ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

(Reported on October 2 and 9, 1995)

Consider the equation

$$u^{(n)}(t) + \sum_{i=1}^{m} \sigma_i(t) \int_{\tau_i(t)}^{t} u(s) d\tau_i(s, t) = 0,$$  \hspace{1cm} (1)

where $n \geq 2$, $m \in \mathbb{N}$, $\tau_i; \sigma_i \in C(R_+; R_+)$, $\tau_i(t) \leq \sigma_i(t) \leq t$ for $t \in R_+$, $\lim_{t \to +\infty} \tau_i(t) = +\infty$ ($i = 1, \ldots, m$), the functions $\tau_i(s, t)$ are measurable, and $\tau_i(s, t)$ is nondecreasing ($i = 1, \ldots, m$).

Let $t_0 \in R_+$. A function $u : [t_0, +\infty] \to R$ is called a proper solution of the equation (1) if it is absolutely continuous along with its derivatives up to the $n - 1$-th order inclusively,

$$\sup \left\{ |u(s)| : s \in [t, +\infty] \right\} > 0 \quad \text{for} \quad t \geq t_0,$$

and there exists $\varphi \in C(R_+; R)$ such that $\varphi(t) = u(t)$ for $t \in [t_0, +\infty]$ and

$$\varphi(t) + \sum_{i=1}^{m} \sigma_i(t) \int_{\tau_i(t)}^{t} \varphi(s) d\tau_i(s, t) = 0$$

almost everywhere on $[t_0, +\infty]$. A proper solution $u : [t_0, +\infty] \to R$ is said to be oscillatory if it has a sequence of zeroes tending to $+\infty$. Otherwise the solution is said to be nonoscillatory.

**Definition.** We say that the equation (1) has the property $A$ if each of its proper solutions is oscillatory when $n$ is even, and either is oscillatory or satisfies $|u^{(i)}| < 0$ for $t \uparrow +\infty$ when $(i = 0, \ldots, n - 1)$ is odd.

**Theorem.** Let for some $i_0 \in \{1, \ldots, m\}$ there exist a nondecreasing $\delta \in C(R_+; R_+)$ such that $\tau_{i_0}(t) \leq \delta(t) \leq \sigma_{i_0}(t)$ for $t \in R_+$,

$$\lim_{t \to +\infty} \int_{\tau_{i_0}(s)}^{t} \int_{\delta(s)}^{\delta(t)} \xi^{n-1} d\xi \tau_{i_0}(\xi, s) ds > 0,$$

and

$$\max \left\{ \frac{\delta(t)}{\tau_{i_0}(t)} : t \in R_+ \right\} < +\infty$$

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and

\[
\lim_{t \to +\infty} \frac{\ln t}{\ln \tau_i(t)} < +\infty \quad (i = 1, \ldots, m). \tag{2}
\]

Then the condition

\[
\inf \left\{ \lim_{t \to +\infty} \ln^\lambda t \int \sum_{i=1}^m \sigma_i(s) \times \right. \\
\left. \times \ln^{-\lambda} \xi \, d\tau_i(\xi, s)ds : \lambda \in [0, k] \right\} > (n - 1)! \text{ for all } k \in N \tag{3}
\]
is sufficient for the equation (1) to have the property A.

**Corollary 1.** Let \( c_i \in [0, +\infty], \alpha_i, \eta_i \in [0, 1], \) and \( \alpha_i < \eta_i \) (\( i = 1, \ldots, m \)). Then for the equation

\[
u^{(n)}(t) + \sum_{i=1}^m c_i t \sum_{i=1}^m \frac{u(s)}{\tau_i(s)\alpha_i} ds = 0
\]
to have the property A, it is sufficient that

\[
\inf \left\{ \frac{1}{\lambda(\lambda + 1)} \sum_{i=1}^m c_i (\alpha_i^{-\lambda-1} - \eta_i^{-\lambda-1}) : \lambda \in [0, +\infty] \right\} > (n - 1)!.
\]

**Corollary 2.** Let \( c_i \in [0, +\infty], \alpha_i \in [0, 1], \) and \( \alpha_i < 1 \) for some \( i_0 \in \{1, \ldots, m\}. \) Then for the equation

\[
u^{(n)}(t) + \frac{1}{t \ln t} \sum_{i=1}^m c_i t \sum_{i=1}^m \frac{u(s)}{\tau_i(s)\alpha_i} ds = 0
\]
to have the property A, it is sufficient that

\[
\inf \left\{ \frac{1}{\lambda} \sum_{i=1}^m c_i \alpha_i^{-\lambda} : \lambda \in [0, +\infty] \right\} > (n - 1)!. 
\]

In the case where the condition

\[
\lim_{t \to +\infty} t \ln \tau_i(t) < +\infty \quad (i = 1, \ldots, m)
\]
holds instead of (2), analogous questions are considered in [1].

**Remark.** Note that the inequality (2) cannot be replaced by

\[
\inf \left\{ \lim_{t \to +\infty} \ln^\lambda t \int \sum_{i=1}^m \sigma_i(s) \times \right. \\
\left. \times \ln^{-\lambda} \xi \, d\tau_i(\xi, s)ds : \lambda \in [0, k] \right\} > (n - 1)! - \varepsilon
\]
for any whatever small \( \varepsilon. \)
References


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