}, \\ \|\Delta(P_n f_1)\|_{L_{2,\rho_1}} &= \|\tau_n^{-1}\delta^{(n)}\| = \|\delta^{(n)}\|, \\ \|\widetilde{w}^{(n)} - w^{(n)}\| &= \|\widetilde{\phi}^{(n)} - \phi^{(n)}\|,\end{aligned}$$

and hence (6.26) yields

$$\|\widetilde{\phi}^{(n)} - \phi^{(n)}\| \leq C_1\bigg(\sum_{i,k=1}^{n} \gamma_{ik}^2\bigg)^{\frac{1}{2}} + C_2\bigg(\sum_{i=1}^{n} \delta_i^2\bigg)^{\frac{1}{2}},$$

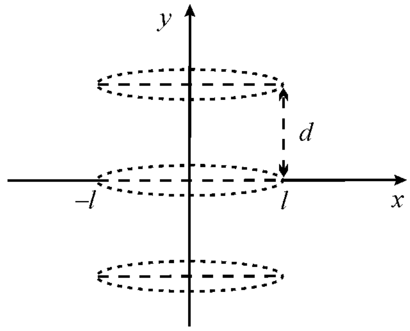
which means that the process is stable.

Note that another approach for approximate solution of the equation (1.1) is available in [21], and the questions dealing with the approximation of a singular integral can be found in [22].

7. Numerical Solutions of Some Singular Integral Equations of the Plane Theory of Elasticity

In this section we present numerical solutions of some problems of the plane theory of cracks. Coefficients of stress intensity are calculated for the following problems: 1) periodic system of parallel cracks; 2) interior crack in a half-plane; 3) interior crack in a non-homogeneous plane consisting of homogeneous half-planes. There are no exact solutions for none of the above-mentioned problems.

1) Periodic system of parallel cracks. Under symmetric distribution of stresses with respect to the crack line the following integral equation has been obtained for the periodic system of parallel cracks [23]:



$$\int_{-l}^{l} v'(t)K_1(t-x)dt = \pi\sigma(x), \quad |x| < l, \tag{7.1}$$

$2l$ is the crack length, d is the distance between the cracks, the function $v'(t)$ defines the coefficients of stress intensity by the formula ([23], p. 132)

$$K_1^{\pm} = \mp \lim_{t\to\pm l}\Big[\Big(\frac{l^2-t^2}{l}\Big)^{\frac{1}{2}} v'(t)\Big],$$

("+" and "−" denote the right and the left end, respectively), the kernel

$$K_1(\tau) = \frac{\pi}{d}\Big(2\coth\frac{\pi\tau}{d} - \frac{\pi\tau}{d}\operatorname{csch}^2\frac{\pi\tau}{d}\Big),$$

$\sigma(x)$ is the load applied to the crack contours. The additional condition

$$\int_{-l}^{l} v'(t)dt = 0 \tag{7.2}$$

is given.

After simple calculations the function $K_1(\tau)$ takes the form

$$K_1(\tau) = \frac{1}{\tau} + \frac{\frac{2\pi\tau}{d}\operatorname{sh}\frac{2\pi\tau}{d} - \operatorname{ch}\frac{2\pi\tau}{d} + 1 - 2(\frac{\pi\tau}{d})^2}{\tau\operatorname{ch}\frac{2\pi\tau}{d} - \tau}.$$

Introduce the following notation:

$$t_1 = \frac{t}{l},\quad v'(t) = v'(lt_1) = \varphi(t_1),\quad x_1 = \frac{x}{l},\quad \sigma(x) = \sigma(lx_1) = f(x_1),$$

the parameter $\lambda = \frac{2l}{d}$. Then we have

$$K_1(x_1,t_1) =$$

$$= \frac{\pi\lambda(t_1-x_1)\operatorname{sh}\pi\lambda(t_1-x_1) - \operatorname{ch}\pi\lambda(t_1-x_1) + 1 - 2[\frac{\pi\lambda(t_1-x_1)}{2}]^2}{(t_1-x_1)\operatorname{ch}\pi\lambda(t_1-x_1) - (t_1-x_1)}.$$

Denote $\xi = \pi\lambda(t_1-x_1)$. Then

$$K_1(x_1,t_1) = \frac{\xi\operatorname{sh}\xi - \operatorname{ch}\xi + 1 - 2(\frac{\xi}{2})^2}{\frac{\xi}{\pi\lambda}(\operatorname{ch}\xi - 1)} = \frac{\xi\frac{e^{\xi}-e^{-\xi}}{2} - \frac{e^{\xi}+e^{-\xi}}{2} + 1 - \frac{\xi^2}{2}}{\frac{\xi}{\pi\lambda}(\frac{e^{\xi}-e^{-\xi}}{2} - 1)} =$$

$$= \frac{\xi[(1+\xi+\frac{\xi^2}{2!}+\cdots) - (1-\xi+\frac{\xi^2}{2!}-\cdots)]}{\frac{\xi}{\pi\lambda}[(1+\xi+\frac{\xi^2}{2!}+\cdots) + (1-\xi+\frac{\xi^2}{2!}-\cdots) - 2]} -$$

$$- \frac{[(1+\xi+\frac{\xi^2}{2!}+\cdots) - (1-\xi+\frac{\xi^2}{2!}-\cdots)] + 2 - \xi^2}{\frac{\xi}{\pi\lambda}[(1+\xi+\frac{\xi^2}{2!}+\cdots) + (1-\xi+\frac{\xi^2}{2!}-\cdots) - 2]} =$$

$$= \frac{\xi(2\xi + 2\frac{\xi^3}{3!}+\cdots) - (2 + 2\frac{\xi^2}{2!}+\cdots) + 2 - \xi^2}{\frac{\xi}{\pi\lambda}(2\frac{\xi^2}{2!} + 2\frac{\xi^4}{4!}+\cdots)} =$$

$$= \frac{2\xi^4[(\frac{1}{3!}-\frac{1}{4!}) + (\frac{1}{5!}-\frac{1}{6!})\xi^2+\cdots]}{\frac{2\xi^3}{\pi\lambda}(\frac{1}{2!}+\frac{\xi^2}{4!}+\cdots)} = \frac{\pi\lambda\xi(\frac{3}{4!}+\frac{5}{6!}\xi^2+\cdots)}{\frac{1}{2!}+\frac{\xi^2}{4!}+\cdots} =$$

$$= \frac{(\pi\lambda)^2(t_1-x_1)\{\frac{3}{4!}+\frac{5}{6!}[\pi\lambda(t_1-x_1)]^2+\cdots\}}{\frac{1}{2!}+\frac{[\pi\lambda(t_1-x_1)]^2}{4!}+\cdots}.$$

Thus the problem (7.1)–(7.2) can be written as follows:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t_1)dt}{t_1-x_1} + \frac{1}{\pi}\int_{-1}^{1}K(x_1,t_1)\varphi(t_1)dt_1 = f(x_1),\quad |x_1|<1,$$

$$\int_{-1}^{1}\varphi(t_1)dt_1 = 0,\quad K(x_1,t_1) = K_1(x_1,t_1).$$

Instead of t_1, x_1 we write t, x. Finally, we have the following singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1}K(x,t)\varphi(t)dt = f(x) \tag{7.3}$$

with the additional condition

$$\int\limits_{-1}^{1} \varphi(t)dt = 0, \tag{7.4}$$

where the kernel

$$K(x,t) = \frac{(n\lambda)^2(t-x)\Big\{\dfrac{3}{4!} + \sum\limits_{n=2}^{\infty} \dfrac{2n+1}{(2n+2)!}[\pi\lambda(t-x)]^{2n-2}\Big\}}{\dfrac{1}{2!} + \sum\limits_{n=2}^{\infty} \dfrac{1}{(2n)!}[\pi\lambda(t-x)]^{2n-2}}.$$

Thus we have the singular integral equation of the first kind (7.3) with the index $\varkappa = 1$ (the condition (7.4)). The kernel $K(x,t)$ is the ratio of two converging power series (for any values of parameters).

The problem (7.3)–(7.4) is solved by the method of collocation. We take the constant load $f(x) = -1$ and seek the fifth approximation

$$\varphi^{(5)}(x) = \sum_{k=1}^{5} a_k \varphi_k(x).$$

The functions $\varphi_k(x)$ have the form

$$\varphi_k(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} T_k(x), \quad k = 1,2,\dots,5,$$

where $T_k(x)$ are the Chebyshev polynomials of the first kind:

$$\begin{aligned} &T_1(x) = x, && T_4(x) = 8x^4 - 8x^2 + 1,\\ &T_2(x) = 2x^2 - 1, && T_5(x) = 16x^5 - 20x^3 + 5x.\\ &T_3(x) = 4x^3 - 3x, && \end{aligned}$$

Moreover, we need the functions

$$\psi_k(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} U_{k-1}(x), \quad k = 1,2,\dots,5,$$

where U_{k-1} are the Chebyshev polynomials of the second kind:

$$\begin{aligned} &U_0(x) = 1, && U_3(x) = 8x^3 - 4x,\\ &U_1(x) = x, && U_4(x) = 16x^4 - 12x^2 + 1.\\ &U_2(x) = 4x^2 - 1, && \end{aligned}$$

The algebraic system (4.3) takes the form

$$\sum_{k=1}^{5} a_k \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} U_{k-1}(x_j) + \sum_{k=1}^{5} a_k (K\varphi_k)(x_j) = -1, \quad j = 1,2,\dots,5, \tag{7.5}$$

where the collocation nodes

$$x_j = \cos\frac{j\pi}{6}, \quad j = 1,2,\dots,5,$$

are the roots of the Chebyshev polynomial

$$U_5(x) = 32x^5 - 32x^3 + 6x:$$

$$x_1 = \frac{\sqrt{3}}{2}, \quad x_2 = \frac{1}{2}, \quad x_3 = 0, \quad x_4 = -\frac{1}{2}, \quad x_5 = -\frac{\sqrt{3}}{2}.$$

To calculate the integrals with weak singularities $K\varphi_k$, $k = 1, 2, \dots, 5$, we use the following Gauss–Chebyshev formula [7]

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} g(t)dt \approx \frac{\pi}{m} \sum_{i=1}^{m} g\left(\cos \frac{2i-1}{2m}\right).$$

Taking $m = 10$, in our case we have

$$g(t) = \frac{1}{\pi}\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} K(x_j, t) T_k(t).$$

In the kernel $K(x,t)$, we take in numerator and denominator as many terms in the series as needed for the error to be less than 10^{-5}. Towards this end, for the value of the parameter $\lambda = 1$ we need the terms up to the twentieth degree.

The main difficulty arises in calculating the expression

$$(K\varphi_k)(x_j) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \frac{1}{10} \sum_{i=1}^{10} K\Big(x_j, \cos\frac{2i-1}{20}\pi\Big) T_k\Big(\cos\frac{2i-1}{20}\pi\Big) \equiv$$
$$\equiv \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} \gamma_{k,j} \quad k, j = 1, 2, \dots, 5.$$

As a result of calculation, we obtain the algebraic system

$$(1+\gamma_{1j})a_1 + (2x_j + \gamma_{2j})a_2 + (4x_j^2 - 1 + \gamma_{3j})a_3 +$$
$$+ (8x_j^3 - 4x_j + \gamma_{4j})a_4 + (16x_j^4 - 12x_j^2 + 1 + \gamma_{5j})a_5 =$$
$$= -\Big(\frac{2}{\pi}\Big)^{-\frac{1}{2}}, \quad j = 1, 2, \dots, 5. \tag{7.6}$$

We take the following values of the parameter: $\lambda = 0.1, 0.2, \dots, 1.1$. The parameter λ affects γ_{kj}, $k, j = 1, 2, \dots, 5$. Using the Gaussian method, we solve the algebraic system (7.6). For the coefficients $a_1, a_2, \dots, a_5$ we obtain the following numerical values.

