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SOME REMARKS ON THE INITIAL PROBLEMS FOR NONLINEAR HYPERBOLIC SYSTEMS

Abstract. For a nonlinear hyperbolic system with two independent variables a priori estimates of solutions of a general initial problem are established. On the basis of these estimates sufficient conditions for the solvability and well-posedness of this problem are found.

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რეზიუმე. ორცვლადიანი არაწრფივი ჰიპერბოლური სისტემებისათვის დადგენილია ზოგადი საწყისი ამოცანის ამონახსნების აპრიორული შეფასებები, რის საფუძველზეც მიღებულია ამ ამოცანის ამოხსნადობისა და კორექტულობის საკმარისი პირობები.

1. Statement of the Problem

Let $0 < a, b < +\infty$, $\gamma_1 : [0, b] \to [0, a]$ be a continuously differentiable and $\gamma_2 : [0, a] \to [0, b]$ be a continuous function such that

$$\gamma_1(y) < a \ \text{ for } \ 0 < y < b \quad \text{and} \quad \gamma_2(x) < b \ \text{ for } \ 0 < x < a. \tag{M_0}$$

Moreover, either

$$\begin{aligned} \gamma_1 \text{ and } \gamma_2 \text{ are nondecreasing}, \ \ \gamma_1(\gamma_2(x)) < x \ \text{ for } \ 0 < x < a, \\ \gamma_2(\gamma_1(y)) < y \ \text{ for } \ 0 < y < b, \end{aligned} \tag{M^*}$$

or

$$\begin{aligned} \gamma_1 \text{ and } \gamma_2 \text{ are nonincreasing}, \ \ \gamma_1(\gamma_2(x)) \leq x \ \text{ for } \ 0 \leq x \leq a, \\ \gamma_2(\gamma_1(y)) \leq y \ \text{ for } \ 0 \leq y \leq b. \end{aligned} \tag{M_*}$$

Then the set

$$G = \Big\{(x, y) \in]0, a[\times]0, b[: \ x > \gamma_1(y), \ y > \gamma_2(x)\Big\}$$

is non-empty and the curves

$$\Gamma_1 = \big\{(\gamma_1(y), y) : \ 0 \leq y \leq b\big\}, \quad \Gamma_2 = \big\{(x, \gamma_2(x)) : \ 0 \leq x \leq a\big\}$$

are the parts of its boundary. In the domain G we consider the nonlinear hyperbolic system

$$\frac{\partial^2 u}{\partial x \partial y} = f\Big(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big) \tag{1.1}$$

with the following initial conditions on Γ_1 and Γ_2:

$$\begin{aligned} \lim_{x \to \gamma_1(y)} u(x, y) = c_1(y) \ \text{ for } \ 0 < y < b, \\ \lim_{y \to \gamma_2(x)} \frac{\partial u(x, y)}{\partial x} = c_2(x) \ \text{ for } \ 0 < x < a, \end{aligned} \tag{1.2}$$

where $c_1 : [0, b] \to R^n$ is a continuously differentiable vector function, $f : \overline{G} \times R^{3n} \to R^n$ and $c_2 : [0, a] \to R^n$ are continuous vector functions and

$$\overline{G} = \Big\{(x, y) \in [0, a] \times [0, b] : \ x \geq \gamma_1(y), \ y \geq \gamma_2(x)\Big\}.$$

The vector function $u : G \to R^n$ is said to be a **solution of the system (1.1)**, if it has uniformly continuous in G partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x \partial y}$ and at every point of that domain satisfies the system (1.1). A solution of the system (1.2) satisfying the initial conditions (1.2) is called **a solution of the problem (1.1), (1.2)**.

Note that the conditions (M_0) and (M^*) are fulfilled, for example, in the cases, where

(i) $\gamma_1(y) \equiv 0$ and $\gamma_2(x) \equiv 0$,

or

(ii) γ_1 is nondecreasing, $\gamma_1(0) = 0$, $\gamma_1(b) > 0$, $\gamma_1(y) < 0$ for $y < b$ and $\gamma_2(x) \equiv 0$,

and the conditions (M_0) and (M_*) are fulfilled in the case, where

(iii) γ_1 is decreasing, $\gamma_1(0) = 0$, $\gamma_1(b) = 0$, and γ_2 is the function, inverse to γ_1.

In the above particular cases the problem (1.1), (1.2) has been investigated thoroughly (see, e.g., [1]–[15] and the references therein). In case (i) it is called the characteristic problem, or the Darboux problem; in case (ii) it is called the Goursat problem[1]; in case (iii) this problem is called the Cauchy problem. In a general case to which we devote the present paper the above-mentioned problem is studied with lesser thoroughness.

Below we obtain an a priori estimate of an arbitrary solution of the problem (1.1), (1.2) on the basis of which we establish sufficient conditions for the solvability and well-posedness of this problem.

Everywhere in the sequel we introduce the following notation.

R^n is the n-dimensional real Euclidean space;

$z_1 \cdot z_2$ is the scalar product of vectors z_1 and $z_2 \in R^n$;

$z = (\zeta_i)_{i=1}^n \in R^n$ is the vector with components $\zeta_1, \dots, \zeta_n$;

$$\|z\| = \sum_{i=1}^n |\zeta_i|, \quad \operatorname{sgn}(z) = \big(\operatorname{sgn}(\zeta_i)\big)_{i=1}^n.$$

Along with (1.1), (1.2) we will consider the perturbed problem

$$\frac{\partial^2 v}{\partial x \partial y} = f\Big(x, y, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\Big) + h(x, y), \tag{1.3}$$

$$\begin{gathered} \lim_{x \to \gamma_1(y)} v(x, y) = c_1(y) + \widetilde{c}_1(y) \ \text{ for } \ 0 < y < b, \\ \lim_{y \to \gamma_2(x)} \frac{\partial v(x, y)}{\partial x} = c_2(x) + \widetilde{c}_2(x) \ \text{ for } \ 0 < x < a, \end{gathered} \tag{1.4}$$

where $\widetilde{c}_1 : [0, b] \to R^n$ is a continuously differentiable vector function, and $\widetilde{c}_2 : [0, a] \to R^n$ and $h : \overline{G} \to R^n$ are continuous functions.

Put

$$\begin{aligned} &\eta(\widetilde{c}_1, \widetilde{c}_2, h) = \\ &= \max\Big\{\|\widetilde{c}_1(y)\| + \|\widetilde{c}_1'(y)\| + \|\widetilde{c}_2(x)\| : \ 0 \le x \le a, \ 0 \le y \le b\Big\} + \\ &+ \max\big\{\|h(x, y)\| : \ (x, y) \in \overline{G}\big\} \end{aligned} \tag{1.5}$$

and introduce the following definition.

