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# ON DISCRETENESS CRITERION FOR THE SPECTRUM OF A DIFFERENTIAL OPERATOR OF EVEN ORDER

**Abstract.** Necessary and sufficient conditions are obtained for the discreteness of the spectrum of the operator

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\,(-1)^m(p(x)u^{(m)}(x))^{(m)}, \ \ x \in I = [0,\infty), \ \ m \geq 1.$$

In the case of $p(x) = x^\nu$, $\nu \in [0,1)$, they are the same:

$$\lim_{s\to\infty} s^{2m-1-\nu} \int\limits_s^\infty \rho(x)\,dx = 0.$$

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**Key words and phrases.** Differential operator, discreteness of the spectrum.

**რეზიუმე.** მიღებულია აუცილებელი და საკმარისი პირობები

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\,(-1)^m(p(x)u^{(m)}(x))^{(m)}, \ \ x \in I = [0,\infty), \ \ m \geq 1,$$

ოპერატორის სპექტრის დისკრეტულობისთვის. იმ შემთხვევაში, როცა $p(x) = x^\nu$, $\nu \in [0,1)$, ისინი ემთხვევა პირობებს:

$$\lim_{s\to\infty} s^{2m-1-\nu} \int\limits_s^\infty \rho(x)\,dx = 0.$$

# 1 M. Sh. Birman's discreteness criterion

We are talking about the discreteness of the spectrum of the differential operator[1]

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\,(-1)^m (p(x)u^{(m)}(x))^{(m)}, \quad x \in I = [0,\infty), \tag{1.1}$$

$m \geq 1$, and its special case which we call *Special case*,

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\,(-1)^m u^{(2m)}(x), \quad x \in I. \tag{1.2}$$

The functions $\rho(x)$, $p(x)$ are assumed to be measurable and positive (almost everywhere) on $(0,\infty)$. We assume that for $s > 0$,

$$\int\limits_0^s \frac{dx}{p(x)} < \infty. \tag{1.3}$$

The operator $\mathcal{L}$ (both in the first and second cases) is considered to be defined on the linear manifold of the space $L_2(I,\rho)$ of functions whose square is Lebesgue integrable on $I$ with weight $\rho$.

For operator (1.2), M. Sh. Birman obtained the following criterion (the necessary and sufficient condition) for the discreteness of the spectrum (given in [3]):

$$\lim_{s\to\infty} s^{2m-1} \int\limits_s^\infty \rho(x)\,dx = 0. \tag{1.4}$$

As is noted in [3], for operator (1.1), the sufficient conditions are far from optimal, and the necessary ones are not considered. In this paper, we obtain a necessary condition and a sufficient condition for the discreteness of the spectrum for operator (1.1). These conditions are different in form, but their application to special cases gives the same result: a necessary and sufficient condition. In particular, let $p(x) = x^\nu$, $\nu \in [0,1)$. Then the condition

$$\lim_{s\to\infty} s^{2m-1-\nu} \int\limits_s^\infty \rho(x)\,dx = 0 \tag{1.5}$$

(generalizing (1.4)) is necessary and sufficient for the discreteness of the spectrum of the operator (1.1) (see example 4 below). In the case of operator (1.2), here we offer a simpler proof of necessity compared to the proof of M. Birman based on choosing a simple test function and obtaining a two-sided estimate.

**Theorem 1.1.** *For the spectrum of operator* (1.1) *to be discrete, it is necessary that*

$$\lim_{s\to\infty} \varphi(s) \int\limits_s^\infty \rho(x)\,dx = 0$$

*and it is sufficient that the equality*

$$\lim_{s\to\infty} \varphi_0(s) \int\limits_s^\infty \rho(x)\,dx = 0$$

*hold. The functions $\varphi$ and $\varphi_0$ are defined in* (3.4) *and* (3.2).