λ	a_1	a_2	a_3	a_4	a_5
0.1	-1.2381020	0.0000000	-0.0000310	0.0000000	0.0000000
0.2	-1.1952743	0.0000000	-0.0004621	0.0000000	0.0000024
0.3	-1.1320088	0.0000000	-0.0020897	0.0000000	0.0000238
0.4	-1.0570586	0.0000000	-0.0057024	0.0000000	0.0001090
0.5	-0.9782156	0.0000000	-0.0117116	0.0000000	0.0003247
0.6	-0.9010623	-0.0000001	-0.0200507	0.0000000	0.0007316
0.7	-0.8288983	-0.0000001	-0.0302837	0.0000000	0.0013545
0.8	-0.7632536	-0.0000001	-0.0417850	0.0000000	0.0021656
0.9	-0.7045076	-0.0000001	-0.0538971	0.0000000	0.0030879
1.0	-0.6523904	-0.0000001	-0.0660312	0.0000000	0.0040109
1.1	-0.6063210	-0.0000001	-0.0777163	0.0000000	0.0048114

The coefficients of stress intensity are as follows

$$K_1^{\pm} = \mp \lim_{x\to\pm 1} [(l(1-x^2))^{\frac{1}{2}}\varphi(x)].$$

Taking $l = 1$, we have

$$K_1^{\pm} = \mp \lim_{x\to\pm 1} [(1-x^2)^{\frac{1}{2}}\varphi^{(5)}(x)] = \mp \lim_{x\to\pm 1} \sum_{k=1}^{5} a_k \left(\frac{2}{\pi}\right)^{\frac{1}{2}} T_k(x).$$

Next, $T_k(1) = 1$, $T_k(-1) = (-1)^k$, hence

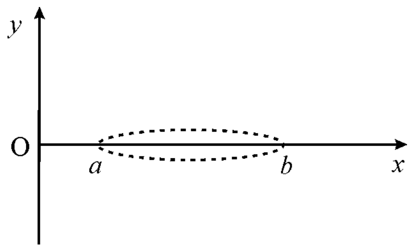
$$K_1^{+} = -\sum_{k=1}^{5} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} a_k, \quad K_1^{-} = \sum_{k=1}^{5} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (-1)^k a_k.$$

Applying these formulas , we calculate the coefficients of stress intensity, K_1^+ and K_1^-. Here we present the following numerical results.

λ	k_1^+	k_1^-
0.1	0.9878877	0.9878878
0.2	0.9540582	0.9540582
0.3	0.9048612	0.9048612
0.4	0.8478740	0.8478740
0.5	0.7895890	0.7895889
0.6	0.7343585	0.7343584
0.7	0.6844478	0.6844477
0.8	0.6406004	0.6406003
0.9	0.6026560	0.6026559
1.0	0.5700177	0.5700176
1.1	0.5419442	0.5419441

The coefficients of stress intensity at the crack ends must be equal; this property remains to within 10^{-6}. The crack length $2l = 2$ ($l = 1$), the parameter $\lambda = \frac{2l}{d} = \frac{2}{d}$, where d is the distance between the cracks. As the distance d increases, the stress coefficient at the crack ends decreases. The obtained numerical results are close to those obtained in [23] (pp. 134–139).

2) Interior crack in a half-plane. Consider the problem of determination of the stressed state of an elastic half-plane with an interior cut perpendicular to the boundary ([23], p. 175) (the crack is on the segment $[a, b]$).



Stresses on the half-plane boundary and at infinity are assumed to be absent, and on the cut contours $y = \pm 0$, $a \le x \le b$, the loads act. This problem is reduced to the singular integral equation ([23], p. 176)

$$\int\limits_a^b \Big[\frac{1}{t-x} + \frac{1}{t+x} + \frac{2t}{(t+x)^2} - \frac{4t^2}{(t+x)^3}\Big] g'(t)dt = \pi P(x), \tag{7.7}$$

$a < x < b$, $P(x)$ is the load, $g(t)$ is the displacement,

$$g(a) = g(b) = 0 \quad (a > 0). \tag{7.8}$$

In the general case, for any function $P(x)$ there is no exact solution of the problem (7.7)–(7.8), but an approximate solution of that problem is obtained in [23] by approximating the kernel of the integral equation. Numerical results for $a > 0$ are not available in [23]; they are given for the limiting case, as $a \to 0$.

We transform the problem (7.7)–(7.8) to the form which will be convenient for our consideration and introduce the new variable

$$x = \frac{a+b}{2} + \frac{b-a}{2}\tau \quad \left(t = \frac{a+b}{2} + \frac{b-a}{2}\xi\right).$$

Denote

$$g'(t) = \varphi(\xi), \quad P(x) = f(\tau).$$

Then the problem (7.7)–(7.8) can be written as

$$\frac{1}{\pi}\int\limits_{-1}^{1} \frac{\varphi(\xi)d\xi}{\xi - \tau} + \frac{b-a}{2\pi} \times$$

$$\times \int\limits_{-1}^{1} \frac{[(\frac{a+b}{2} + \frac{b-a}{2}\tau)^2 + 4(\frac{a+b}{2} + \frac{b-a}{2}\xi)(\frac{a+b}{2} + \frac{b-a}{2}\tau) - (\frac{a+b}{2} + \frac{b-a}{2}\xi)^2]}{[a+b+\frac{b-a}{2}(\xi+\tau)]^3} \varphi(\xi)d\xi =$$

$$= f(\tau), \tag{7.9}$$

$$\int\limits_{-1}^{1} \varphi(\xi)d\xi = 0. \tag{7.10}$$

Consider two cases:

1) if $a = 1$, $b = 3$, then $x = 2 + \tau$ $(t = 2 + \xi)$;

2) if $a = 0.5$, $b = 2.5$, then $x = 1.5 + \tau$ $(t = 1.5 + \xi)$.

Instead of the variables τ and ξ we use again x and t. As a result, we obtain two problems: for the case 1) we have

$$\frac{1}{\pi}\int\limits_{-1}^{1} \frac{\varphi(t)dt}{t-x} + \frac{1}{\pi}\int\limits_{-1}^{1} K_1(x,t)\varphi(t)dt = f(x), \tag{7.11}$$

$$\int_{-1}^{1} \varphi(t)dt = 0, \tag{7.12}$$

where the kernel

$$K_1(x,t) = \frac{(2+x)^2 + 4(2+t)(2+x) - (2+t)^2}{(4+t+x)^3}.$$

For the case (2) we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi(t)dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1} K_2(x,t)\varphi(t)dt = f(x), \tag{7.13}$$

$$\int_{-1}^{1} \varphi(t)dt = 0, \tag{7.14}$$

where the kernel

$$K_2(x,t) = \frac{(1,5+x)^2 + 4(1,5+t)(1,5+x) - (1,5+t)^2}{(3+t+x)^3}\,.$$

We solve the problems (7.11)–(7.12) and (7.13)–(7.14) by using the method of collocation.We take the constant load $f(x) \equiv -1$. In this case we need the same functions as in the problem of parallel cracks. An approximate solution is sought in the form

$$\varphi^{(n)}(x) = \sum_{k=1}^{n} a_k \varphi_k.$$

To define the coefficients $a_1, a_2, \dots, a_n$, we obtain for the problem (7.11)–(7.12) the following algebraic system:

$$\sum_{k=1}^{n} \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} a_k \Big\{U_{k-1}(x_j) + \frac{1}{m}\sum_{i=1}^{m}\Big[K_1(x_j, \cos\frac{2i-1}{2m}\pi)T_k\Big(\cos\frac{2i-1}{2m}\pi\Big)\Big]\Big\} = \\ = -1, \quad j = 1,2,\dots,n.$$

The collocation nodes

$$x_j = \cos\frac{j\pi}{n+1}, \quad j = 1,2,\dots,n.$$

For the problem (7.13)–(7.14) we replace $K_1(x,t)$ by the function $K_2(x,t)$. The kernels $K_1(x,t)$ and $K_2(x,t)$ are rational functions and hence the corresponding integrals can be easily calculated.

We calculate the coefficients of stress intensity by using the formulas of [23] (p. 128)

$$K_1^+ = -\lim_{x\to b}[(2(b-x))^{\frac{1}{2}} g'(x)] \quad \text{and} \quad K_1^- = -\lim_{x\to a}[(2(x-a))^{\frac{1}{2}} g'(x)].$$

In our case, for $a = 1$, $b = 3$ ($x = 2 + \tau$) we have

$$\begin{cases} K_1^+ = -\lim\limits_{\tau\to 1}[(2(1-\tau))^{\frac{1}{2}}\varphi(\tau)], \\ K_1^- = \lim\limits_{\tau\to -1}[(2(1+\tau))^{\frac{1}{2}}\varphi(\tau)]. \end{cases} \tag{7.15}$$

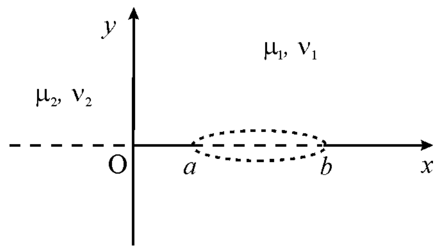
For $a = 0.5$ and $b = 2.5$, the formulas (7.15) remain valid.
From (7.15) we find that

$$K_1^+ = -\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\sum_{k=1}^{n} a_k, \quad K_1^- = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\sum_{k=1}^{n}(-1)^k a_k.$$

For the coefficients of stress intensity we have obtained the following numerical values.

Number of Approximation	Segment	k_1^+	k_1^-
$n = 5$	$a = 1, \quad b = 3$	1.05386	1.09139
	$a = 0,5, \; b = 2.5$	1.09645	1.20493
$n = 10$	$a = 1, \quad b = 3$	1.05390	1.09131
	$a = 0.5, \; b = 2.5$	1.09669	1.20355

It can be seen that the coefficient of stress intensity on the left end is greater than on the right one ($K_1^- > K_1^+$), which corresponds to the physical content. Moreover, when the crack is shifting to the boundary, the coefficients of stress intensity increase, which also corresponds to the physical content. When an analogous crack takes place on an infinite plane, the coefficients of stress intensity at the ends are equal to unity [23]. As the half-plane recedes, the coefficients of stress intensity increase. Numerical solutions likewise satisfy physical meaning. The fifth and tenth approximations for $a = 1$, $b = 3$ coincide to within 10^{-3}, and for $a = 0.5$, $b = 2.5$, to within 10^{-2}.



3) Interior crack in a non-homogeneous plane consisting of homogeneous half-planes [24]. Let there be two half-planes ($x \geq 0$ and $x \leq 0$) of different materials which are rigidly clamped along the line $x = 0$. One of the half-planes ($x \geq 0$) has interior crack on the segment $[a, b]$ ($a > 0$). Symmetric normal stresses $P(x)$ are applied to the crack contours. This problem is reduced to the following singular integral equation

$$\frac{1}{\pi}\int\limits_a^b \frac{g'(t)dt}{t-x}+\frac{1}{\pi}\int\limits_a^b\Big[-\frac{1}{2}\frac{1-\alpha}{\alpha+\varkappa_1}+\frac{\varkappa_1-\alpha\varkappa_2}{1+\alpha\varkappa_2}\frac{1}{t+x}+$$

$$+\frac{2(\alpha-1)}{\alpha+\varkappa_1}\frac{tx-t^3}{(t+x)^3}\Big]g'(t)dt=P(x); \tag{7.16}$$

$$\alpha=\frac{\mu_1}{\mu_2},\quad \varkappa_k=\frac{3-\nu_k}{1+\nu_k},\quad k=1,2,$$

μ_1, μ_2 are Lamé constants and ν_1, ν_2 are Poisson coefficients. The right half-plane is taken to be aluminum and the left one is steel. Poisson's coefficients $\nu_1=0.34$ and $\nu_2=0.26$. Young's moduli are:

$$E_1=0,7\cdot 10^6\frac{K_2}{cm^2}\quad\text{and}\quad E_2=2,1\cdot 10^6\frac{K_2}{cm^2}.$$

Lamé's constant

$$\mu=\frac{E}{2(1+\nu)}.$$

Equation (7.16) takes the form

$$\frac{1}{\pi}\int\limits_a^b \frac{g'(t)dt}{t-x}+\frac{1}{\pi}\int\limits_a^b K(x,t)g'(t)dt=P(x), \tag{7.17}$$

where

$$K(x,t)=-0.54\cdot\frac{1}{t+x}-0.60\cdot\frac{tx-t^3}{(t+x)^3}\,,$$

$$g(a)=g(b)=0.$$

We consider two cases.

1) $a=1$, $b=3$ and 2) $a=0.5$, $b=2.5$.