Definition 1.1. The problem (1.1), (1.2) is said to be **well-posed** if there exist positive constants r and η_0 such that for arbitrary continuous vector functions $h : \overline{G} \to R^n$, $\widetilde{c}_2 : [0, a] \to R^n$ and continuously differentiable vector function $\widetilde{c}_1 : [0, a] \to R^n$ satisfying the condition

$$\eta(\widetilde{c}_1, \widetilde{c}_2, h) \le \eta_0, \tag{1.6}$$

[1] Sometimes the problem (1.1), (1.2) in case (i) is called the Goursat problem and in case (ii) the Darboux problem.

the problem (1.3), (1.4) is uniquely solvable and in the domain G the inequality

$$\big\|u(x,y)-v(x,y)\big\|+\Big\|\frac{\partial(u(x,y)-v(x,y))}{\partial x}\Big\|+\Big\|\frac{\partial(u(x,y)-v(x,y))}{\partial y}\Big\|\leq \\ \leq r\eta(\widetilde{c}_1,\widetilde{c}_2,h) \tag{1.7}$$

is fulfilled, where u and v are the solutions of the problems (1.1), (1.2) and (1.1), (1.3), respectively.

2. A Priori Estimates

Theorem 2.1. *Let there exist positive numbers δ and λ and a continuous non-decreasing function $\varphi : [0,+\infty[\to[0,+\infty[$ such that*

$$\varphi(\tau)>0 \ \text{ for } \ \tau>0, \quad \lim_{\tau\to+\infty}\psi_\delta(\tau)>\lambda(a+b), \quad \text{where } \ \psi_\delta(\tau)=\int_\delta^\tau \frac{ds}{\varphi(s)}\,, \tag{2.1}$$

and let respectively on the domains G and $G\times R^{3n}$ the inequalities

$$\|c_1(y)\|+\|c_2(x)\|+\int_{\gamma_1(y)}^{x}\|c_2(s)\|\,ds+ \\ +\big\|c_1'(y)+\gamma_1'(y)c_2(\gamma_1(y))\big\|\leq\delta, \quad 2+y+\|\gamma'(y)\|\leq\lambda \tag{2.2}$$

and

$$\big\|f(x,y,z_1,z_2,z_3)\big\|\leq\varphi\big(\|z_1\|+\|z_2\|+\|z_3\|\big) \tag{2.3}$$

be fulfilled. Then an arbitrary solution of the problem (1.1), (1.2) *in the domain G admits the estimate*

$$\|u(x,y)\|+\Big\|\frac{\partial u(x,y)}{\partial x}\Big\|+\Big\|\frac{\partial u(x,y)}{\partial y}\Big\|\leq\psi_\delta^{-1}(\lambda(x+y)), \tag{2.4}$$

where ψ_δ^{-1} is the function inverse to ψ_δ.

To prove the theorem, we need two auxiliary propositions.

Lemma 2.1. *If $(x,y)\in G$ and*

$$\gamma_1(y)<s<x, \quad \gamma_2(x)<t<y, \tag{2.5}$$

then

$$(s,y)\in G, \quad (x,t)\in G. \tag{2.6}$$

Proof. If the condition (M^*) is fulfilled, then by virtue of (2.5) we have

$$y>\gamma_2(x)\geq\gamma_2(s), \quad x>\gamma_1(y)\geq\gamma_1(t).$$

Hence the inclusions (2.6) are valid.

Now we consider the case where the condition (M_*) is fulfilled. By (2.5), there exist

$$y_0\in\,]0,y[\,, \quad x_0\in\,]0,x[$$

such that

$$\gamma_1(y_0) < s, \quad \gamma_2(x_0) < t,$$

whence, owing to (M_*), we find

$$\gamma_2(s) \le \gamma_2(\gamma_1(y_0)) \le y_0 < y, \quad \gamma_1(t) \le \gamma_1(\gamma_2(x_0)) \le x_0 < x.$$

Consequently, the inclusions (2.6) are valid. □

On a closed set $\overline{G}$ let us consider the integral inequality

$$\begin{aligned}
\xi(x,y) \le \delta + \lambda_0(x,y) \int\limits_{\gamma_1(y)}^{x} ds \int\limits_{\gamma_2(s)}^{y} \varphi(\xi(s,t))\, dt + \\
+ \lambda_1(x,y) \int\limits_{\gamma_1(y)}^{x} \varphi(\xi(s,y))\, ds + \lambda_2(x,y) \int\limits_{\gamma_2(x)}^{y} \varphi(\xi(x,t))\, dt + \\
+ \lambda_3(x,y) \int\limits_{\gamma_2(\gamma_1(y))}^{y} \varphi\big(\xi(\gamma_1(y),t)\big)\, dt + \\
+ \lambda_4(x,y) \int\limits_{\gamma_1(\gamma_2(x))}^{x} \varphi\big(\xi(s,\gamma_2(x))\big)\, ds,
\end{aligned} \tag{2.7}$$

where $\delta > 0$, $\lambda_k : \overline{G} \to [0,+\infty[$ $(k = 0,1,2,3,4)$ are continuous functions, and $\varphi : [0,+\infty[\to [0,+\infty[$ is a continuous nondecreasing function.

The continuous function $\xi : \overline{G} \to [0,+\infty[$ is said to be a **solution of the integral inequality (2.7)** if it satisfies this inequality at every point of the set $\overline{G}$.

Lemma 2.2. *Let there exist a non-negative constant λ such that*

$$y\lambda_0(x,y) + \sum_{k=1}^{4} \lambda_k(x,y) \le \lambda \ \textit{ for } \ (x,y) \in \overline{G}, \tag{2.8}$$

and let the condition (2.1) *be fulfilled. Then an arbitrary solution of the integral inequality* (2.7) *admits the estimate*

$$\xi(x,y) \le \psi_\delta^{-1}(\lambda(x+y)) \ \textit{ for } \ (x,y) \in \overline{G}, \tag{2.9}$$

where ψ_δ^{-1} is the function inverse to ψ_δ.

Proof. Set

$$\begin{aligned}
&\tau_0 = \min\big\{x+y : \ (x,y) \in \overline{G}\big\}, \\
&\zeta(\tau) = \max\big\{\xi(x,y) : \ (x,y) \in \overline{G}, \ x+y \le \tau\big\} \ \text{ for } \ \tau_0 \le \tau \le a+b, \\
&\zeta(\tau) = \zeta(\tau_0) \ \text{ for } \ 0 \le \tau \le \tau_0.
\end{aligned}$$

It is clear that $\zeta : [0, a+b] \to [0, +\infty[$ is a continuous non-decreasing function, and

$$\xi(x,y) \le \zeta(x+y) \ \text{ for } \ (x,y) \in \overline{G}.$$

If, moreover, we take into account the condition (M^*) (the condition (M_*)) and the inequality (2.8), then from (2.7) we obtain

$$\begin{aligned}
\xi(x,y) \le \delta + y\lambda_0(x,y) \int\limits_{\gamma_1(y)}^{x} \varphi(\zeta(s+y))\,ds + \\
&+ \lambda_1(x,y) \int\limits_{\gamma_1(y)}^{x} \varphi(\zeta(s+y))\,ds + \lambda_2(x,y) \int\limits_{\gamma_2(x)}^{y} \varphi(\zeta(x+t))\,dt + \\
&+ \lambda_3(x,y) \int\limits_{\gamma_2(\gamma_1(y))}^{y} \varphi(\zeta(x+t))\,dt + \lambda_4(x,y) \int\limits_{\gamma_1(\gamma_2(x))}^{x} \varphi(\zeta(s+y))\,ds \le \\
&\le \delta + \Big(y\lambda_0(x,y) + \sum_{k=1}^{4} \lambda_k(x,y) \Big) \int\limits_{0}^{x+y} \varphi(\zeta(s))\,ds \le \\
&\le \delta + \lambda \int\limits_{0}^{x+y} \varphi(\zeta(s))\,ds \ \text{ for } \ (x,y) \in \overline{G}.
\end{aligned}$$