[1] The symbol := means *equals by definition.*

## 1.1 Problem research scheme

For ease of reading, we present here the scheme of the study in full. The reader can find the general results, for example, in [3]. A simple scheme [4] is proposed that does not require a deep study of the spectral theory of unbounded operators. The differential operator $\mathcal{L}$ will be obtained as a solution to the variational problem. Let

$$[u,v] := \int_0^\infty p(x)u^{(m)}(x)v^{(m)}(x)\,dx, \tag{1.6}$$

and let $W$ be the space of functions with absolutely continuous derivative $u^{(m-1)}$ on each finite interval from $I=[0,\infty)$, finite value $[u,u]$, and satisfying the boundary conditions

$$u(0)=\cdots=u^{(m-1)}(0)=0. \tag{1.7}$$

The form $[u,v]$ is an inner product in the Hilbert space $W$ (Lemma 5.1).

The operator $T$ defined by the equality $Tu(x)=u(x)$ acts from $W$ to $L_2(I,\rho)$ and is bounded (Lemma 5.2) under the condition

$$\sup_{s\in I}\Phi(s)<\infty, \tag{1.8}$$

where the function $\Phi(s)$ is defined in (3.6). This condition is apparently also necessary for the operator $T$ to be bounded. In any case, this is true in the case of $p(x)\equiv 1$ (for the operator (1.2)) (Remark 2.1).

The problem of minimizing the functional $0.5[u,u]-(f,Tu)$, $u\in W$, leads to an equation for $u\in W$ in the variational form $[u,v]=(f,Tv)$, $v\in W$. This equation has a solution $u=T^*f$.

Since $\overline{T(W)}=L_2(I,\rho)$ (Lemma 5.4), the operator $T^*$ is an injection. Denote $\mathcal{L}=(T^*)^{-1}$. Thus, the domain of the operator $\mathcal{L}$ lies in $W$, and the equation $\mathcal{L}u=f$ has a solution $u=T^*f$. The representation of the operator $\mathcal{L}$ in form (1.2) is constructed as the Euler equation in a variational problem (Lemma 5.5).

This equation is the boundary value problem $\mathcal{L}u=f$, (1.7),

$$(pu^{(m)})(\infty)=\cdots=(pu^{(m)})^{(m-1)}(\infty)=0. \tag{1.9}$$

In spectral theory, the operator $\mathcal{L}$ is assumed to be unbounded with domain of definition belonging to $L_2(I,\rho)$. It is also convenient to consider the domain of $\mathcal{L}$ as an independent space, which is what we do. In this case, the spectrum will be expressed by using the operators $\mathcal{L}$ and $T$. A number $\lambda$ is called regular if the operator $(\mathcal{L}-\lambda T)^{-1}$ is defined on the entire space $L_2(I,\rho)$ and is continuous. The spectrum of the operator $\mathcal{L}$ is the set of values $\lambda$ that are not regular. The number $\lambda$ will be eigenvalue if the equation $(\mathcal{L}-\lambda T)u=0$ has nonzero solutions. A spectrum is called discrete if it consists only of eigenvalues of finite multiplicity.

A necessary and sufficient condition for the discreteness of the spectrum is the compactness of the operator $T$. Sufficiency follows from the fact that the equation $\mathcal{L}u=\lambda Tu$ is equivalent to $u=\lambda T^*Tu$. Necessity follows from the fact that the compactness of operator $T^*T$ coincides with the compactness of $T$ [1, Chapter 10, Theorem 5].

**Theorem 1.2** (Gelfand [2])**.** *For the set $U$ to be relatively compact in the Banach space $E$, it is necessary and sufficient that for any sequence of linear bounded functionals converging on any element $z\in E$, the convergence is uniform on the set $U$.*

**Lemma 1.1.** *The compactness criterion for the operator $T$ is the equality*

$$\lim_{s\to\infty}\sup_{[u,u]=1}(Tu,Tu)_{[s,\infty)}=0, \tag{1.10}$$

*where*

$$(Tu,Tu)_{[s,\infty)} := \int_s^\infty u(x)^2\rho(x)\,dx. \tag{1.11}$$

*Proof. Necessity.* Let $T$ be compact, but for some $\sigma > 0$, there are sequences $s_k \to \infty$ and $u_k$ with the norm $\|u_k\| = 1$ such that $(Tu_k, Tu_k)_{[s_k,\infty)} \geq \sigma$. Let

$$f_n = \frac{\chi_{[s_n,\infty)}}{\|Tu_n\|_{[s_n,\infty)}} Tu_n,$$

where $\chi_{[s_n,\infty)}$ is the characteristic function of the set $[s_n, \infty)$. Since