For the first case we introduce new variables τ, ξ; $x=2+\tau$, $t=2+\xi$. Denote $g'(2+\xi)\equiv\varphi(\xi)$, $P(2+\tau)\equiv f(\tau)$, $K(2+\tau,2+\xi)\equiv K_1(\tau,\xi)$. Then from (7.17) we have

$$\frac{1}{\pi}\int\limits_{-1}^1 \frac{\varphi(\xi)d\xi}{\xi-\tau}+\frac{1}{\pi}\int\limits_{-1}^1 K_1(\tau,\xi)\varphi(\xi)d\xi=f(\tau),$$

$$\int\limits_{-1}^1 \varphi(\xi)d\xi=0.$$

Instead of the variables τ, ξ we again write x,t and obtain the singular integral equation

$$\frac{1}{\pi}\int\limits_{-1}^1 \frac{\varphi(t)dt}{t-x}+\frac{1}{\pi}\int\limits_{-1}^1 K_1(x,t)\varphi(t)dt=f(x), \tag{7.18}$$

where

$$K_1(x,t) = -0.54. \cdot \frac{1}{4+t+x} - 0.60 \cdot \frac{(2+t)(2+x)-(2+t)^3}{(4+t+x)^3},$$
$$\int_{-1}^{1} \varphi(t)dt = 0. \tag{7.19}$$

Analogously, for the second case ($a = 0.5$, $b = 2.5$) we obtain the following singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\varphi(t)dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1} K_2(x,t)\varphi(t)dt = f(x), \tag{7.20}$$

where

$$K_2(x,t) = -0.54 \cdot \frac{1}{3+t+x} - 0.60 \cdot \frac{(1,5+t)(1,5+x)-(1,5+t)^3}{(3+t+x)^3},$$
$$\int_{-1}^{1} \varphi(t)dt = 0. \tag{7.21}$$

As before, using the method of collocation, we seek for an approximate solution

$$\varphi^{(n)}(x) = \sum_{k=1}^{n} a_k \varphi_k(x).$$

The coefficients of stress intensity can be calculated as in problem (2) by the formulas

$$K_1^+ = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{k=1}^{n} a_k, \quad \text{and} \quad K_1^- = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{k=1}^{n} (-1)^k a_k.$$

For the coefficients of stress intensity we obtain the following numerical values:

Approximation	Segment	k_1^+	k_1^-
$n = 5$	$a = 1, \quad b = 3$	0.96250	0.93435
	$a = 0.5, \; b = 2.5$	0.94798	0.88489
$n = 10$	$a = 1, \quad b = 3$	0.96281	0.93469
	$a = 0.5, \; b = 2.5$	0.94798	0.88501

It can be seen that: 1) the coefficient of stress intensity is smaller at the left end than at the right one ($K_1^- < K_1^+$); 2) when the crack is shifting to the boundary, the coefficients of stress intensity decrease; 3) coefficients of stress intensity are less then unity. The above-mentioned facts correspond to physical content of the problem (steel is more rigid than aluminum). The fifth and tenth approximations coincide to within 10^{-3}.

In Section 7 we have presented the results obtained in [25].

8. Projective-Iterative Method of Solution of Singular Integral Equations

In this section we will consider 1) the scheme of the projective-iterative method for the operator equation of the second kind; 2) projective-iterative method for the singular integral equation; 3) numerical results.

1) Scheme of the projective-iterative method for the equation of the second kind. Let in the Banach space E the operator equation of the second kind

$$u + Tu = f, \quad u, f \in E, \tag{8.1}$$

be given, where T is a linear bounded operator in E. An approximate solution u_n of the equation (8.1) satisfies the equation ([9], p. 199)

$$u_n + P_n T u_n = P_n f, \quad u_n \in E_n, \tag{8.2}$$

where $\{E_n\}$ and $\{P_n\}$ are subspaces and projectors, respectively: $P_n^2 = P_n$, $P_n(D(P_n)) = E_n$, $E_n \subset D(P_n)$, $T(E) \subset D(P_n)$, $f \in D(P_n)$, $P_n T$ is bounded in E.

In [9], under the conditions that there exists the inverse bounded operator $(I+T)^{-1} : E \to E$ and $\|P^{(n)}T\|_{n\to\infty} \to 0$ ($P^{(n)} \equiv I - P_n$) it has been proved that

$$\|(I + P_n T)^{-1}\| \le C_1, \quad n \ge n_0. \tag{8.3}$$

Moreover, $\|P^{(n)} f\| \to 0$ as $n \to \infty$, and $\|u - u_n\| \to 0$, $n \to \infty$.

Suppose that an approximate solution u_n of the equation (8.2) is found and the iteration

$$\widetilde{u}_n = -Tu_n + f \tag{8.4}$$

is taken.

The projective-iterative scheme proposed in [26] looks as follows:

1) calculation of the residual

$$r_0 \equiv f - \widetilde{u}_n - T\widetilde{u}_n;$$

2) solution of the Galerkin equations

$$u_n^{(1)} + P_n T u_n^{(1)} = P_n r_0, \quad u_n^{(1)} \in E_n;$$

3) taking the iteration

$$\widetilde{u}_n^{(1)} = -T u_n^{(1)} + r_0;$$

4) finding the sum

$$\widetilde{u}_{n,1} \equiv \widetilde{u}_n + \widetilde{u}_n^{(1)}.$$

Cycles (1)–(4) can be repeated l times after which we obtain an approximate solution

$$\widetilde{u}_{n,l} \equiv \widetilde{u}_n + \widetilde{u}_n^{(1)} + \cdots + \widetilde{u}_n^{(l)}. \tag{8.5}$$

It is shown in [26] that if there exists the inverse bounded operator $(I + T)^{-1}$ and $\|TP^{(n)}\|_{n\to\infty} \to 0$, then for $n \ge n_0$

$$u - \widetilde{u}_{n,l} = (I + TP_n)^{-1}(-TP^{(n)}) \cdots (I + TP_n)^{-1}(-TP^{(n)}u), \tag{8.6}$$

and for $q_n \equiv \|(I+TP_n)^{-1}TP^{(n)}\| \le q < 1$ the estimate

$$\|u - \widetilde{u}_{n,l}\| \le q^{l+1}\|P^{(n)}u\| \tag{8.7}$$

is valid.

In [27], the projective-iterative scheme has been applied to the equation of the type $Au + Ku = f$, $u \in D(A)$, $f \in H$, where H is a Hilbert space ([28], p. 426). When Green's function of the operator A, $u \in D(A)$ is known, one can obtain the error estimates showing the order of convergence of the projective-iterative method as $n \to \infty$ (l is fixed) both for a version of the Bubnov–Galerkin method and for the method of finite elements.

In this subsection we will consider projective-iterative methods for singular integral equations and prove theorems showing the order of convergence when the number of cycles l is fixed, and approximation number $n \to \infty$.

a) Projective-iterative method for the equation of the first kind

$$Su + Ku = f, \tag{8.8}$$

where

$$Su \equiv \frac{1}{\pi}\int\limits_{-1}^{1}\frac{u(t)dt}{t-x}, \quad -1 < x < 1, \quad Ku \equiv \frac{1}{\pi}\int\limits_{-1}^{1}K(x,t)u(t)dt.$$

For the index $\varkappa = 1$ we take the additional condition

$$\int\limits_{-1}^{1}u(t)dt = P, \tag{8.9}$$

where P is a given number. In Section 2 the use has been made of the weight space L_{2,ρ_1} with the weight $\rho_1 = (1-x^2)^{\frac{1}{2}}$, $L_{2,\rho_1} = L_{2,\rho_1}^{(1)} \oplus L_{2,\rho_1}^{(2)}$ and S an isometric operator mapping $L_{2,\rho_1}^{(2)}$ onto L_{2,ρ_1};

$$S^{-1}(L_{2,\rho_1}) = L_{2,\rho_1}^{(2)}, \quad S^*u = -(1-t^2)^{-\frac{1}{2}}S(1-x^2)^{\frac{1}{2}}u, \quad S^{-1} = S^*.$$

The problem (8.8)–(8.9) has been replaced by one functional equation

$$S\phi + K\phi = f_1, \quad \phi \in L_{2,\rho_1}^{(2)}, \quad f_1 \in L_{2,\rho_1}, \tag{8.10}$$

where

$$f_1 \equiv f - P\pi^{-1}K(1-t^2)^{-\frac{1}{2}} \quad \phi(t) = u(t) - p\pi^{-1}(1-t^2)^{-\frac{1}{2}}.$$

It was required of the kernel $K(x,t)$ that

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1}K^2(x,t)\frac{\rho_1(x)}{\rho_1(t)}\,dtdx < +\infty.$$

We took two orthonormal complete systems:

1) $\varphi_k(x) \equiv (1-x^2)^{-\frac{1}{2}}\widehat{T}_k(x)$, $k = 0,1,\ldots$, $\widehat{T}_0 = \left(\frac{1}{\pi}\right)^{\frac{1}{2}}T_0$, $\widehat{T}_{k+1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}T_{k+1}(x)$, $k = 0,1,\ldots$, where $T_k(x)$, $k = 0,1,\ldots$, are the Chebyshev polynomials of the first kind;

2) $\psi_{k+1} \equiv \left(\frac{2}{\pi}\right)^{\frac{1}{2}} U_k(x),\ k = 0,1,\dots,$ where $U_k,\ k = 0,1,\dots,$ are the Chebyshev polynomials of the second kind.

We had

$$S\varphi_0 = 0, \quad S\varphi_k = \psi_k, \quad k = 1,2,\dots. \tag{8.11}$$

An approximate solution of the equation (8.10) is sought in the form

$$\phi_n = \sum_{k=1}^{n} a_k \varphi_k.$$

We compose the algebraic system by using the Galerkin–Petrov method

$$[S\phi_n + K\phi_n - f_1, \psi_i] = 0, \quad i = 1,2,\dots,n,$$

which with regard for (8.11) results in

$$a_i + \sum_{k=1}^{n} [K\varphi_k, \psi_i] a_k = [f_1, \psi_i], \quad i = 1,2,\dots,n. \tag{8.12}$$

Introducing the notation $w \equiv S\phi$, $w_n \equiv S\phi_n$, we write the equations (8.10) and (8.12), respectively, in the form

$$w + KS^{-1}w = f_1, \quad w, f_1 \in L_{2,\rho_1}, \tag{8.13}$$

$$w_n + P_n K S^{-1} w_n = P_n f_1, \tag{8.14}$$

where the projector

$$P_n v = \sum_{k=1}^{n} [v, \psi_k] \psi_k.$$

Thus we have obtained the operators of the type (8.1) and (8.2) in which $T = KS^{-1}$.

One iteration is as follows:

$$\widetilde{w}_n = -KS^{-1}w_n + f_1 = -K\phi_n + f_1.$$

Further, we find that

$$\widetilde{\phi}_n = S^{-1}\widetilde{w}_n = \frac{(1-t^2)^{-\frac{1}{2}}}{\pi} \int_{-1}^{1} \frac{(1-x)^{\frac{1}{2}} \widetilde{w}_n(x)}{t-x}\, dx =$$

$$= \frac{(1-t^2)^{-\frac{1}{2}}}{\pi} \int_{-1}^{1} \frac{(1-x^2)^{\frac{1}{2}} (-K\phi_n + f_1)}{t-x}\, dx. \tag{8.15}$$

Finally,

$$\widetilde{u}_n = \widetilde{\phi}_n + P\pi^{-1}(1-x^2)^{-\frac{1}{2}}. \tag{8.16}$$

In [29], under certain conditions for $\widetilde{u}_n$ the error estimate is established. We will establish an analogous estimate for the cyclic projective-iterative scheme.

Cycle 1)–4) has the following form:

1) $r_0 = f_1 - \widetilde{w}_n - KS^{-1}\widetilde{w}_n = f_1 - (-K\phi_n + f_1) - K\widetilde{\phi}_n = K\phi_n - K\widetilde{\phi}_n$;

2) We find scalar products $[r_0, \psi_i]$, $i = 1, 2, \ldots, n$, and then solve the algebraic system (the left-hand side of the algebraic system remains unchanged)

$$a_i^{(1)} + \sum_{k=1}^{n} a_k^{(1)} [K\varphi_k, \psi_i] = [r_0, \psi_i], \quad i = 1, 2, \ldots, n.$$

We obtain

$$\phi_n^{(1)} = \sum_{k=1}^{n} a_k^{(1)} \varphi_k, \quad w_n^{(1)} = \sum_{k=1}^{n} a_k^{(1)} \psi_k, \quad w_n^{(1)} + P_n K S^{-1} w_n^{(1)} = P_n r_0;$$

3) Taking the iteration

$$\widetilde{w}_n^{(1)} = -KS^{-1} w_n^{(1)} + r_0 = -K\phi_n^{(1)} + r_0,$$

we arrive at

$$\widetilde{\phi}_n^{(1)} = S^{-1} \widetilde{w}_n^{(1)} = \frac{(1-t^2)^{-\frac{1}{2}}}{\pi} \int_{-1}^{1} \frac{(1-x^2)^{\frac{1}{2}} (-K\phi_n^{(1)} + r_0)}{t - x} \, dx;$$

4) Taking now the sum

$$\widetilde{\phi}_{n,1} = \widetilde{\phi}_n + \widetilde{\phi}_n^{(1)} \ \big(\widetilde{w}_{n,1} = \widetilde{w}_n + \widetilde{w}_n^{(1)} = (-K\phi_n + f_1) + (-K\phi_n^{(1)} + r_0)\big),$$

we finally get

$$\widetilde{u}_{n,1} = \widetilde{\phi}_{n,1} + P\pi^{-1}(1-x^2)^{-\frac{1}{2}}.$$

If we repeat the cycle 1)–4) l times, we will obtain

$$\widetilde{\phi}_{n,l} = \widetilde{\phi}_n + \widetilde{\phi}_n^{(1)} + \cdots + \widetilde{\phi}_n^{(l)}, \quad \widetilde{u}_{n,l} = \widetilde{\phi}_{n,l} + P\pi^{-1}(1-x^2)^{-\frac{1}{2}}.$$

Our aim is to obtain an estimate for $\|u - \widetilde{u}_{n,l}\|$.