Therefore,

$$\zeta(\tau) \le \delta + \lambda \int\limits_{0}^{\tau} \varphi(\zeta(s))\,ds \ \text{ for } \ 0 < \tau \le a+b.$$

Assuming that

$$\zeta_0(\tau) = \delta + \lambda \int\limits_{0}^{\tau} \varphi(\zeta(s))\,ds,$$

from the last inequality we have

$$\zeta(\tau) \le \zeta_0(\tau) \ \text{ for } \ 0 < \tau \le a+b,$$

$$\zeta_0(0) = \delta, \quad 0 < \frac{\zeta_0'(\tau)}{\varphi(\zeta_0(\tau))} = \frac{\lambda\varphi(\zeta(\tau))}{\varphi(\zeta_0(\tau))} \le \lambda \ \text{ for } \ 0 < \tau \le a+b.$$

Therefore,

$$\psi_\delta(\zeta_0(\tau)) = \int\limits_{0}^{\tau} \frac{\zeta_0'(s)}{\varphi(\zeta_0(s))}\,ds \le \lambda\tau \ \text{ for } \ 0 < \tau \le a+b,$$

and hence

$$\zeta_0(\tau) \le \psi_\delta^{-1}(\lambda\tau) \ \text{ for } \ 0 < \tau \le a+b.$$

Consequently, the estimate (2.9) is valid since

$$\xi(x,y) \le \zeta(x+y) \le \zeta_0(x+y) \ \text{ for } \ (x,y) \in \overline{G}. \qquad \square$$

Proof of Theorem 2.1. Owing to the fact that u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are uniformly continuous in G, we can continuously extend these vector functions on $\overline{G}$. Then by Lemma 2.1 we find

$$u(x,y) = c_1(y) + \int\limits_{\gamma_1(y)}^{x} c_2(s)\,ds + \int\limits_{\gamma_1(y)}^{x} ds \int\limits_{\gamma_2(s)}^{y} \frac{\partial^2 u(s,t)}{\partial s \partial t}\,dt, \tag{2.10}$$

$$\frac{\partial u(x,y)}{\partial x} = c_2(x) + \int\limits_{\gamma_2(x)}^{y} \frac{\partial^2 u(x,t)}{\partial x \partial t}\,dt, \tag{2.11}$$

$$\begin{aligned}\frac{\partial u(x,y)}{\partial y} &= c_1'(y) + \gamma_1'(y) c_2(\gamma_1(y)) + \int\limits_{\gamma_1(y)}^{x} \frac{\partial^2 u(s,y)}{\partial s \partial y}\,ds + \\ &\quad + \gamma_1'(y) \int\limits_{\gamma_2(\gamma_1(y))}^{y} \frac{\partial^2 u(s,t)}{\partial s \partial t}\Big|_{s=\gamma_1(y)}\,dt.\end{aligned} \tag{2.12}$$

If we assume that

$$\xi(x,y) = \|u(x,y)\| + \Big\|\frac{\partial u(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial u(x,y)}{\partial y}\Big\|,$$

then by virtue of the condition (2.3) we obtain

$$\Big\|\frac{\partial^2 u(x,y)}{\partial x \partial y}\Big\| \le \varphi(\xi(x,y)) \ \text{ for } \ (x,y) \in \overline{G}.$$

If along with the above-said we take into account the inequalities (2.2) and (2.3), then the identities (2.10)–(2.12) result in the inequality (2.7), where the functions λ_k $(k = 0,1,2,3,4)$ are given by the equalities

$$\lambda_0(x,y) = 1, \ \ \lambda_1(x,y) = 1, \ \ \lambda_2(x,y) = 1, \ \ \lambda_3(x,y) = |\gamma_1'(y)|, \ \ \lambda_4(x,y) = 0$$

and satisfy the condition (2.8). By Lemma 2.2, the function ξ admits the estimate (2.9). Consequently, the estimate (2.9) is valid. $\square$

Theorem 2.2. *Let there exist a positive number δ and a continuous non-decreasing function $\varphi_0 : [0,+\infty[\to [0,+\infty[$ such that*

$$\varphi_0(\tau) > 0 \ \textit{ for } \ \tau > 0,$$

$$\lim_{\tau \to +\infty} \psi_{0\delta}(\tau) > (1+b)(a+b), \ \ \textit{where} \ \ \psi_{0\delta}(\tau) = \int\limits_{\delta}^{\tau} \frac{ds}{\varphi_0(s)}, \tag{2.13}$$

$$\|c_1(y)\| + \|c_2(x)\| + \int\limits_{\gamma_1(y)}^{x} \|c_2(s)\|\,ds \le \delta \ \textit{ for } \ (x,y) \in G \tag{2.14}$$

and

$$f(x,y,z_1,z_2,z_3)\cdot \operatorname{sgn}(z_2) \le \varphi_0\big(\|z_1\|+\|z_2\|\big) \tag{2.15}$$
$$\text{for } (x,y)\in G, \ \ (z_1,z_2,z_3)\in R^{3n}$$

are fulfilled. Then an arbitrary solution of the problem (1.1), (1.2) *in the domain* G *admits the estimate*

$$\|u(x,y)\| + \Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \le \psi_{0\delta}^{-1}\big((1+b)(x+y)\big), \tag{2.16}$$

where $\psi_{0\delta}^{-1}$ *is the function inverse to* $\psi_{0\delta}$. *If, moreover, along with* (2.13)–(2.15) *the inequalities*

$$\big\|c_1'(y)+\gamma_1'(y)c_2(\gamma_1(y))\big\| \le \delta, \ \ 1+|\gamma_1'(y)| \le \lambda \ \text{ for } \ 0\le y\le b \tag{2.17}$$

and

$$\big\|f(x,y,z_1,z_2,z_3)\big\| \le \varphi(\|z_3\|) \tag{2.18}$$
$$\text{for } (x,y)\in G, \ \ \|z_1\|+\|z_2\| \le \psi_{0\delta}^{-1}\big((1+b)(a+b)\big), \ \ z_3\in R^n$$

are fulfilled, where $\lambda = \text{const}$, *and* $\varphi : [0,+\infty[\, \to [0,+\infty[$ *is a continuous nondecreasing function satisfying the condition* (2.1), *then an arbitrary solution of the problem* (1.1), (1.2) *in the domain* G *together with* (2.16) *admits the estimate*

$$\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \le \psi_\delta^{-1}(\lambda(x+y)). \tag{2.19}$$