$$(f_n, z)^2 \leq \frac{1}{\|Tu_n\|^2_{[s_n,\infty)}} \int\limits_{s_n}^{\infty} u_n^2 \rho\, dx \int\limits_{s_n}^{\infty} z^2 \rho\, dx = \int\limits_{s_n}^{\infty} z^2 \rho\, dx,$$

we have $(f_n, z) \to 0$ for any $z \in L_2(I, \rho)$. However, the inequality

$$(f_n, Tu_n) = \frac{1}{\|Tu_n\|_{[s_n,\infty)}} \int\limits_{s_n}^{\infty} u_n^2 \rho\, dx = \sqrt{\int\limits_{s_n}^{\infty} u_n^2 \rho\, dx} \geq \sqrt{\sigma}$$

contradicts Theorem 1.2.

*Sufficiency.* Let $f_n \in L_2(I, \rho)$ be a sequence converging on any $z \in L_2(I, \rho)$, i.e., $(f_n, z) \to 0$. Let us show that $(f_n, Tu) \to 0$ uniformly on $[u, u] \leq 1$. From the inequalities

$$\Bigg(\int\limits_{s}^{\infty} f_n u \rho\, dx\Bigg)^2 \leq \int\limits_{s}^{\infty} f_n^2 \rho\, dx \int\limits_{s}^{\infty} u^2 \rho\, dx \leq C \int\limits_{s}^{\infty} u^2 \rho\, dx,$$

due to (1.10), it follows that

$$\lim_{s\to\infty} \int\limits_{s}^{\infty} f_n u \rho\, dx = 0$$

uniformly on $[u, u] \leq 1$. Therefore, it suffices to show the uniform convergence of $\int\limits_0^c f_n u \rho\, dx$ on $[u, u] \leq 1$ for any $c > 0$. We have

$$\begin{aligned}
\Bigg(\int\limits_{0}^{c} f_n u \rho\, dx\Bigg)^2 &= \Bigg(\int\limits_{0}^{c} f(x)\rho(x) \int\limits_{0}^{x} \frac{(x-s)^{m-1}}{(m-1)!} u^{(m)}(s)\, ds\, dx\Bigg)^2 \\
&= \Bigg(\int\limits_{0}^{c} u^{(m)}(s)\, ds \int\limits_{s}^{c} \frac{(x-s)^{m-1}}{(m-1)!} f_n(x)\rho(x)\, dx\Bigg)^2 \\
&\leq \int\limits_{0}^{c} p(s)(u^{(m)}(s))^2 ds \int\limits_{0}^{c} \varphi_n(s)^2\, ds \leq \int\limits_{0}^{c} \varphi_n(s)^2\, ds,
\end{aligned}$$

where

$$\varphi_n(s) = \frac{1}{\sqrt{p(s)}} \int\limits_{s}^{c} \frac{(x-s)^{m-1}}{(m-1)!} f_n(x)\rho(x)\, dx.$$

Note that $\varphi_n(s) = (f_n, z_s)$, where $z_s(x) = 0$ if $x \notin [s, c]$, and

$$z_s(x) = \frac{1}{\sqrt{p(s)}} \frac{(x-s)^{m-1}}{(m-1)!}$$

if $x \in [s, c]$. Since $z_s \in L_2(I, \rho)$, we have $\varphi_n(s) \to 0$ for any $s \in [0, c]$. Since

$$\varphi_n(s)^2 \leq \frac{1}{p(s)} \int\limits_{s}^{c} \frac{x^{2m-2}}{(m-1)!^2} \rho(x)\, dx \int\limits_{s}^{c} f_n(x)^2 \rho(x)\, dx \leq \frac{C}{p(s)}\,,$$

by the Lebesgue theorem,

$$\int_0^c \varphi_n(s)^2\,ds \to 0. \qquad \square$$

# 2 Estimates. *Special case*

As follows from Lemma 1.1, we need two-sided estimates of the expression $\sup\limits_{[u,u]=1} (Tu,Tu)_{[s,\infty)}$. To do this, we need auxiliary inequalities, which we will obtain first for a simpler particular case, and then in a general form.