For the sake of clearness, we write out the cycle for $l = 2$. We have

1) $r_1 = f_1 - \widetilde{w}_{n,1} - KS^{-1}\widetilde{w}_{n,1} = f_1 - [(-K\phi_n + f_1) - (-K\phi_n^{(1)} + r_0)] - K(\widetilde{\phi}_n + \widetilde{\phi}_n^{(1)}) = K\phi_n + K\phi_n^{(1)} - K(\widetilde{\phi}_n + \widetilde{\phi}_n^{(1)}) - r_0 = K(\phi_n + \phi_n^{(1)}) - K(\widetilde{\phi}_n + \widetilde{\phi}_n^{(1)}) - r_0$.

The right-hand side of the algebraic system is a column matrix with elements $[r_1, \psi_i]$, $i = 1, 2, \ldots, n$. We find $a_1^{(2)}, a_2^{(2)}, \ldots, a_n^{(2)}, \phi_n^{(2)}, w_n^{(2)}$. Taking the iteration $\widetilde{u}_{n,2} = \widetilde{\phi}_{n,2} + P\pi^{-1}(1-x^2)^{-\frac{1}{2}}$, we get $\widetilde{\Phi}_n^{(2)} = S^{-1}\widetilde{w}_n^{(2)}$, $\widetilde{\phi}_{n,2} = \widetilde{\phi}_n + \widetilde{\phi}_n^{(1)} + \widetilde{\phi}_n^{(2)}$, and finally, $\widetilde{u}_{n,2} = \widetilde{\phi}_{n,2} + P\pi^{-1}(1-x^2)^{-\frac{1}{2}}$.

For our case, the formula (8.6) takes the form

$$w - \widetilde{w}_{n,l} = (I + KS^{-1}P_n)^{-1}(-KS^{-1}P^{(n)}) \cdots \\ \cdots (I + KS^{-1}P_n)^{-1}(-KS^{-1}P^{(n)} w), \tag{8.17}$$

where $w = S\phi$, $\widetilde{w}_{n,l} = S\widetilde{\phi}_{n,l}$.

Theorem 8.1. *Let the following conditions be fulfilled:* 1) *there exists the inverse operator* $(I + KS^{-1})^{-1}$ *mapping* L_{2,ρ_1} *onto itself;* 2) *the derivative* $w^{(m)} \in \operatorname{Lip}_M \alpha$*;* $0 < \alpha \leq 1$*;* 3) *the derivative of order* r *with respect to* t *of the kernel* $K_t^{(r)}(x,t) \in \operatorname{Lip}_{M_1} \alpha_1$, $0 < \alpha_1 \leq 1$, $\forall x \in [-1,1]$. *Then the estimate*

$$\|u - \widetilde{u}_{n,l}\|_{L_{2,\rho_1}} = 0(n^{-(m+\alpha)-(l+1)(r+\alpha_1)}) \quad as \quad n \to \infty \tag{8.18}$$

is valid, where l *is fixed.*

Proof. We have $u - \widetilde{u}_{n,l} = \phi - \widetilde{\phi}_{n,l} = S^{-1}(w - \widetilde{w}_{n,l})$, $S(L_{2,\rho_1}^{(2)}) = L_{2,\rho_1}$, $S^{-1}(L_{2,\rho_1}) = L_{2,\rho_1}^{(2)}$. The norm $\|S\|_{L_{2,\rho_1}^{(2)} \to L_{2,\rho_1}} = \|S^{-1}\|_{L_{2,\rho_1} \to L_{2,\rho_1}^{(2)}} = 1$. Therefore from (8.17) it follows that

$$\begin{aligned} \|u - \widetilde{u}_{n,l}\| = \|w - \widetilde{w}_{n,l}\| \leq \\ \leq \|(I + KS^{-1}P_n)^{-1}\|^{l+1} \|KS^{-1}P^{(n)}\|^{l+1} \|P^{(n)}u\|. \end{aligned} \tag{8.19}$$

Estimate now the norms $\|KS^{-1}P^{(n)}\|$, $\|P^{(n)}u\|$. We have

$$\|KS^{-1}P^{(n)}v\|^2 = \left\| KS^{-1} \sum_{k=1}^{\infty} [v, \psi_k]\psi_k \right\|^2 = \left\| K \sum_{k=n+1}^{\infty} [v, \psi_k]\varphi_k \right\|^2 =$$

$$= \frac{1}{\pi^2} \left\| \sum_{k=n+1}^{\infty} [v, \psi_k]((K(x,t), \varphi_k(t))) \right\|^2 \leq$$

$$\leq \frac{1}{\pi^2} \left\| \left\{ \sum_{k=n+1}^{\infty} [v, \psi_k]^2 \right\}^{\frac{1}{2}} \times \left\{ \sum_{k=n+1}^{\infty} (K(x,t), \varphi_k(t))^2 \right\}^{\frac{1}{2}} \right\|^2 \leq$$

$$\leq \frac{\|v\|^2}{\pi^2} \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} (K(x,t), \varphi_k(t)) \right)^2 dx =$$

$$= \frac{\|v\|^2}{\pi^2} \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} (K(x,t), (1 - t^2)^{-\frac{1}{2}} \widehat{T}_k(t))^2 \right) dx =$$

$$= \frac{\|v\|^2}{\pi^2} \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \left\| \sum_{k=n+1}^{\infty} [K(x,t), \widehat{T}_k(t)]_{L_{2,\rho_1^{-1}}} \widehat{T}_k(t) \right\|_{L_{2,\rho_1^{-1}}} dx,$$

$$\left\| \sum_{k=n+1}^{\infty} [K(x,t), \widehat{T}_k(t)]_{L_{2,\rho_1^{-1}}} \widehat{T}_k(t) \right\|_{L_{2,\rho_1^{-1}}}^2 =$$

$$= \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} [K(x,t), \widehat{T}_k(t)]_{L_{2,\rho_1^{-1}}} \widehat{T}_k(t) \right)^2 dt =$$

$$= \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} \left(K(x,t) - \sum_{k=0}^{n} [K(x,t), \widehat{T}_k(t)]_{L_{2,\rho_1^{-1}}} \widehat{T}_k(t) \right)^2 dt \leq$$

$$\leq \int_{-1}^{1} (1-t^2)^{\frac{1}{2}} (K(x,t) - P_n(x,t))^2 dt \leq \pi (E_n^t(K(x,t)))^2,$$

where x is a parameter, $P_n(x,t)$ is the polynomial of the best uniform approximation with respect to t, and $E_n^t(K(x,t))$ is the corresponding derivation

$$|K(x,t) - P_n(x,t)| \leq E_n^t(K(x,t)), \quad -1 \leq x, t \leq 1.$$

If $K_t^{(r)}(x,t) \in \mathrm{Lip}_M \, \alpha_1$, $0 < \alpha_1 \leq 1$, $\forall x \in [-1,1]$, and is continuous with respect to x in $[-1,1]$, then ([8], Ch. XIV, §4)

$$E_n^t(K(x,t)) = O(n^{(-r+\alpha_1)}).$$

Furthermore,

$$\|KS^{-1}P^{(n)}v\|^2 \leq \frac{\|v\|^2}{\pi^2} \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} \, \pi \, (E_n^t(K(x,t)))^2 dx =$$

$$= \frac{\|v\|^2}{\pi^2} \, \pi \, \frac{\pi}{2} \, (E_n^t(K(x,t)))^2.$$

Thus $\|KS^{-1}P^{(n)}\| = O(n^{-(r+\alpha_1)})$.

Moreover,

$$\|P^{(n)}w\|^2 = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} (w - P_n w)^2 dx =$$

$$= \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} \left(w - \sum_{k=1}^{n} [w, \psi_k] \psi_k \right)^2 dx \leq$$

$$\leq \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} (w - P_{n-1})^2 dx \leq \frac{\pi}{2} \|w - P_{n-1}\|_c^2,$$

where P_{n-1} is the polynomial of the best uniform approximation.

Due to the condition 2) from the statement of the above theorem, by Jackson's theorem [30] we have

$$\|w - P_{n-1}\|_c \leq \frac{C(w)}{(n-1)^{m+\alpha}}, \quad n > 1,$$

with the constant $C(w)$ depending on w and its derivatives, i.e., if $w^{(m)} \in \mathrm{Lip}_m \, \alpha$, $0 < \alpha \leq 1$ then $\|P^{(n)}w\| = O(n^{-(m+\alpha)})$.

Finally, we get

$$\|w - \widetilde{w}_{n,l}\| = O(n^{-(m+\alpha)}) O(n^{-(l+1)(r+\alpha_1)}),$$

i.e., we obtain the estimate (8.18). □

The index $\varkappa = -1$. The weight space $L_{2,\rho_2}[-1,1]$ has the weight $\rho_2 = (1-x^2)^{-\frac{1}{2}}$. The homogeneous equation $Su = 0$ has in the space L_{2,ρ_2} only zero solution $u = 0$, and the conjugate homogeneous equation $S^*u = -(1-t^2)^{\frac{1}{2}} S(1-x^2)^{-\frac{1}{2}} u = 0$ has in L_{2,ρ_2} the nonzero solution $u = 1$.

For the equation

$$Su + Ku = f, \quad u, f \in L_{2,\rho_2}, \tag{8.20}$$

to have a solution $u \in L_{2,\rho_2}$, it is necessary that

$$[Ku - f, 1] = 0. \tag{8.21}$$

This can be fulfilled if

$$K(L_{2,\rho_2}) \perp 1 \quad \text{and} \quad f \perp 1. \tag{8.22}$$

The condition $K(L_{2,\rho_2}) \perp 1$ means that

$$\int_{-1}^{1} K(x,t)(1-x^2)^{-\frac{1}{2}}\, dx = 0.$$

If the conditions (8.22) are not fulfilled for the equation (8.20), then we can always transform it in such a way that it will be fulfilled for the transformed equation [20]. One have to take the function $\psi_0(x) \equiv -\pi^{-\frac{1}{2}}$ and then find $\widetilde{K}(t) = [K(x,t), \psi_0]$, $K_0(x,t) - \widetilde{K}(t)\psi_0$, $[f, \psi_0]$ and $f_1 = f - [f, \psi_0]\psi_0$. The equation (8.20) can now be written as

$$Su + K_0 u = f_1, \quad u \in L_{2,\rho_2}, \quad f_1 \in L^{(2)}_{2,\rho_2}, \tag{8.23}$$

where $L^{(2)}_{2,\rho_2}$ is the orthogonal supplement of the one-dimensional subspace $L^{(1)}_{2,\rho_2}$ of the function $\psi_0 = -\pi^{-\frac{1}{2}}$, $L_{2,\rho_2} = L^{(1)}_{2,\rho_2} \dot{+} L^{(2)}_{2,\rho_2}$. It has been shown in [20] (just as in Section 6) that the procedure described above is not necessary in practice; an approximate solution u_n remains unchanged. We will need the equation (8.23) only in order to prove that the process converges.

In the space L_{2,ρ_2} the following two systems of functions are orthonormal and complete:

1) $\varphi_{k+1} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} U_k(x)$, $k = 0, 1, \dots$;

2) $\psi_0 = -\pi^{-\frac{1}{2}}$, $\psi_{k+1} = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} T_{k+1}(x)$, $k = 0, 1, \dots$,

where U_k, T_{k+1} are the Chebyshev polynomials of the second and first kind, respectively.

In Section 2 we had $S\varphi_k = \psi_k$, $k = 1, 2, \dots$. Under the conditions (8.22) we have the equation

$$Su + Ku = f, \quad u \in L_{2,\rho_2}, \quad f \in L^{(2)}_{2,\rho_2}. \tag{8.24}$$

Its approximate solution is sought in the form

$$u_n = \sum_{k=1}^{n} a_k \varphi_k(x).$$

By the Galerkin–Petrov method,

$$[Su_n + Ku_n - f, \psi_i] = 0, \quad i = 1, 2, \dots, n,$$

with regard for which $S\varphi_k = \psi_k$ results in the algebraic system

$$a_i + \sum_{k=1}^{n} [K\varphi_k, \psi_i] a_k = [f, \psi_i], \quad i = 1, 2, \dots, n. \tag{8.25}$$

We denote $w = Su, w_n = Su_n = \sum_{k=1}^{n} a_k \psi_k$. Then the algebraic system (8.25) can be written by means of the orthoprojector P_n on the span of the functions $\psi_1, \psi_2, \dots, \psi_n$ as follows:

$$w_n + P_n K S^{-1} w_n = P_n f. \tag{8.26}$$

Let an approximate solution u_n (w_n) be found. Taking the iteration

$$\widetilde{w}_n = -KS^{-1} w_n + f = -Ku_n + f,$$

we have to find

$$\widetilde{u}_n = S^{-1} \widetilde{w}_n = \frac{(1-t^2)^{\frac{1}{2}}}{\pi} \int_{-1}^{1} \frac{(1-x^2)^{-\frac{1}{2}} \widetilde{w}_n(x)}{t-x} \, dx =$$

$$= \frac{(1-t^2)^{\frac{1}{2}}}{\pi} \int_{-1}^{1} \frac{(1-x^2)^{-\frac{1}{2}} (-Ku_n + f)}{t-x} \, dx.$$

Cycle 1)–4) looks as follows:

1) We find

$$r_0 = f - \widetilde{w}_n - KS^{-1} \widetilde{w}_n = f - (-Ku_n + f) - K\widetilde{u}_n = Ku_n - K\widetilde{u}_n$$

and then take the exact equation $Su^{(1)} + Ku^{(1)} = r_0$ whose approximate solution is sought in the form

$$u_n^{(1)} = \sum_{k=1}^{n} a_k^{(1)} \varphi_k.$$

2) We find scalar products $[r_0, \psi_i]$, $i = 1, 2, \dots, n$, and solve the algebraic system

$$a_i^{(1)} + \sum_{k=1}^{n} [K\varphi_k, \psi_i] a_k^{(1)} = [r_0, \psi_i], \quad i = 1, 2, \dots, n,$$

and then we find $u_n^{(1)}$.