Proof. Just as above, the vector functions u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be assumed to be continuously extendable on $\overline{G}$. Suppose

$$\xi_0(x,y) = \|u(x,y)\| + \Big\|\frac{\partial u(x,y)}{\partial x}\Big\|.$$

According to the condition (2.15), for an arbitrarily fixed $(x,y)\in G$ we have

$$\begin{aligned}\frac{\partial}{\partial t}\Big\|\frac{\partial u(x,t)}{\partial x}\Big\| &= \frac{\partial^2 u(x,t)}{\partial x\partial t}\cdot \operatorname{sgn}\Big(\frac{\partial u(x,t)}{\partial x}\Big) = \\ &= f\Big(x,t,u(x,t),\frac{\partial u(x,t)}{\partial x},\frac{\partial u(x,t)}{\partial t}\Big)\cdot \operatorname{sgn}\Big(\frac{\partial u(x,t)}{\partial x}\Big) \le \\ &\le \varphi_0(\xi_0(x,t)) \ \text{ for almost all } \ t\in\,]\gamma_2(x),y[\,.\end{aligned}$$

Integrating both parts of the above inequality from $\gamma_2(x)$ to y, we obtain

$$\Big\|\frac{\partial u(x,y)}{\partial x}\Big\| \le \|c_2(x)\| + \int\limits_{\gamma_2(x)}^{y} \varphi(\xi_0(x,t))\,dt.$$

Therefore,

$$\|u(x,y)\| \le \|c_1(y)\| + \int\limits_{\gamma_1(y)}^{x} \|c_2(s)\|\,ds + \int\limits_{\gamma_1(y)}^{x} ds \int\limits_{\gamma_2(s)}^{y} \varphi(\xi_0(s,t))\,dt.$$

The last two inequalities with regard for (2.14) yield

$$\xi_0(x,y)\le\delta+\int\limits_{\gamma_1(y)}^{x}ds\int\limits_{\gamma_2(s)}^{y}\varphi(\xi_0(s,t))\,dt+\int\limits_{\gamma_2(x)}^{y}\varphi(\xi_0(x,t))\,dt\ \text{ for }\ (x,y)\in\overline{G}.$$

By virtue of the condition (2.13) and Lemma 2.2 it follows that

$$\xi_0(x,y)\le\psi_{0\delta}^{-1}\big((1+b)(x+y)\big)\ \text{ for }\ (x,y)\in\overline{G}.$$

Consequently, the estimate (2.16) is valid.

Assume now that along with (2.13)–(2.15) the conditions (2.1), (2.17) and (2.18) are fulfilled. We put

$$\xi(x,y)=\Big\|\frac{\partial u(x,y)}{\partial y}\Big\|.$$

Then by the inequalities (2.16)–(2.18), from the representation (2.12) we find

$$\xi(x,y)\le\delta+\int\limits_{\gamma_1(y)}^{x}\varphi(\xi(s,y))\,ds+(\lambda-1)\int\limits_{\gamma_2(\gamma_1(y))}^{y}\varphi(\xi(\gamma_1(y),t))\,dt\ \text{ for }\ (x,y)\in\overline{G},$$

whence, owing to the condition (2.1) and Lemma 2.2, we obtain the inequality (2.9). Consequently, the estimate (2.19) is valid. □

3. Solvability and Well-Posedness

First we give two auxiliary propositions on solvability and well-posedness of the problem (1.1), (1.2) in the case where f is bounded in $G\times R^{3n}$.

Lemma 3.1. *Let there exist a positive constant ρ_0 such that on the domain $G\times R^{3n}$ the conditions*

$$\big\|f(x,y,z_1,z_2,z_3)\big\|\le\rho_0 \tag{3.1}$$

and

$$\big\|f(x,y,z_1,z_2,z_3)-f(x,y,z_1,\overline{z}_2,\overline{z}_3)\big\|\le\rho_0\big(\|z_2-\overline{z}_2\|+\|z_3-\overline{z}_3\|\big) \tag{3.2}$$

are fulfilled. Then the problem (1.1), (1.2) *has at least one solution.*

Proof. Without loss of generality, ρ_0 can be assumed to be so large that

$$\|c_1(y)\|+\|c_2(x)\|+\int\limits_{\gamma_1(y)}^{x}\|c_2(s)\|\,ds+\big\|c_1'(y)+\gamma_1'(y)\cdot c_2(\gamma_1(y))\big\|\le$$
$$\le\rho_0\ \text{ for }\ (x,y)\in\overline{G},\ \ |\gamma_1'(y)|\le\rho_0\ \text{ for }\ 0\le y\le b. \tag{3.3}$$

Suppose

$$\rho=1+a+b+(3+a+2b+ab+\rho_0)\rho_0, \quad c_3(y)=c_1'(y)+\gamma_1'(y)c_2(\gamma_1(y)),$$

$$\omega(s)=\max\Big\{\|c_2(x)-c_2(\overline{x})\|+|\gamma_2(x)-\gamma_2(\overline{x})| : \ 0\le x,\overline{x}\le a, \ |x-\overline{x}|\le s\Big\}+$$
$$+\max\Big\{\|c_3(y)-c_3(\overline{y})\|+|\gamma_1(y)-\gamma_1(\overline{y})|+|\gamma_1'(y)-\gamma_1'(\overline{y})|+$$
$$+\big\|\gamma_2(\gamma_1(y))-\gamma_2(\gamma_1(\overline{y}))\big\| : \ 0\le y,\overline{y}\le b, \ |y-\overline{y}|\le s\Big\}+$$
$$+\max\Big\{\big\|f(x,y,z_1,z_2,z_3)-f(\overline{x},\overline{y},\overline{z}_1,\overline{z}_2,\overline{z}_3)\big\| : \ (x,y)\in\overline{G}, \ (\overline{x},\overline{y})\in\overline{G},$$
$$\sum_{i=1}^{3}\|z_i\|\le\rho, \ \sum_{i=1}^{3}\|\overline{z}_i\|\le\rho, \ |x-\overline{x}|+|y-\overline{y}|+\sum_{i=1}^{3}\|z_i-\overline{z}_i\|\le 3\rho^2 s\Big\}.$$

From the continuity of the functions c_2, c_3, γ_1, γ_1', γ_2 and f it follows that the function $\omega : [0,+\infty[\to[0,+\infty[$ is likewise continuous and $\omega(0)=0$.

Let C be the Banach space of vector functions $z=(z_1,z_2,z_3):\overline{G}\to R^{3n}$ with the norm

$$\|z\|_C=\max\Big\{\sum_{k=1}^{3}\|z_k(x,y)\| : \ (x,y)\in\overline{G}\Big\}.$$

By D we denote the set of all $z=(z_1,z_2,z_3)\in C$ satisfying in $\overline{G}$ the conditions

$$\|z\|_C\le\rho, \quad \big\|z_1(x,y)-z_1(\overline{x},\overline{y})\big\|\le\rho\big(|x-\overline{x}|+|y-\overline{y}|\big),$$
$$\big\|z_2(x,y)-z_2(x,\overline{y})\big\|\le\rho|y-\overline{y}|, \quad \big\|z_2(x,y)-z_2(\overline{x},y)\big\|\le\rho\exp(\rho y)\omega(|x-\overline{x}|),$$
$$\big\|z_3(x,y)-z_3(\overline{x},y)\big\|\le\rho|x-\overline{x}|,$$
$$\big\|z_3(x,y)-z_3(x,\overline{y})\big\|\le\rho\exp(\rho(x+b))\omega(|y-\overline{y}|).$$

Obviously, D is a convex, compact subset of the space C.