3) We take the iteration

$$\widetilde{w}_n^{(1)} = -KS^{-1} w_n^{(1)} + r_0 = -Ku_n^{(1)} + r_0,$$

and find

$$\widetilde{u}_n^{(1)} = S^{-1}\widetilde{w}_n^{(1)} = \frac{(1-t^2)^{\frac{1}{2}}}{\pi}\int\limits_{-1}^{1}\frac{(1-x^2)^{-\frac{1}{2}}(-Ku_n+r_0)}{t-x}\,dx.$$

4) We take the sum

$$\widetilde{u}_{n,1} = \widetilde{u}_n + \widetilde{u}_n^{(1)}.$$

This cycle can be repeated. Using the projective-iterative method, we obtain an approximate solution

$$\widetilde{u}_{n,l} = \widetilde{u}_n + \widetilde{u}_n^{(1)} + \cdots + \widetilde{u}_n^{(l)}.$$

Theorem 8.2. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *which maps* $L^{(2)}_{2,\rho_2}$ *onto itself, the derivatives* $w^{(m)} \in \mathrm{Lip}_M\,\alpha$, $0<\alpha\le 1$, $K_t^{(r)}(x,t) \in \mathrm{Lip}_{M_1}\,\alpha_1$, $0<\alpha_1$, $0<\alpha_1\le 1$, $\forall x\in[-1,1]$, *then the estimate*

$$\|u-\widetilde{u}_{n,l}\|_{L_{2,\rho_2}} = O(n^{-(m+\alpha)-(l+1)(r+\alpha_1)}) \quad as \quad n\to\infty \tag{8.27}$$

is valid, where l is fixed.

This theorem is proved analogously to Theorem 8.1.

2) The index $\varkappa = 0$. Here we may have two cases $\rho_3^{(1)} = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$ and $\rho_3^{(2)} = (\rho_3^{(1)})^{-1}$.

Let us consider the case $\rho_3 = \rho_3^{(1)}$. The homogeneous equations $Su=0$ and $S^*u=0$ have only the trivial solution $u=0$. We have the equation

$$Su + Ku = f, \quad u, f \in L_{2,\rho_3}. \tag{8.28}$$

In Section 2, we had two orthonormal complete systems of functions:

1) $\varphi_k \equiv C_k(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}P_k^{(-\frac{1}{2},\frac{1}{2})}(x)$, $k=0,1,\dots$, $C_k = (h_k^{(-\frac{1}{2},\frac{1}{2})})^{-\frac{1}{2}}$, $h_k^{(\frac{1}{2},-\frac{1}{2})} = h_k^{(-\frac{1}{2},\frac{1}{2})} = 2\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)[(2k+1)(k!)^2]^{-1}$, where $P_k^{(-\frac{1}{2},\frac{1}{2})}(x)$, $k=0,1,\dots$, are the Jacobi polynomials;

2) $\psi_k \equiv -C_kP_k^{(\frac{1}{2},-\frac{1}{2})}(x)$, $k=0,1,\dots$,

$$S\varphi_k = \psi_k, \quad k=0,1,\dots . \tag{8.29}$$

An approximate solution of the equation (8.28) is sought in the form

$$u_n = \sum_{k=0}^{n} a_k\varphi_k.$$

The Galerkin-Petrov method yields

$$[Su_n + Ku_n - f, \psi_i] = 0, \quad i=0,1,\dots,n.$$

The latter with regard for (8.29) results in the algebraic system

$$a_i + \sum_{k=0}^{n}[K\varphi_k,\psi_i]a_k = [f,\psi_i], \quad i=0,1,\dots,n, \tag{8.30}$$

which can be written by means of the orthoprojector P_n on the span of the functions $\psi_0, \psi_1, \dots, \psi_n$ as follows:

$$w_n + P_n K S^{-1} w_n = P_n f, \quad w_n \equiv S u_n = \sum_{k=0}^{n} a_k \psi_k. \tag{8.31}$$

The singular operator $S : L_{2,\rho_3} \to L_{2,\rho_3}$ is unitary, $S(L_{2,\rho_3}) = L_{2,\rho_3}$, $S^{-1}(L_{2,\rho_3}) = L_{2,\rho_3}$.

The iteration has the form

$$\widetilde{w}_n = -KS^{-1} w_n + f = -K u_n + f.$$

We have to find

$$\widetilde{u}_n = S^{-1} \widetilde{w}_n = \frac{1}{\pi} \Big(\frac{1+t}{1-t}\Big)^{\frac{1}{2}} \int_{-1}^{1} \Big(\frac{1-x}{1+x}\Big)^{\frac{1}{2}} \frac{\widetilde{w}_n(x)}{t-x}\, dx.$$

Cycle 1)–4) looks as follows:

1) We find

$$r_0 = f - \widetilde{w}_n - KS^{-1}\widetilde{w}_n = f - (-Ku_n + f) - K\widetilde{u}_n = Ku_n - K\widetilde{u}_n.$$

2) We calculate $[r_0, \psi_i]$, $i = 0, 1, \dots, n$, solve the algebraic system

$$a_i^{(1)} + \sum_{k=0}^{n} [K\varphi_k, \psi_i] a_k^{(1)} = [r_0, \psi_i], \quad i = 0, 1, \dots, n,$$

and then find $u_n^{(1)} = \sum_{k=0}^{n} a_k^{(1)} \varphi_k(x)$, $w_n^{(1)} = \sum_{k=0}^{n} a_k^{(1)} \psi_k(x)$.

3) We take the iteration

$$\widetilde{w}_n^{(1)} = -KS^{-1} w_n^{(1)} + r_0 = -K u_n^{(1)} + r_0,$$

and find

$$\widetilde{u}_n^{(1)} = S^{-1} \widetilde{w}_n^{(1)} = \frac{1}{\pi} \Big(\frac{1+t}{1-t}\Big)^{\frac{1}{2}} \int_{-1}^{1} \Big(\frac{1-x}{1+x}\Big)^{\frac{1}{2}} \frac{(-Ku_n^{(1)} + r_0)}{t-x}\, dx.$$

4) We take the sum

$$\widetilde{u}_{n,1} = \widetilde{u}_n + \widetilde{u}_n^{(1)}.$$

Repeating the cycle 1)–4) l times, we obtain an approximate solution of the equation (8.28),

$$\widetilde{u}_{n,l} = \widetilde{u}_n + \widetilde{u}_n^{(1)} + \dots + \widetilde{u}_n^{l}.$$

The following theorem is valid.

Theorem 8.3. *If there exists the inverse operator $(I + KS^{-1})^{-1}$ which maps L_{2,ρ_3} onto itself, the derivatives $u^{(m)} \in \mathrm{Lip}_M\, \alpha$, $0 < \alpha \le 1$, $K_t^{(r)}(x,t) \in \mathrm{Lip}_{M_1}\, \alpha_1$, $0 < \alpha_1 \le 1$, $\forall x \in [-1, 1]$, then the estimate*

$$\|u - \widetilde{u}_{n,l}\|_{L_{2,\rho_3}} = O(n^{-(m+\alpha)-(l+1)(r+\alpha_1)}) \tag{8.32}$$

is valid as $n \to \infty$, where l is fixed.

This theorem is proved analogously to Theorem 8.1.

Consider now the collocation-iterative method.

1) $\varkappa = 1$, $\alpha = \beta = -\frac{1}{2}$. In Section 4, we had the approximate solution

$$\phi_n = \sum_{k=1}^{n} a_k \varphi_k,$$

the algebraic system

$$\sum_{k=1}^{n} a_k \psi_k(x_j) + \sum_{k=1}^{n} a_k (K\varphi_k)(x_j) = f_1(x_j), \quad j = 1, 2, \dots, n,$$

and the nodes were the roots of the Chebyshev polynomial $U_n(x)$. The algebraic system by means of the projector $\Pi_m v \equiv L_m(v)$, $v \in C[-1,1]$, where $L_m(v)$ is the Lagrange interpolation polynomial, can be written in the form

$$w^{(n)} + \Pi_{n-1} K S^{-1} w^{(n)} = \Pi_{n-1} f, \quad w^{(n)} \in \overline{L}_{2,\rho_1}^{(n)}, \tag{8.33}$$

where $\overline{L}_{2,\rho_1}^{(n)}$ is the linear span of the functions $\psi_1, \psi_2, \dots, \psi_n$.

The initial equation is as follows:

$$w + K S^{-1} w = f_1, \quad w, f_1 \in L_{2,\rho_1}. \tag{8.34}$$

The cyclic projective-iterative method provides us with

$$w - \widetilde{w}_{n,l} = (I + KS^{-1}\Pi_{n-1})^{-1}(-KS^{-1}\Pi^{(n-1)}) \cdots (I + KS^{-1}\Pi_{n-1})^{-1} \times$$
$$\times (-KS^{-1}\Pi^{(n-1)} w), \quad \Pi^{(n-1)} = I - \Pi_{n-1}, \tag{8.35}$$

whence

$$\|w - \widetilde{w}_{n,l}\| \le \|(I + KS^{-1}\Pi_{n-1})^{-1}\|^{l+1} \|KS^{-1}\Pi^{(n-1)}\|^{l+1} \|\Pi^{(n-1)} w\|. \tag{8.36}$$

Estimate the following norms: $\|\Pi^{(n-1)} w\|$ and $\|KS^{-1}\Pi^{(n-1)}\|$.

If the function $w(x)$ is m times continuously differentiable on the segment $[-1,1]$, and $w^{(m)}(x) \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, then

$$\|w(x) - L_n(x,w)\|_c \le \frac{c_2 \ln n}{n^{m+\alpha}} \tag{8.37}$$

(see [30], p. 288). Moreover, $\|\Pi^{(n)} w\| = \|w - L_n\| \le \|\Pi^{(n)} w\|_c \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$.

The residual term of the Newton and Lagrange interpolation formulas is expressed by the formula ([5], p. 109)

$$R_n(x) = w(x) - L_n(x,w) = \left[\prod_{k=0}^{n} (x - x_k)\right] w(x; x_0; \dots, x_n),$$

where $w(x; x_0; \dots; x_n)$ is the divided difference of the function $w(x)$.

Denote $K_1(x,t) \equiv (1-t^2)^{-\frac{1}{2}} K(x,t)$. In our case

$$\Pi^{(n-1)} v = v - L_{n-1}(v) = \prod_{k=0}^{n-1} (x - x_k) v(x, x_0 \dots; x_{n-1}) =$$

$$= \prod_{k=0}^{n-1}(x-x_k)\delta^{(n)}v(x) = \omega(x)\delta^{(n)}v(x),$$

where $\delta^{(n)}v(t) \equiv v(t;t_0;\dots;t_{n-1})$ is the divided difference.

If as interpolation nodes we take the zeros of the Chebyshev polynomial $U_n(x)$, then

$$\Pi^{(n-1)}v = \Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}v.$$

Thus we have

$$\begin{aligned}\|KS^{-1}\Pi^{(n-1)}v\| &= \pi^{-1}\|(K(x,t),S^{-1}\Pi^{(n-1)}v)\| = \\ &= \pi^{-1}\|[K_1(x,t),S^{-1}\Pi^{(n-1)}v]_t\| = \\ &= \pi^{-1}\|[SK_1(x,\tau)(x,t),\Pi^{(n-1)}v(t)]_t\|_x = \\ &= \pi^{-1}\Big\|\Big[SK_1(x,\tau)(x,t),\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\frac{\widehat{U}_n(t)}{2^n}\delta^{(n)}v(t)\Big]_t\Big\| = \\ &= \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\frac{1}{2^n}\|[\delta^{(n)}v(SK_1)(x,t),\widehat{U}_n(t)]_t\| = \\ &= \frac{1}{\pi}\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\frac{1}{2^n}\|[\delta^{(n)}v(SK_1)(x,t),P^{(n-1)}\widehat{U}_n]\| = \\ &= \Big(\frac{1}{2\pi}\Big)^{\frac{1}{2}}\frac{1}{2^n}\|[P^{n-1}(\delta^{(n)}v(SK_1)(x,t),\widehat{U}_n(t)]_t\|. \end{aligned} \tag{8.38}$$

Further,

$$\|KS^{-1}\Pi^{(n-1)}v\| \le \Big(\frac{1}{2\pi}\Big)^{\frac{1}{2}}\frac{1}{2^n}\|P^{(n-1)}(\delta^{(n)}v(SK_1)(x,t))\|,$$

$\|(I-P_{n-1})v\| \le \|v-P_{n-1}v\|$, where $P_{n-1}v$ is the polynomial of the best uniform approximation. If the derivative $v^{(m)} \in \operatorname{Lip}\alpha$, $0<\alpha\le 1$, then by Jackson's theorem,

$$\|v-P_{n-1}v\|_c = O\Big(\frac{1}{n-1}\Big)^{m+\alpha}.$$

Thus the following theorem is valid.

Theorem 8.4. *If there exists the inverse operator* $(I+KS^{-1})^{-1}$ *mapping* L_{2,ρ_1} *onto itself, the derivative* $w^{(m)} \in \operatorname{Lip}_M\alpha$, $0<\alpha_1\le 1$, *and the derivative* $(\delta^{(n)}v(t)(SK_1)(x,t))_t^{(r)} \in \operatorname{Lip}_{M_1}\alpha_1$, $0<\alpha_1\le 1$, $\forall x\in[-1,1]$, *then the estimate*

$$\|u-\widetilde{u}_{n,l}\|_{L_{2,\rho_1}} = O\Big(\frac{\ln\pi}{n^{m+\alpha}}\frac{1}{2^n}\frac{1}{n^{r+\alpha_1}}\Big) \tag{8.39}$$

is valid.

This estimate has been obtained under rigid restrictions imposed on the functions Su, $(SK_1)(x,t)$ and $\delta^{(n)}(Su)$, where $\delta^{(n)}(Su)$ is the divided difference of the function Su.

Collocation-iterative schemes for the indices $\varkappa = -1$ and $\varkappa = 0$ can be considered analogously.

Remark. The result obtained for the equation $(S+K)u = f$ is likewise valid for the general equation $(a+bS+K)u = f$, where a and b are real numbers, $a^2 + b^2 = 1$. The operator $S^{-1}v = S^*v = -(1-t^2)^{-\frac{1}{2}}S(1-x^2)^{\frac{1}{2}}v$ of the projective-iterative method in the general case is replaced by the operator

$$(a+bS)^{-1}v = (a+bS)^*v = a - b(1-t)^{\alpha}(1+t)^{\beta}S(1-x)^{-\alpha}(1+x)^{-\beta}.$$

Numerical Example. For the index $\varkappa = -1$ we consider the singular integral equation

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{u(t)dt}{t-x} + \frac{1}{\pi}\int\limits_{-1}^{1}(t^8x^8 + t^7x^7)u(t)dt = f(x), \quad x \in]-1,1[, \tag{8.40}$$

where $f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{3}{128}x^7 - 32x^6 + 48x^4 - 18x^2 + 1\right)$, with the exact solution

$$u(x) = \varphi_6(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(32x^5 - 32x^3 + 6x)(1-x^2)^{\frac{1}{2}},$$

where φ_k, $k = 0,1,\dots$, is the orthonormal system of functions in L_{2,ρ_2}, $\rho_2 = (1-x^2)^{-\frac{1}{2}}$.

We find the fifth approximation

$$u_5(x) = \sum_{k=1}^{5} a_k\varphi_k(x), \quad \varphi_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}U_{k-1}(x),$$

where $u_{k-1}(x)$, $k = 1,2,\dots$, are the Chebyshev polynomials of the second kind.

The Petrov–Galerkin method provides us with the algebraic system

$$a_i + \sum_{k=1}^{5}[K\varphi_k, \psi_i]a_k = [f, \psi_i], \quad i = 1,2,\dots,5, \tag{8.41}$$

$\psi_{k+1} = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}T_{k-1}(x)$, $k = 0,1,\dots$, $T_k(x)$ are the Chebyshev polynomials of the first kind.

Solution of the above system (calculations are performed to within 10^{-7}) yields $a_1 = -0.0128344$, $a_2 = -0.0003810$, $a_3 = -0.0077006$, $a_4 = -0.0001905$, $a_5 = -0.0025327$.

$$w_5 \equiv Su_5 = \sum_{k=1}^{5} a_k\psi_k \quad (S\varphi_k = \psi_k, \quad k = 1,2,\dots).$$

Then we take the iteration

$$\widetilde{w}_5 = -KS^{-1}w_5 + f = -Ku_5 + f,$$

and calculate

$$\widetilde{u}_5 = S^{-1}\widetilde{w}_5 = \frac{(1-t^2)^{\frac{1}{2}}}{\pi} \int\limits_{-1}^{1} \frac{(1-x^2)^{-\frac{1}{2}}\widetilde{w}_5(x)}{t-x}\,dx =$$

$$= -0.0128344\varphi_1 - 0.0003810\varphi_2 - 0.0077006\varphi_3 - 0.0001905\varphi_4 - \\ -0.0025668\varphi_5 + 0.9999456\varphi_6 - 0.0003666\varphi_7 - 0.0000068\varphi_8.$$

Next, we calculate the residual

$$r_0(x) = f(x) - \widetilde{w}_5(x) - KS^{-1}\widetilde{w}_5(t) = \\ = f(x) - (-Ku_5 + f(x)) - K\widetilde{u}_5 = Ku_5 - K\widetilde{u}_5,$$

by means of which we find $[r_0, \psi_i]$, $i = 1, 2, \ldots, 5$, and then compose the algebraic system

$$a_i^{(1)} + \sum_{k=1}^{n} a_k^{(1)}[K\varphi_k, \psi_i] = [r_0, \psi_i], \quad i = 1, 2, \ldots, 5. \tag{8.42}$$

(The equations (8.41) and (8.42) differ by their right-hand sides only.)

Solving the system (8.42), we arrive at

$$a_1^{(1)} = 0.0128327, \quad a_2^1 = 0.0003564, \quad a_3^{(1)} = 0.0076996, \\ a_4^{(1)} = 0.0001782, \quad a_5^{(1)} = 0.0012529.$$

An approximate solution is

$$u_5^{(1)}(x) = \sum_{k=1}^{5} a_k^{(1)}\varphi_k(x).$$

Again, we take the iteration

$$\widetilde{w}_5^{(1)} = -KS^{-1}w_5^{(1)} + r_0 = -Ku_5^{(1)} + r_0, \quad w_5^{(1)} = Su_5^{(1)} = \sum_{k=1}^{5} a_k^{(1)}\psi_k$$

and calculate

$$\widetilde{u}_5^{(1)} = S^{-1}\widetilde{w}_5^{(1)} = \frac{(1-t^2)^{\frac{1}{2}}}{\pi} \int\limits_{-1}^{1} \frac{(1-x^2)^{-\frac{1}{2}}\widetilde{w}_5^{(1)}(x)dx}{t-x} =$$

$$= 0.0128376\varphi_1 + 0.0003564\varphi_2 + 0.0077026\varphi_3 + 0.0001782\varphi_4 + \\ +0.0025675\varphi_5 + 0.0000509\varphi_6 + 0.0003668\varphi_7 + 0.0000069\varphi_8.$$

The projective-iterative method for the cycle $l = 1$ provides us with the approximate solution

$$\widetilde{u}_{5,1} = \widetilde{u}_5 + \widetilde{u}_5^{(1)} = 0.0000032\varphi_1 - 0.0000246\varphi_2 + 0.0000020\varphi_3 - \\ -0.0000123\varphi_4 + 0.0000007\varphi_5 + 0.9999547\varphi_6 + \\ +0.0000002\varphi_7 + 0.0000001\varphi_8.$$

In the norm of the space L_{2,ρ_2}, for the relative error we obtain

1) $\frac{\|\Delta u_5(x)\|}{\|u\|} \approx 100.01151\%$;
2) $\frac{\|\Delta \widetilde{u}_5(x)\|}{\|u\|} \approx 1.51855\%$;
3) $\frac{\|\Delta \widetilde{u}_{5.1}(x)\|}{\|u\|} \approx 0.00476\%$, $\|u\| = 1$,

$\|\cdot\|$ is the norm of the space L_{2,ρ_2}, $\rho_2 = (1-x^2)^{-\frac{1}{2}}$.

The above example can be solved by means of the collocation-iterative method as well. An approximate solution is again sought in the form

$$u_5(x) = \sum_{k=1}^{5} a_k \varphi_k(x), \quad \varphi_k(x) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}} U_{k-1}(x),$$

and the algebraic system (4.20) yields

$$a_0(-\widehat{T}_0) + \sum_{k=1}^{5} a_k(-\widehat{T}_k(x_j)) + \sum_{k=1}^{5} a_k (K\varphi_k)(x_j) = f(x_j), \qquad (8.43)$$
$$j = 0, 1, \dots, 5,$$

where the collocation points $x_0, x_1, \dots, x_5$ are the roots of the Chebyshev polynomial $T_{n+1}(x)$ $(n=5)$,

$$x_j = \cos \frac{2j+1}{2(n+1)}, \quad j = 0, 1, \dots, 5.$$

Solving the system (8.43), we find $u_5(x)$. Repeating the cycle, we find $\widetilde{u}_5(x)$ and $\widetilde{u}_{5,1}(x)$. We obtain relative errors in the norm of the space L_{2,ρ_2}:

1) $\frac{\|\Delta u_5\|}{\|u\|} \approx 100.02931\%$;
2) $\frac{\|\Delta \widetilde{u}_5\|}{\|u\|} \approx 1.59781\%$;
3) $\frac{\|\Delta \widetilde{u}_{5,1}\|}{\|u\|} \approx 0.06325\%$.

The relative errors $\frac{\|u-u_5\|}{\|u\|}$ of the projective and collocation methods are approximately equal to 100%. This is natural, because an approximate solution is sought in the form of a combination of $\varphi_1, \varphi_2, \dots, \varphi_5$, and the exact solution $u = \varphi_6$; the system $\{\varphi_k\}$ is orthonormal.

Numerical calculations were performed by G. G. Khvedelidze. The projective-iterative methods quoted in Section 8 are presented in [29] and [31].

9. Stability of the Projective-Iterative Method

In this section we investigate the stability of the projective-iterative method for the singular integral equation

$$Su + Ku = f. \qquad (9.1)$$

The scheme of approximate solution is proposed and investigated in Section 8.

The projective-iterative method has been considered in [26] for the operator equation of the second kind

$$u + Tu = f, \quad u, f \in E, \qquad (9.2)$$

where T is a completely continuous operator in the Banach space E.

An approximate solution u_n by the Galerkin method satisfies the equation

$$u_n + P_n T u_n = P_n f, \quad u_n \in E_n; \tag{9.3}$$

here $\{P_n\}$ and $\{E_n\}$ are the sequences of projectors and subspaces, respectively. An approximate solution is sought in the form

$$u_n = \sum_{k=1}^{n} a_k \varphi_k,$$

where $\varphi_1, \varphi_2, \dots$ is the basis system. If E is a Hilbert space, then the equation (9.3) can be written as

$$\sum_{k=1}^{n} ((I+T)\varphi_k, \varphi_i) a_k = (f, \varphi_i), \quad i = 1, 2, \dots, n. \tag{9.4}$$

In practice, when defining scalar products $((I+T)\varphi_k, \varphi_i)$ and (f, φ_i), k, $i = 1, 2, \dots, n$ there may take place the errors γ_{ik} and δ_i. Naturally, there arises the question as to how these errors affect the approximate solution u_n. For the projective method, this question has been investigated by many authors (see, e.g., [32], [33], [34]). For the projective-iterative scheme, the question on the stability has been considered by the author in [35].

We introduce the following notation: H is a Hilbert space, $\Gamma_n \equiv (\gamma_{ik})_{i,k=1}^{n}$ is the error matrix, and $\delta^{(n)} \equiv (\delta_1, \dots, \delta_n)$ is the error vector. In practice, instead of (9.3) we have a perturbed approximate equation

$$(I + P_n T + \Delta_n) v_n = P_n f + \Delta(P_n f), \quad v_n \in H_n, \tag{9.5}$$

where the operator Δ_n corresponds to the error matrix Γ_n, and $\Delta(P_n f)$ to the error vector $\delta^{(n)}$, H_n is the linear span of the elements $\varphi_1, \dots, \varphi_n$.