In the space C we consider the operator $p=(p_1,p_2,p_3)$, which for an arbitrary $z=(z_1,z_2,z_3)\in C$ is defined by the equalities

$$p_1(z)(x,y)=c_1(y)+\int\limits_{\gamma_1(y)}^{x}c_2(s)\,ds+$$
$$+\int\limits_{\gamma_1(y)}^{x}ds\int\limits_{\gamma_2(s)}^{y}f\big(s,t,z_1(s,t),z_2(s,t),z_3(s,t)\big)\,dt, \tag{3.4}$$

$$p_2(z)(x,y)=c_2(x)+\int\limits_{\gamma_2(x)}^{y}f\big(x,t,z_1(x,t),z_2(x,t),z_3(x,t)\big)\,dt, \tag{3.5}$$

$$p_3(z)(x,y) = c_3(y) + \int\limits_{\gamma_1(y)}^{x} f\big(s,y,z_1(s,y),z_2(s,y),z_3(s,y)\big)\,ds +$$

$$+\gamma_1'(y) \int\limits_{\gamma_2(\gamma_1(y))}^{y} f\big(\gamma_1(y),t,z_1(\gamma_1(y),t),z_2(\gamma_1(y),t),z_3(\gamma_1(y),t)\big)\,dt. \quad (3.6)$$

Owing to the continuity of $f : \overline{G}\times R^{3n} \to R^n$, it is evident that the operator $p : C \to C$ is continuous.

For an arbitrarily fixed $z = (z_1, z_2, z_3) \in D$ we put

$$\widetilde{z}_k(x,y) = p_k(z)(x,y) \ \ (k=1,2,3), \quad \widetilde{z}(x,y) = \big(\widetilde{z}_1(x,y), \widetilde{z}_2(x,y), \widetilde{z}_3(x,y)\big).$$

Then by virtue of the conditions (3.1)–(3.3), from (3.4)–(3.6) we find

$$\|\widetilde{z}\|_C \le (1 + ab + b + a + \rho_0 b)\rho_0 \le \rho,$$

$$\big\|\widetilde{z}_1(x,y) - \widetilde{z}_1(\overline{x},\overline{y})\big\| = \Bigg\| \int\limits_{x}^{\overline{x}} z_2(s,y)\,ds + \int\limits_{y}^{\overline{y}} z_2(\overline{x},t)\,dt \Bigg\| \le$$

$$\le \Bigg| \int\limits_{x}^{\overline{x}} \|z_2(s,y)\|\,ds \Bigg| + \Bigg| \int\limits_{y}^{\overline{y}} \|z_2(\overline{x},t)\|\,dt \Bigg| \le \rho\big(|x-\overline{x}| + |y-\overline{y}|\big),$$

$$\big\|\widetilde{z}_2(x,y) - \widetilde{z}_2(x,\overline{y})\big\| =$$

$$= \Bigg\| \int\limits_{y}^{\overline{y}} f\big(x,t,z_1(x,t),z_2(x,t),z_3(x,t)\big)\,dt \Bigg\| \le \rho|y-\overline{y}|,$$

$$\big\|\widetilde{z}_2(x,y) - \widetilde{z}_2(\overline{x},y)\big\| \le \|c_2(x) - c_2(\overline{x})\| + \rho_0\|\gamma_2(x) - \gamma_2(\overline{x})\| +$$

$$+\int\limits_{\gamma_2(x)}^{y} \Big\| f\big(x,t,z_1(x,t),z_2(x,t),z_3(x,t)\big) - f\big(\overline{x},t,z_1(\overline{x},t),z_2(x,t),z_3(\overline{x},t)\big) \Big\|\,dt +$$

$$+\int\limits_{\gamma_2(x)}^{y} \Big\| f\big(x,t,z_1(\overline{x},t),z_2(x,t),z_3(\overline{x},t)\big) - f\big(\overline{x},t,z_1(\overline{x},t),z_2(\overline{x},t),z_3(\overline{x},t)\big) \Big\|\,dt \le$$

$$\le (1 + \rho_0 + b)\omega(|x-\overline{x}|) + \rho_0 \int\limits_{\gamma_2(x)}^{y} \big\|z_2(x,t) - z_2(\overline{x},t)\big\|\,dt \le$$

$$\le (1 + \rho_0 + b)\omega(|x-\overline{x}|) + \rho_0\rho\omega(|x-\overline{x}|) \int\limits_{0}^{y} \exp(\rho t)\,dt \le$$

$$\le \big(1 + \rho_0 + b + \rho_0 \exp(\rho y)\big)\omega(|x-\overline{x}|) \le \rho \exp(\rho y)\omega(|x-\overline{x}|),$$

$$\big\|\widetilde{z}_3(x,y) - \widetilde{z}_3(\overline{x},y)\big\| \le$$

$$= \left\| \int\limits_{x}^{\overline{x}} f\big(s, y, z_1(s,y), z_2(s,y), z_3(s,y)\big)\, ds \right\| \leq \rho|y-\overline{y}|,$$

and

$$\big\|\widetilde{z}_3(x,y) - \widetilde{z}_3(x,\overline{y})\big\| \leq \|c_3(y) - c_3(\overline{y})\| + \rho_0\|\gamma_1(y) - \gamma_1(\overline{y})\| +$$
$$+ \int\limits_{\gamma_1(y)}^{x} \Big\| f\big(s, y, z_1(s,y), z_2(s,y), z_3(s,y)\big) - f\big(s, \overline{y}, z_1(s,\overline{y}), z_2(s,\overline{y}), z_3(s,y)\big)\Big\|\, ds +$$
$$+ \int\limits_{\gamma_1(y)}^{x} \Big\| f\big(s, \overline{y}, z_1(s,\overline{y}), z_2(s,\overline{y}), z_3(s,y)\big) - f\big(s, \overline{y}, z_1(s,\overline{y}), z_2(s,\overline{y}), z_3(s,\overline{y})\big)\Big\|\, ds +$$
$$+ \rho_0 b|\gamma_1'(y) - \gamma_1'(\overline{y})| + \rho_0^2\big|\gamma_2(\gamma_1(y)) - \gamma_2(\gamma_1(\overline{y}))\big| +$$
$$+ \rho_0 \int\limits_{\gamma_2(\gamma_1(y))}^{y} \Big\| f\big(\gamma_1(y), t, z_1(\gamma_1(y),t), z_2(\gamma_1(y),t), z_3(\gamma_1(y),t)\big) -$$
$$- f\big(\gamma_1(\overline{y}), t, z_1(\gamma_1(\overline{y}),t), z_2(\gamma_1(y),t), z_3(\gamma_1(\overline{y}),t)\big)\Big\|\, dt +$$
$$+ \rho_0 \int\limits_{\gamma_2(\gamma_1(y))}^{y} \Big\| f\big(\gamma_1(\overline{y}), t, z_1(\gamma_1(\overline{y}),t), z_2(\gamma_1(y),t), z_3(\gamma_1(\overline{y}),t)\big) -$$
$$- f\big(\gamma_1(\overline{y}), t, z_1(\gamma_1(\overline{y}),t), z_2(\gamma_1(\overline{y}),t), z_3(\gamma_1(\overline{y}),t)\big)\Big\|\, dt \leq$$
$$\leq (1 + \rho_0 + a)\omega(|y-\overline{y}|) + \rho_0\rho\omega(|y-\overline{y}|) \int\limits_{0}^{x} \exp(\rho(s+b))\, ds +$$
$$+ (2\rho_0 b + \rho_0^2)\omega(|y-\overline{y}|) + \rho_0\rho\omega(|y-\overline{y}|) \int\limits_{0}^{y} \exp(\rho t)\, dt \leq$$
$$\leq \Big(1 + \rho_0 + a + 2\rho_0 b + \rho_0^2 + \rho_0 \exp(\rho(x+b)) + \rho_0\exp(\rho b)\Big)\omega(|y-\overline{y}|) \leq$$
$$\leq \rho\exp(\rho(x+b))\omega(|y-\overline{y}|).$$