In the projective-iterative method, for every cycle j we have the exact equation

$$(I+T) u^j = r_{j-1}, \quad u^j, r_{j-1} \in H, \quad j = 0, 1, \dots, l, \tag{9.6}$$

where

$$r_{j-1} \equiv f - \widetilde{u}_{n,j-1} - T\widetilde{u}_{n,j-1} = (I+T)(u - \widetilde{u}_{n,j-1}),$$
$$\widetilde{u}_{n,j-1} = \widetilde{u}_n + \widetilde{u}_n^1 + \dots + \widetilde{u}_n^{j-1}, \quad \widetilde{u}_n^j = -T u_n^j + r_{j-1},$$

and the corresponding approximate equation

$$u_n^j + P_n T u_n^j = P_n r_{j-1}, \quad u_n^j \in H_n, \quad j = 0, 1, \dots, l \tag{9.7}$$
$$(r_{-1} = f, \quad u_n^0 = u_n).$$

For perturbation we have

$$(I + P_n T + \Delta_n) v_n^j = P_n \widetilde{r}_{j-1} + \Delta(P_n \widetilde{r}_{j-1}) = P_n r_{j-1} +$$
$$+ P_n(\widetilde{r}_{j-1} - r_{j-1}) + \Delta(P_n \widetilde{r}_{j-1}), \quad v_n^j \in H_n, \quad j = 0, 1, \dots, l, \tag{9.8}$$

where

$$\widetilde{r}_{j-1} \equiv f - \widetilde{v}_{n,j-1} - T\widetilde{v}_{n,j-1}, \quad \widetilde{v}_{n,j-1} = \widetilde{v}_n + \widetilde{v}_n^1 + \cdots + \widetilde{v}_n^{j-1},$$
$$\widetilde{v}_n^j = -Tv_n^j + \widetilde{r}_{j-1}, \quad j = 0, 1, \dots, l \quad (\widetilde{r}_{-1} = f).$$

Indeed, the perturbation Δ_n of the operator $I + P_nT$ for every cycle remains unchanged, but the right-hand side with the additional term $P_n(\widetilde{r}_{j-1} - r_{j-1})$ varies. Thus

$$\widetilde{r}_{j-1} - r_{j-1} = (f - \widetilde{v}_{n,j-1} - T\widetilde{v}_{n,j}) - (f - \widetilde{u}_{n,j-1}) - T\widetilde{u}_{n,j-1} =$$
$$= (I + T)(\widetilde{v}_{n,j-1} - \widetilde{u}_{n,j-1}) \neq 0, \quad j = 1, 2, \dots, l.$$

The equations (9.7) and (9.8) result in

$$(I + P_nT + \Delta_n)(v_n^j - u_n^j) = P_n(\widetilde{r}_{j-1} - r_{j-1}) +$$
$$+ \Delta(P_n\widetilde{r}_{j-1}) - \Delta_n u_n^j, \quad j = 0, 1, \dots, l \quad (\widetilde{r}_{-1} = r_{-1} = f). \quad (9.9)$$

If the operator $I + T$ is continuously invertible and $\|P^{(n)}T\|_{n\to\infty} \to 0$ ($P^{(n)} \equiv I - P_n$), then $\|(I + P_nT)^{-1}\| \leq C$, $n \geq n_0$.

Let $A_n \equiv I + P_nT + \Delta_n$. If $C\|\Delta_n\| \leq \beta$, $\beta \in (0, 1)$, i.e., $\|\Delta_n\| \leq \beta/C \equiv r$, we have

$$\|A_n^{-1}\| \leq \frac{C}{1 - \beta}, \quad n \geq n_0. \quad (9.10)$$

From the equation (9.9) we have

$$v_n^j - u_n^j = A_n^{-1}[P_n(\widetilde{r}_{j-1} - r_{j-1}) + \Delta(P_n\widetilde{r}_{j-1}) - \Delta_n u_n^j], \quad (9.11)$$
$$j = 0, 1, \dots, l, \quad n \geq n_0.$$

To determine the stability of the projective-iterative method and to find the conditions of stability, we present here auxiliary lemmas. Lemmas 1, 2 and 4 are proved by the method of mathematical induction.

Lemma 1. *The formula*

$$\widetilde{v}_{nl} - \widetilde{u}_{n,l} = \sum_{k=0}^{l} (-T)^{l+1-k}(v_n^k - u_n^k), \quad l = 0, 1, \dots, \quad (9.12)$$

is valid.

Proof. For $l = 0$ we have $\widetilde{v}_{n_0} = \widetilde{v}_n$, $\widetilde{u}_{n_0} = \widetilde{u}_n$, $v_n^0 = v_n$, $u_n^0 = u_n$, $\widetilde{v}_n = -Tv_n + f$, $\widetilde{u}_n = -Tu_n + f$, and hence

$$\widetilde{v}_{n_0} - \widetilde{u}_{n,0} = -T(v_n^0 - u_n^0).$$

Suppose that the equation (9.12) is valid and prove it for $l + 1$. We have

$$\widetilde{v}_{n,l+1} - \widetilde{u}_{n,l+1} = \widetilde{v}_{nl} - \widetilde{u}_{n,l} + \widetilde{v}_n^{l+1} - \widetilde{u}_n^{l+1} =$$
$$= \widetilde{v}_{nl} - \widetilde{u}_{nl} + (-Tv_n^{l+1} + \widetilde{r}_l) - (-Tu_n^{l+1} + r_l) =$$
$$= \widetilde{v}_{nl} - \widetilde{u}_{nl} - T(v_n^{l+1} - u_n^{l+1}) + (\widetilde{r}_l - r_l).$$

Further,

$$\widetilde{r}_l - r_l = (f - \widetilde{v}_{nl} - T\widetilde{v}_{nl}) - (f - \widetilde{u}_{nl} - T\widetilde{u}_{nl}) = -(I+T)(\widetilde{v}_{nl} - \widetilde{u}_{nl}).$$

Therefore

$$\widetilde{v}_{n,l+1} - \widetilde{u}_{n,l+1} = -T(\widetilde{v}_{nl} - \widetilde{u}_{nl}) - T(v_n^{l+1} - u_n^{l+1}) =$$
$$= -T\sum_{k=0}^{l}(-T)^{l+1-k}(v_n^k - u_n^k) - T(v_n^{l+1} - u_n^{l+1}) =$$
$$= \sum_{k=0}^{l+1}(-T)^{l+2-k}(v_n^k - u_n^k).$$

The lemma is proved. □

Lemma 2. *The formula*

$$\widetilde{r}_l - r_l = L\sum_{k=-1}^{l-1}(-T + A_n^{-1}P_nL)^{l-1-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k - \Delta_n u_n^{k+1})] \qquad (9.13)$$
$$l = 0, 1, \dots \quad (L \equiv T + T^2, \quad \widetilde{r}_{-1} = r_{-1} = 0),$$

is valid.

Proof. For $l = 0$ we have

$$\widetilde{r}_0 - r_0 = (f - \widetilde{v}_n) - T\widetilde{v} - (f - \widetilde{u}_n - T\widetilde{u}_n) = -(I+T)(\widetilde{v}_n - \widetilde{u}_n) =$$
$$= -(I+T)(-T)(v_n - u_n) = L(v_n - u_n) = LA_n^{-1}[\Delta(P_nf) - \Delta_n u_n].$$

For $l+1$,

$$\widetilde{r}_{l+1} - r_{l+1} = [f - (I+T)\widetilde{v}_{n,l+1}] - [f - (I+T)\widetilde{u}_{n,l+1}] =$$
$$= -(I+T)(\widetilde{v}_{n,l+1} - \widetilde{u}_{n,l+1}) = -(I+T)(\widetilde{v}_{nl} - \widetilde{u}_{nl}) -$$
$$-(I+T)(\widetilde{v}_n^{l+1} - \widetilde{u}_n^{l+1}).$$

Taking into account (9.11) and the type of the expression $\widetilde{v}_n^{l+1} - \widetilde{u}_n^{l+1}$, we get

$$\widetilde{r}_{l+1} - r_{l+1} = \widetilde{r}_l - r_l - (I-T)[(-T)(v_n^{l+1} - u_n^{l+1}) + (\widetilde{r}_l - r_l)] =$$
$$= L(v_n^{l+1} - u_n^{l+1}) - T(\widetilde{r}_l - r_l) = LA_n^{-1}[P_n(\widetilde{r}_l - r_l) +$$
$$+\Delta(P_n\widetilde{r}_l) - \Delta_n u_n^{l+1}] - T(\widetilde{r}_l - r_l).$$

Thus we obtain the recurrence formula

$$\widetilde{r}_{l+1} - r_{l+1} = (-T + LA_n^{-1}P_n)(\widetilde{r}_l - r_l) + LA_n^{-1}[\Delta(P_n\widetilde{r}_l) - \Delta_n u_n^{l+1}].$$

Next, we have

$$\widetilde{r}_{l+1} - r_{l+1} =$$
$$= (-T + LA_n^{-1}P_n)L\sum_{k=-1}^{l-1}(-T + A_n^{-1}P_nL)^{l-1-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k) - \Delta_n u_n^{k+1}] +$$

$$+LA_n^{-1}[\Delta(P_n\widetilde{r}_l)-\Delta_n u_n^{l+1}]=$$

$$=L\sum_{k=-1}^{l}(-T+A_n^{-1}P_nL)^{l-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]$$

(the operators L and T are commutative). □

Lemma 3. *The following formula is valid:*

$$v_n^l-u_n^l=A_n^{-1}P_nL\sum_{k=-1}^{l-2}(-T+A_n^{-1}P_nL)^{l-2-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]+$$
$$+A_n^{-1}[\Delta(P_n\widetilde{r}_{l-1})-\Delta_n u_n^l], \tag{9.14}$$

$l=0,1,\dots$ (*for* $l=0$ *only the second summand remains*).

Proof. On the basis of the formula (9.11) and Lemma 2, we have

$$v_n^l-u_n^l=A_n^{-1}[P_n(\widetilde{r}_{l-1}-r_{l-1})+\Delta(P_n\widetilde{r}_{l-1})-\Delta_n u_n^l]=$$
$$=A_n^{-1}P_nL\sum_{k=-1}^{l-2}(-T+A_n^{-1}P_nL)^{l-2-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]+$$
$$+A_n^{-1}[\Delta(P_n\widetilde{r}_{l-1})-\Delta_n u_n^l].$$

The lemma is proved. □

Lemma 4. *The formula*

$$\widetilde{v}_{nl}-\widetilde{u}_{nl}=-T\sum_{k=-1}^{l-1}(-T+A_n^{-1}P_nL)^{l-1-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}], \tag{9.15}$$
$$l=0,1,\dots\quad (L=T+T^2),$$

is valid.

Proof. For $l=0$ we have

$$\widetilde{v}_{n_0}-\widetilde{u}_{n_0}=-T(v_n-u_n)=-TA_n^{-1}[\Delta(P_n\widetilde{r}_{-1})-\Delta_n u_n]\quad(\widetilde{r}_{-1}=f).$$

For $l+1$, relying on Lemma 3, we obtain

$$\widetilde{v}_{n,l+1}-\widetilde{u}_{n,l+1}=\widetilde{v}_{nl}-\widetilde{u}_{nl}-T(v_n^{l+1}-u_n^{l+1})+(\widetilde{r}_l-r_l)=$$
$$=\widetilde{v}_{nl}-\widetilde{u}_{nl}-T(v_n^{l+1}-u_n^{l+1})-(I+T)(\widetilde{v}_{nl}-\widetilde{u}_{nl})=$$
$$=-T(\widetilde{v}_{nl}-\widetilde{u}_{nl})-T(v_n^{l+1}-u_n^{l+1})=$$
$$=-T\Big\{-T\sum_{k=-1}^{l-1}(-T+A_n^{-1}P_nL)^{l-1-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]\Big\}-$$
$$-T\Big\{A_n^{-1}P_nL\sum_{k=-1}^{l-1}(-T+A_n^{-1}P_nL)^{l-1-k}A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]+$$
$$+A_n^{-1}[\Delta(P_n\widetilde{r}_l)-\Delta_n u_n^{l+1}]\Big\}=$$

$$= -T \sum_{k=-1}^{l-1} (-T + A_n^{-1} P_n L)^{l-k} A_n^{-1} [\Delta(P_n \widetilde{r}_k) - \Delta_n u_n^{k+1}] -$$

$$-T A_n^{-1} [\Delta(P_n \widetilde{r}_l) - \Delta_n u_n^{l+1}] =$$

$$= -T \sum_{k=-1}^{l} (-T + A_n^{-1} P_n L)^{l-k} A_n^{-1} [\Delta(P_n \widetilde{r}_k) - \Delta_n u_n^{k+1}].$$

Lemma is proved. □

Lemma 5. *The solutions u_n^j, $j = 0, 1, \dots, l$, $n \geq n_0$, of the equation* (9.7) *are uniformly bounded.*

Proof. For $n \geq n_0$, from the equation (9.7) we have

$$u_n^j = (I + P_n T)^{-1} P_n r_{j-1} = (I + P_n T)^{-1} P_n (I + T)(u - \widetilde{u}_{n,j-1}),$$

$$\|u_n^j\| \leq C \|I + T\| \, \|u - \widetilde{u}_{n,l-1}\|_{n\to\infty} \to 0 \quad \text{as} \quad \|P^{(n)} u\|_{n\to\infty} \to 0.$$

Therefore

$$\|u_n^j\| \leq D(u), \quad n \geq n_0, \quad j = 0, 1, \dots, l; \tag{9.16}$$

here D is a constant depending on u. □

It is assumed that the following conditions for the convergence are fulfilled: a) invertibility of the operator $I + T$; b) completeness of the coordinate system $\varphi_1, \varphi_2, \dots$; c) linear independence of $\varphi_1, \varphi_2, \dots, \varphi_n$; d) $\|P^{(n)} T\|_{n\to\infty} \to 0$ ($P^{(n)} \equiv I - P_n$). If these conditions are fulfilled, we can formulate the problem of stability.

The solutions obtained by means of the projective-iterative scheme do not, in general, belong to the linear span H_n of the elements $\varphi_1, \dots, \varphi_n$. For convergence of the sequence $\{\widetilde{u}_{n,l}\}$, $n = 1, 2, \dots, l$, l is fixed, of approximate solutions of the projective-iterative method, it is necessary that $\|T P^{(n)}\|_{n\to\infty} \to 0$.

Consider first the stability of the projective-iterative method from the theoretical point of view. The norm on H_n is induced, $\|w_n\|_{H_n} = \|w_n\|_H$, $w_n \in H_n$.

Definition 1. The projective-iterative method is said to be stable from the space H_n to H, if there exist independent of n constants $r > 0$, $C_1^{(l)} > 0$ and $C_2^{(l)} > 0$ such that for $\|\Delta_n\| \leq r$ the perturbed approximate equations (9.8) have unique solutions v_n^j, $j = 0, 1, \dots, l$, and the estimate

$$\|\widetilde{v}_{nl} - \widetilde{u}_{nl}\|_H \leq C_1^{(l)} \max_{-1 \leq k \leq l-1} \|\Delta(P_n \widetilde{r}_k)\|_{H_n} + C_2^{(l)} \|\Delta_n\|_{H_n} \tag{9.17}$$

is valid.

Theorem 9.1. *For the stability of the projective-iterative method in terms of Definition* 1 *the conditions of its convergence are sufficient.*

Proof. From the estimates (9.10), (9.16) and Lemma 4 we obtain (9.17), where

$$C_1^{(l)} \equiv \frac{\|T\|C}{1-\beta} \sum_{k=-1}^{l-1} \Big(\|T\| + \frac{C}{1-\beta} \|T+T^2\| \Big)^{l-1-k}, \quad C_2^{(l)} \equiv C_1^{(l)} D(u).$$

Consider now the stability from the Euclidean space R_n ($l_2^{(n)}$) to H. The norm of the vector $\tau^{(n)} \in l_2^{(n)}$ has the form $\|\tau^{(n)}\| = \Big(\sum_{k=1}^n \tau_k^2 \Big)^{\frac{1}{2}}$.

We write the equations (9.3) and (9.5) in matrix form and suppose

$$B_n \equiv ((I+T)\varphi_k, \varphi_i)_{i,k=1}^n, \quad \Gamma_n \equiv (\gamma_{ik})_{i,k=1}^n, \quad a^{(n)} \equiv (a_1, \dots, a_n)^T,$$
$$f^{(n)} \equiv [(f,\varphi_1), \dots, (f,\varphi_n)]^T, \quad \delta^{(n)} \equiv (\delta_1, \dots, \delta_n)^T,$$
$$b^{(n)} \equiv (b_1, \dots, b_n)^T, \quad v_n = \sum_{k=1}^n b_k \varphi_k.$$

We have

$$B_n a^{(n)} = f^{(n)}, \quad a^{(n)}, f^{(n)} \in l_2^{(n)}, \tag{9.18}$$
$$(B_n + \Gamma_n) b^{(n)} = f^{(n)} + \delta^{(n)}, \quad b^{(n)}, f^{(n)}, \delta^{(n)} \in l_2^{(n)}. \tag{9.19}$$

In the case of the Bubnov–Galerkin method, the passage from (9.18), (9.19) to (9.3), (9.5) can be realized as follows: the orthoprojectors $P_n v = \sum_{k=1}^n \beta_k \varphi_k$, where $\beta_1, \dots, \beta_n$, are defined from the condition

$$\|v - \sum_{k=1}^n \beta_k \varphi_k\|_H^2 = \min.$$

Thus we obtain the linear algebraic system

$$\sum_{k=1}^n \beta_k (\varphi_k, \varphi_i) = (v, \varphi_i), \quad i = 1, \dots, n.$$

We denote by $\phi_n \equiv (\varphi_k, \varphi_i)_{i,k=1}^n$ the matrix of the above system. Hence we have

$$\phi_n \beta^{(n)} = v^{(n)}, \quad \beta^{(n)} \equiv (\beta_1, \dots, \beta_n)^T, \quad v^{(n)} = [(v,\varphi_1), \dots, (v,\varphi_n)]^T.$$

Introduce the operator

$$S_n v_n \equiv S_n \sum_{k=1}^n \tau_k \varphi_k = \tau^{(n)}, \quad \tau^{(n)} \in l_2^{(n)}, \quad v_n \in H_n.$$

When the elements $\varphi_1, \dots, \varphi_n$ are linearly independent, there exist ϕ_n^{-1} and S_n^{-1}. Thus

$$P_n v = S_n^{-1} \phi_n^{-1} v^{(n)}, \quad v^{(n)} \in l_2^{(n)}, \quad P_n v \in H_n.$$

Using this notation we can write the equations (9.3) and (9.5) in the form

$$S_n^{-1} \phi_n^{-1} B_n S_n u_n = S_n^{-1} \phi_n^{-1} f^{(n)}, \tag{9.20}$$

$$S_n^{-1}\phi_n^{-1}(B_n+\Gamma_n)S_n v_n = S_n^{-1}\phi_n^{-1}(f^{(n)}+\delta^{(n)}), \tag{9.21}$$

and the operators $I+P_nT = S_n^{-1}\phi_n^{-1}B_nS_n$, $A_n \equiv I+P_nT+\Delta_n = S_n^{-1}\phi_n^{-1}(B_n+\Gamma_n)S_n$.

If the coordinate system $\varphi_1,\varphi_2,\dots$ is strongly minimal in H, i.e., the least eigenvalue $\lambda_1^{(n)}$ of the symmetric matrix ϕ_n satisfies the condition $\lambda_1^{(n)} \geq \lambda_0 > 0$, then

$$\|\phi_n^{-1}\|_{l_2^{(n)}\to l_2^{(n)}} \leq \lambda_0^{-1}, \quad \|S_n\|_{H_n\to l_2^{(n)}} \leq \lambda_0^{-\frac{1}{2}}. \tag{9.22}$$

Indeed,

$$(w_n,w_n) = \sum_{i,k=1}^{n} \tau_i\tau_k(\varphi_k,\varphi_i) = (\phi_n\tau^{(n)},\tau^{(n)}) \geq \lambda_1^{(n)}\|\tau^{(n)}\|^2,$$

$$\|\phi_n\tau^{(n)}\| \geq \lambda_1^{(n)}\|\tau^{(n)}\| \geq \lambda_0\|\tau^{(n)}\|,\ \|\phi_n^{-1}\| \leq \lambda_0^{-1},\ \|w_n\| \geq \lambda_0^{\frac{1}{2}}\|\tau^{(n)}\|,$$

$$w_n = S_n^{-1}\tau^{(n)}, \quad \|S_n\|_{H_n\to l_2^{(n)}} \leq \lambda_0^{-\frac{1}{2}}.$$

If the eigenvalues of the matrix ϕ_n are bounded from above by a number Λ_0 not dependent on n, then

$$\|\phi_n\|_{l_2^{(n)}\to l_2^{(n)}} \leq \Lambda_0, \quad \|S_n^{-1}\|_{l_2^{(n)}\to H_n} \leq \Lambda_0^{\frac{1}{2}}. \tag{9.23}$$

If the eigenvalues of the matrix ϕ_n are bounded from above and from below by positive numbers $\Lambda_0>0$ and $\lambda_0>0$ not dependent on n, then the basis system $\varphi_1,\varphi_2,\dots,$ is called uniformly linear independent, or almost orthonormal in H ([32], [33]). In [34] the uniformly linear independence is defined by the conditionality number of the matrix

$$K(\phi_n) = \frac{\lambda_{\max}(\phi_n)}{\lambda_{\min}(\phi_n)}, \quad K(\phi_n) \leq \text{const}, \quad \forall n \in N.$$

The theorem is proved. □

Definition 2. The projective-iterative method is said to be stable from the space $l_2^{(n)}$ to H, if there exist independent of n constants $r>0$, $C_1^{(l)}>0$ and $C_2^{(l)}>0$ such that for $\|\Gamma_n\| \leq r$ the perturbed approximate equations (9.8) have the unique solutions v_n^j, $j=0,1,\dots,l$, and the estimate

$$\|\widetilde{v}_{nl}-\widetilde{u}_{nl}\|_H \leq C_1^{(l)} \max_{-1\leq k\leq l-1} \|\delta^{(nk)}\|_{l_2^{(n)}} + C_2^{(l)}\|\Gamma_n\|_{l_2^{(n)}} \tag{9.24}$$

is valid.

Theorem 9.2. *The uniform linear independence* (*almost orthonormality*) *of the coordinate system* $\varphi_1,\varphi_2,\dots$ *in* H *is sufficient for the projective-iterative method to be stable in terms of Definition* 2.

Proof. We have

$$B_n^{-1} = S_n(I+P_nT)^{-1}S_n^{-1}\phi_n^{-1}, \quad n \geq n_0.$$

Taking into account (9.22), (9.23) and the fact that $\|(I-P_nT)^{-1}\| \le C$, $n \ge n_0$, we find that

$$\|B_n^{-1}\| \le C\lambda_0^{-\frac{3}{2}}\Lambda_0^{\frac{1}{2}} \equiv C_1, \quad n \ge n_0.$$

For $C_1\|\Gamma_n\| \le \beta < 1$, i.e., when $\|\Gamma_n\| \le \frac{\beta}{C_1} \equiv r$, we obtain

$$\|(B_n+\Gamma_n)^{-1}\| \le \|B_n^{-1}\|\,\|(I+B_n^{-1}\Gamma_n)^{-1}\| \le \frac{C_1}{1-\beta}, \quad n \ge n_0. \tag{9.25}$$

Further, taking into account the type of the operator A_n, we find that

$$A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}] = S_n^{-1}(B_n+\Gamma_n)^{-1}(\widetilde{\delta}^{(nk)}-\Gamma_n S_n u_n^{k+1}),$$

whence, on the basis of (9.22), (9.23) and (9.25), we have

$$\|A_n^{-1}[\Delta(P_n\widetilde{r}_k)-\Delta_n u_n^{k+1}]\|_H \le \frac{C_1\Lambda_0^{\frac{1}{2}}}{1-\beta}(\|\widetilde{\delta}^{(nk)}\| + \lambda_0^{-\frac{1}{2}}\|\Gamma_n\|\,\|u_n^{k+1}\|). \tag{9.26}$$

From (9.15), (9.16) and (9.26) it follows the estimate (9.24), where

$$C_1^{(l)} \equiv \frac{\Lambda_0^{\frac{1}{2}}C_1}{1-\beta}\|T\|\sum_{k=-1}^{l-1}\|-T+A_n^{-1}P_nL\|^{l-1-k},$$

$$C_2^{(l)} \equiv C_1^{(l)}\lambda_0^{-\frac{1}{2}}D(u), \quad \|\Gamma_n\| \le \left(\sum_{i,k=1}^{n}\gamma_{ik}^2\right)^{\frac{1}{2}};$$

here $\widetilde{\delta}^{nk}$ is the error of the vector $[(\widetilde{r}_k,\varphi_1),\dots,(\widetilde{r}_k,\varphi_n)]$. □

Definition 2 and Theorem 9.2 are used directly for the singular integral equations $Su+Ku=f$ and $(a+bS+K)u=f$ when they are solved by the projective-iterative method proposed in Section 8. Approximate schemes regularize singular integral equations, and the matter reduces to the operator equation of the second kind $w+KS^{-1}w=f$, where $w=Su$, and $w+KS^{-1}w=f$, where $w=(a+bS)^{-1}u$. Weight spaces $L_{2,\rho}[-1,1]$ are Hilbert ones. Systems of the functions $\{\phi_k\}$ and $\{\psi_k\}$ are orthonormal; $\lambda_0=\Lambda_0=1$. One should take into account different values of the index $\varkappa=1,-1,0$. For $\varkappa=1$, the domain of definition of the operator $S+K$ $(a+bS+K)$ gets narrower. For $\varkappa=-1$, it is the domain of values of the operator $S+K$ $(a+bS+K)$ that gets narrower.

Lemmas 1–4 are essential in the problem of stability of the projective-iterative method. In the perturbed equation, unlike the projective method, where only the operator and the right-hand side perturb, there appear additional perturbation caused by mutual connection between the projective scheme and iteration.

The problem on the stability of the collocation and collocation- iterative methods still remains unsolved.

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(Received 15.07.2004)

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