Consequently, $\widetilde{z} \in D$. Thus we have proved that the continuous operator p maps the convex, compact set D into itself. This, according to Schauder's principle, implies that there exists the vector function $z = (z_1, z_2, z_3) \in D$ such that $p(z) = z$, i.e.,

$$z_i(x,y) = p_i(z)(x,y) \ \text{ for } \ (x,y) \in \overline{G} \ \ (i = 1,2,3).$$

Suppose

$$u(x,y) = z_1(x,y).$$

Then by virtue of the equalities (3.4)–(3.6) we have

$$\frac{\partial u(x,y)}{\partial x} = z_2(x,y), \quad \frac{\partial u(x,y)}{\partial y} = z_3(x,y),$$

and u is a solution of the problem (1.1), (1.2). □

Lemma 3.2. *Let there exist positive constants ρ_0 and ℓ such that on the domain $G \times R^{3n}$ along with* (3.1) *the condition*

$$\begin{aligned} &\big\|f(x,y,z_1,z_2,z_3) - f(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3)\big\| \le \\ &\le \ell\big(\|z_1-\overline{z}_1\| + \|z_2-\overline{z}_2\| + \|z_3-\overline{z}_3\|\big) \end{aligned} \tag{3.7}$$

is fulfilled. Then the problems (1.1), (1.2) *and* (1.3), (1.4) *are uniquely solvable, and the difference of their solutions admits the estimate* (1.7), *where*

$$\begin{aligned} &r = \Big(1+a+\lambda+\frac{1}{\ell}\Big)\exp\big(\lambda\ell(a+b)\big), \\ &\lambda = \max\big\{2+y+|\gamma_1'(y)| : \ 0 \le y \le b\big\}. \end{aligned} \tag{3.8}$$

Proof. By Lemma 3.1, the problems (1.1), (1.2) and (1.3), (1.4) are solvable. Let u and v be arbitrary solutions of these problems and

$$w(x,y) = u(x,y) - v(x,y).$$

Then w is a solution of the problem

$$\begin{gathered} \frac{\partial^2 w}{\partial x \partial y} = f_0\Big(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\Big), \\ \lim_{x\to\gamma_1(y)} w(x,y) = \widetilde{c}_1(y) \ \text{ for } \ 0<y<b, \\ \lim_{y\to\gamma_2(x)} \frac{w(x,y)}{\partial y} = \widetilde{c}_2(x) \ \text{ for } \ 0<x<a, \end{gathered}$$

where

$$\begin{aligned} f_0(x,y,z_1,z_2,z_3) = f\Big(x,y,v(x,y)+z_1,\frac{\partial v(x,y)}{\partial x}+z_2,\frac{\partial v(x,y)}{\partial y}+z_3\Big) - \\ - f\Big(x,y,v(x,y),\frac{\partial v(x,y)}{\partial x},\frac{\partial v(x,y)}{\partial y}\Big) + h(x,y). \end{aligned}$$

On the other hand, owing to (1.5) and (3.7), we have

$$\big\|f_0(x,y,z_1,z_2,z_3)\big\| \le \varphi\big(\|z_1\|+\|z_2\|+\|z_3\|\big),$$

$$\begin{aligned} \Big\|\widetilde{c}_1(y) + \int\limits_{\gamma_1(y)}^{x} \widetilde{c}_2(s)\,ds\Big\| + \|\widetilde{c}_2(x)\| + \big\|c_1'(y)+\gamma_1'(y)c_2(\gamma_1(y))\big\| \le \\ \le (1+a+\lambda)\eta(\widetilde{c}_1,\widetilde{c}_2,h), \end{aligned}$$

where

$$\varphi(\tau) = \eta(\widetilde{c}_1,\widetilde{c}_2,h) + \ell\tau.$$

This, by Theorem 2.1, implies that for an arbitrary

$$\delta > (1+a+\lambda)\eta(\widetilde{c}_1,\widetilde{c}_2,h)$$

the function w in the domain G admits the estimate

$$\|w(x,y)\| + \Big\|\frac{\partial w(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial w(x,y)}{\partial y}\Big\| \le$$
$$\le \big(\delta + \eta(\widetilde{c}_1,\widetilde{c}_2,h)/\ell\big)\exp(\ell\lambda(x+y)) - \eta(\widetilde{c}_1,\widetilde{c}_2,h)/\ell.$$

Passing in the above inequality to the limit as $\delta \to (1+a+\lambda)\eta(\widetilde{c}_1,\widetilde{c}_2,h)$, we obtain

$$\|w(x,y)\| + \Big\|\frac{\partial w(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial w(x,y)}{\partial y}\Big\| \le$$
$$\le \Big(1+a+\lambda+\frac{1}{\ell}\Big)\eta(\widetilde{c}_1,\widetilde{c}_2,h)\exp(\ell\lambda(x+y)) - \eta(\widetilde{c}_1,\widetilde{c}_2,h)/\ell.$$

Consequently, the estimate (1.7), where r is the number given by equalities (3.8), is valid.

If $\widetilde{c}_1(y) \equiv 0$, $\widetilde{c}_2(x) \equiv 0$ and $h(x,y) \equiv 0$, then it follows from (1.7) that $u(x,y) = v(x,y)$. Thus the problem (1.1), (1.2) has a unique solution. Obviously, the problem (1.3), (1.4) is likewise uniquely solvable. ☐

We say that the vector function f **satisfies the local Lipschitz condition with respect to the last** $2n$ **variables** if for an arbitrary positive number $\rho > 0$ there exists $\ell(\rho) > 0$ such that

$$\big\|f(x,y,z_1,z_2,z_3) - f(x,y,z_1,\overline{z}_2,\overline{z}_3)\big\| \le \ell(\rho)\big(\|z_2-\overline{z}_2\| + \|z_3-\overline{z}_3\|\big) \tag{3.9}$$
$$\text{for } (x,y)\in G,\ \ \|z_i\|\le\rho,\ \ \|\overline{z}_k\|\le\rho\ \ (i=1,2,3;\ k=2,3).$$

If, however, instead of (3.9) is fulfilled the condition

$$\big\|f(x,y,z_1,z_2,z_3) - f(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3)\big\| \le$$
$$\le \ell(\rho)\big(\|z_1-\overline{z}_1\| + \|z_2-\overline{z}_2\| + \|z_3-\overline{z}_3\|\big) \tag{3.10}$$
$$\text{for } (x,y)\in G,\ \ \|z_i\|\le\rho,\ \ \|\overline{z}_i\|\le\rho\ \ (i=1,2,3),$$

then we say that the vector function f **satisfies the local Lipschitz condition with respect to the last** $3n$ **variables**.

Theorem 3.1. *Let there exist positive numbers δ, λ and a continuous nondecreasing function $\varphi : [0,+\infty[\to [0,+\infty[$ satisfying the condition* (2.1), *such that the inequalities* (2.2) *and* (2.3) *are fulfilled respectively on the domains G and $G\times R^{3n}$. Let, moreover, the vector function f satisfy the local Lipschitz condition with respect to the last $2n$ variables* (*with respect to the last $3n$ variables*). *Then the problem* (1.1), (1.2) *has at least one solution* (*the problem* (1.1), (1.2) *is well-posed*).

Proof. According to (2.1), there exist positive constants δ_0 and η_0 such that

$$(1+a+\lambda)\eta_0 \le \delta_0 - \delta, \tag{3.11}$$

$$\lim_{\tau\to+\infty}\psi(\tau) > \lambda(a+b), \ \ \text{where } \ \psi(\tau) = \int_{\delta_0}^{\tau}\frac{ds}{\eta_0+\varphi(s)}\,. \tag{3.12}$$

Let ψ^{-1} be the function inverse to ψ, and

$$\rho = \psi^{-1}(\lambda(a+b)). \tag{3.13}$$

Suppose

$$p(z) = \begin{cases} z & \text{for } \|z\| \le \rho \\ \dfrac{\rho}{\|z\|}\, z & \text{for } \|z\| > \rho \end{cases}, \qquad \widetilde{f}(x,y,z_1,z_2,z_3) =$$
$$= f\big(x,y,p(z_1),p(z_2),p(z_3)\big), \tag{3.14}$$

and consider the hyperbolic systems

$$\frac{\partial^2 u}{\partial x \partial y} = \widetilde{f}\Big(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big) \tag{3.15}$$

and

$$\frac{\partial^2 v}{\partial x \partial y} = \widetilde{f}\Big(x,y,v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\Big) + h(x,y) \tag{3.16}$$

with the initial conditions (1.2) and (1.4), where $h : \overline{G} \to R^n$ and $\widetilde{c}_2 : [0,a] \to R^n$ are continuous vector functions, and $\widetilde{c}_1 : [0,b] \to R^n$ is a continuously differentiable vector function satisfying the inequality (1.6).

Consider first the case where the condition (3.9) is fulfilled. Then by the inequalities (1.6), (2.2), (2.3), (3.11) and the identity (3.1), respectively in the domains G and $G \times R^{3n}$ the conditions

$$\big\|c_1(y) + \widetilde{c}_1(y)\big\| + \Big\| \int\limits_{\gamma_1(y)}^{x} \big(c_2(s) + \widetilde{c}_2(s)\big)\, ds \Big\| + \big\|c_2(x) + \widetilde{c}_2(x)\big\| +$$
$$+ \Big\| c_1'(y) + \widetilde{c}_1'(y) + \gamma_1'(y)\big(c_2(\gamma_1(y)) + \widetilde{c}_2(\gamma_1(y))\big)\Big\| \le$$
$$\le \delta + (1+a+\lambda)\eta(\widetilde{c}_1,\widetilde{c}_2,h) \le \delta_0, \tag{3.17}$$

$$\big\|\widetilde{f}(x,y,z_1,z_2,z_3)\big\| + \|h(x,y)\| \le \eta_0 + \varphi(3p), \tag{3.18}$$

$$\big\|\widetilde{f}(x,y,z_1,z_2,z_3)\big\| + \|h(x,y)\| \le \eta_0 + \varphi\big(\|z_1\| + \|z_2\| + \|z_3\|\big), \tag{3.19}$$

$$\big\|\widetilde{f}(x,y,z_1,z_2,z_3) - \widetilde{f}(x,y,z_1,\overline{z}_2,\overline{z}_3)\big\| \le \ell(\rho)\big(\|z_2 - \overline{z}_2\| + \|z_3 - \overline{z}_3\|\big) \tag{3.20}$$

are fulfilled.

By virtue of Lemma 3.1 and the conditions (3.18) and (3.20), the problems (3.15), (1.2) and (3.16), (1.4) have at least one solution. On the other hand, by Theorem 2.1 and the conditions (3.12), (3.13), (3.17) and (3.19), an arbitrary solution u of the problem (3.15), (1.2) and an arbitrary solution v of the problem (3.16), (1.4) in the domain G admit the estimates

$$\|u(x,y)\| + \Big\|\frac{\partial u(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial u(x,y)}{\partial y}\Big\| \le \rho, \tag{3.21}$$

$$\|v(x,y)\| + \Big\|\frac{\partial v(x,y)}{\partial x}\Big\| + \Big\|\frac{\partial v(x,y)}{\partial y}\Big\| \le \rho. \tag{3.22}$$

This, with regard for (3.14), implies that u and v are respectively the solutions of the problems (1.1), (1.2) and (1.3), (1.4). Thus we have proved that the problems (1.1), (1.2) and (1.3), (1.4) are solvable.

It should be noted here that by virtue of Theorem 3.1 and the conditions (2.2), (2.3), (3.12) and (3.13) an arbitrary solution u of the problem (1.1), (1.2) and an arbitrary solution v of the problem (1.3), (1.4) admit the estimates (3.21), (3.22). This, according to (3.14), implies that u is a solution of the problem (3.15), (1.2), and v is a solution of the problem (3.16), (1.4). Hence the problem (1.1), (1.2) is equivalent to the problem (3.15), (1.2), and the problem (1.3), (1.4) is equivalent to the problem (3.16), (1.4).

We now pass to the consideration of the case where the condition (3.10) is fulfilled. Then the vector function $\widetilde{f}$ in the domain $G \times R^{3n}$ along with (3.18) and (3.19) satisfies as well the condition

$$\begin{aligned} &\big\|\widetilde{f}(x,y,z_1,z_2,z_3) - \widetilde{f}(x,y,\overline{z}_1,\overline{z}_2,\overline{z}_3)\big\| \le \\ &\le \ell(\rho)\big(\|z_1-\overline{z}_1\| + \|z_2-\overline{z}_2\| + \|z_3-\overline{z}_3\|\big). \end{aligned}$$

Then by Lemma 3.2, the problems (3.15), (1.2) and (3.16), (1.4) are uniquely solvable, and the difference of their solutions admits the estimate (1.7), where

$$r = \big(1 + a + \lambda + 1/\ell(\rho)\big)\exp\big(\lambda\ell(\rho)(a+b)\big)$$

is the number independent of $\widetilde{c}_1$, $\widetilde{c}_2$ and h. This, according to the abovesaid, implies that the problems (1.1), (1.2) and (1.3), (1.4) are also uniquely solvable, and the difference of their solutions admits the estimate (1.7). Consequently, the problem (1.1), (1.2) is well-posed. $\square$

Theorem 3.2. *Let there exist positive constants ε, δ, λ and continuous non-decreasing functions $\varphi : [0,+\infty[\, \to [0,+\infty[$, $\varphi_0 : [0,+\infty[\, \to [0,+\infty[$ such that the conditions* (2.1), (2.13)–(2.15), (2.17) *and*

$$\big\|f(x,y,z_1,z_2,z_3)\big\| \le \varphi(\|z_3\|) \tag{3.23}$$

$$\text{for } (x,y)\in G, \ \ \|z_1\|+\|z_2\| \le \varepsilon + \psi_{0\delta}^{-1}(1+a)(a+b), \ \ z_3 \in R^n,$$

are fulfilled. Let, moreover, the vector function f satisfy the local Lipschitz condition with respect to the last $2n$ variables (*with respect to the last $3n$ variables*). *Then the problem* (1.1), (1.2) *has at lest one solution* (*the problem* (1.1), (1.2) *is well-posed*).

Proof. Owing to (2.1), (2.13), there exist positive constants δ_0 and η_0 such that along with (3.11) and (3.12) we have

$$\begin{gathered} \lim_{\tau\to+\infty} \psi_0(\tau) > (1+b)(a+b), \\ \psi_0^{-1}\big((1+b)(a+b)\big) < \varepsilon + \psi_{0\delta}^{-1}\big((1+b)(a+b)\big), \end{gathered} \tag{3.24}$$

where

$$\psi_0(\tau) = \int\limits_{\delta_0}^{\tau} \frac{ds}{\eta_0 + \varphi_0(s)}\,, \ \ \psi_0^{-1} \text{ is the function inverse to } \psi_0.$$

Let ψ^{-1} be the function inverse to ψ,

$$\rho = \psi_0^{-1}\big((1+b)(a+b)\big) + \psi^{-1}(\lambda(a+b)),$$

and let $\widetilde{f}$ be the vector function given by equalities (3.14).

By Theorem 2.2, the conditions (2.1), (2.13)–(2.15), (2.17), (3.23) and (3.24) guarantee the equivalence of the problems $(1.1),(1.2)$ and $(3.15),(1.2)$ and also of the problems $(1.3),(1.4)$ and $(3.16),(1.4)$ in the case where $\widetilde{c}_1$, $\widetilde{c}_2$ and h satisfy the inequality (1.6). If we apply now Lemmas 3.1 and 3.2, the validity of Theorem 3.2 becomes evident. □

Theorems 3.1 and 3.2, respectively, imply the following corollaries.

Corollary 3.1. *Let the vector function f satisfy the local Lipschitz condition with respect to the last $2n$ variables (with respect to the last $3n$ variables). Let, moreover, there exist a positive constant ℓ_0 such that in the domain $G \times R^{3n}$ the inequality*

$$\big\|f(x,y,z_1,z_2,z_3)\big\| \le \ell_0\big(1+\|z_1\|+\|z_2\|+\|z_3\|\big) \tag{3.25}$$

holds. Then the problem $(1.1),(1.2)$ has at least one solution (the problem $(1.1),(1.2)$ is well-posed).

Corollary 3.2. *Let the vector function f satisfy the local Lipschitz condition with respect to the last $2n$ variables (with respect to the last $3n$ variables). Let, moreover, there exist a positive constant ℓ_0 and a continuous function $\ell : R^{2n} \to [0,+\infty[$ such that in the domain $G\times R^{3n}$ the inequalities*

$$f(x,y,z_1,z_2,z_3)\cdot \operatorname{sgn}(z_2) \le \ell_0\big(1+\|z_1\|+\|z_2\|\big)$$

and

$$\big\|f(x,y,z_1,z_2,z_3)\big\| \le \ell(z_1,z_2)\big(1+\|z_3\|\big)$$

hold. Then the problem $(1.1),(1.2)$ has at least one solution (the problem $(1.1),(1.2)$ is well-posed).

From Corollary 3.1, in particular, follow the theorems by A. Alexiewicz and Orlicz [1], P. Hartman and A. Wintner [9], and W. Walter [15, p. 161], concerning the solvability of a characteristic problem and the Cauchy problem for the system (1.1).

In contrast to Corollary 3.1, Corollary 3.2 covers equations with rapidly growing with respect to phase variables right-hand members. As an example, in the case $n=1$ we consider the differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = f_0(x,y)\exp\Big(|u|+\Big|\frac{\partial u}{\partial x}\Big|\Big)\Big|\frac{\partial u}{\partial y}\Big|\Big(\frac{\partial u}{\partial x}\Big)^{2m-1} + f_1(x,y), \tag{3.26}$$

where $f_0 : \overline{G} \to]-\infty,0[$ and $f_1 : \overline{G} \to R$ are continuous functions. For this equation the condition (3.25) of Corollary 3.1 is violated. Nevertheless, by Corollary 3.2, the problem $(3.26),(1.2)$ is well-posed.

References

1. A. Alexiewicz and W. Orlicz, Some remarks on the existence and uniqueness of solutions of the hyperbolic equation $\frac{\partial^2 z}{\partial x \partial y} = f\big(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\big)$. *Stud. Math.* **15**(1956), No. 2, 201–215.
2. A. V. Bitsadze, To the question on the statement of a characteristic problem for second order hyperbolic systems. (Russian) *Dokl. AN SSSR* **223**(1976), No. 6, 1289–1291.
3. K. Deimling, A Carathéodory theory for systems of integral equations. *Ann. di Mat. Pura ed Appl.* **86**(1970), 217–260.
4. M. Grigolia, On the uniqueness of a solution of the Goursat problem for a quasilinear hyperbolic system. (Russian) *Differentsial'nye Uravneniya* **11**(1975), No. 12, 2210–2219; English transl.: *Differ. Equations* **11**(1975), No. 12, 1641–1647.
5. M. Grigolia, On the solvability of the Goursat problem. (Russian) *Trudy Tbiliss. Gos. Univ.* **204**(1978), 91–105.
6. M. Grigolia, On the well-posedness of the Goursat problem. (Russian) *Trudy Tbiliss. Gos. Univ.* **232–233**(1982), 30–48.
7. M. Grigolia, On a generalization of a characteristic problem for hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 4, 678–686; English transl.: *Differ. Equations* **21**(1985), No. 4, 458–465.
8. M. Grigolia, On the existence and uniqueness of solutions of the Goursat problem for systems of functional partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Physics* **16**(1999), 154–158.
9. P. Hartman and A. Wintner, On hyperbolic partial differential equations. *Amer. J. Math.* **74**(1952), No. 4, 834–864.
10. M. Meredov, On the Goursat and Darboux problem for a class of hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **9**(1973), No. 7, 1326–1333.
11. V. N. Vragov, On the Goursat and Darboux problems for a class of hyperbolic equations. (Russian) *Differentsial'nye Uravneniya* **8**(1972), No. 1, 7–16.
12. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. I. Eindeutigkeitssätze für die charakteristische anfangswertproblem. *Math. Z.* **71**(1959), 308–324.
13. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. II. Existenzsätze für das charakteristische anfangswertproblem. *Math. Z.* **71**(1959), 436–453.
14. W. Walter, Über die differentialgleichung $u_{xy} = f(x, y, u, u_x, u_y)$. III. Die nichtcharakteristische anfangswertaufgabe. *Math. Z.* **73**(1960), 268–279.
15. W. Walter, Differential and integral inequalities. *Springer-Verlag, Berlin, Heidelberg, New York*, 1970.

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