Here, we consider the case $p(x) \equiv 1$, i.e., operator (1.2). In connection with criterion (1.10), estimates of the expression $\sup\limits_{[u,u]=1} (Tu,Tu)_{[s,\infty)}$ above are given in [3]. This makes it possible to obtain sufficient conditions for the discreteness of the spectrum. Below, we give two-sided estimates, which give a necessary and sufficient condition, i.e., a criterion. The proof is given not only for readability. The evaluation scheme is used below for the general operator (1.1).

## 2.1 Inequalities

Let a function $u(x)$ satisfy conditions (1.7), have an absolutely continuous $u^{(m-1)}(x)$ locally on $I = [0,\infty)$, and $\int\limits_I \left(u^{(m)}(x)\right)^2\,dx < \infty$. Let

$$A_k(u) := \int_I \frac{|u^{(k)}(s)u^{(k+1)}(s)|}{s^{2m-2k-1}} ds, \quad k = 0,\dots,m-1,$$

$$B_k(u) := \int_I \frac{u^{(k)}(s)^2}{s^{2m-2k}}\,ds, \quad k = 0,\dots,m.$$

From the Cauchy inequality $A_k^2 \le B_k B_{k+1}$,

$$\begin{aligned} B_k = \int_I \frac{ds}{s^{2m-2k}} 2\int_0^s u^{(k)}(x)u^{(k+1)}(x)\,dx &= 2\int_I u^{(k)}(x)u^{(k+1)}(x)\,dx \int_x^\infty \frac{ds}{s^{2m-2k}} \\ &= \frac{2}{2m-2k-1}\int_I \frac{u^{(k)}(x)u^{(k+1)}(x)}{x^{2m-2k-1}}\,dx \le \frac{2}{2m-2k-1}A_k. \end{aligned}$$

That's why

$$A_k \le \frac{2}{2m-2k-1}B_{k+1}, \quad B_k \le \frac{4}{(2m-2k-1)^2}B_{k+1},$$

$$B_0 \le \frac{4^m}{((2m-1)!!)^2}B_m, \quad B_1 \le \frac{2^{2m-2}}{((2m-3)!!)^2}B_m, \quad A_0 \le \frac{2^{2m-1}}{(2m-1)!!(2m-3)!!}B_m.$$

So,

$$A_0 \le C_m[u,u], \tag{2.1}$$

where

$$C_m := \frac{2^{2m-1}}{(2m-1)!!(2m-3)!!}, \tag{2.2}$$

since $B_m = [u,u]$.

## 2.2 Upper and lower estimates

Here, we consider the lower and upper bounds for the expression $(Tu, Tu)_{[r,\infty)}$ in the case $p(x) = 1$.

Let

$$\Phi(s) := s^{2m-1} \int_s^\infty \rho(x)\, dx. \tag{2.3}$$

**Lemma 2.1.** *There are the estimates*

$$(Tu, Tu)_{[r,\infty)} \le 4C_m \sup_{s\in[r,\infty)} \Phi(s) \cdot [u, u] \tag{2.4}$$

*and*

$$\sup_{[u,u]=1} (Tu, Tu)_{[s,\infty)} \ge \frac{1}{(m!)^2}\, s^{2m-1} \int_s^\infty \rho(x)\, dx = \frac{1}{(m!)^2}\, \Phi(s). \tag{2.5}$$

*Proof.*

$$(Tu, Tu)_{[r,\infty)} = \int_r^\infty u^2 \rho\, dx = \int_r^\infty \rho(x)\, dx \Big( u(r)^2 + 2 \int_r^x u(s)u'(s)\, ds \Big)$$
$$= u(r)^2 \int_r^\infty \rho(x)\, dx + 2 \int_r^\infty u(s)u'(s)\, ds \int_s^\infty \rho(x)\, dx = S_1 + 2S_2,$$
$$S_2 = \int_r^\infty u(s)u'(s)\, ds \int_s^\infty \rho(x)\, dx = \int_r^\infty \frac{u(s)u'(s)}{s^{2m-1}}\, \Phi(s)\, ds \le C_m \sup_{s\in[r,\infty)} \Phi(s) \cdot [u, u],$$
$$S_1 = 2 \int_0^r u(s)u'(s)\, ds \int_r^\infty \rho(x)\, dx \le 2 \int_0^r \frac{u(s)u'(s)}{s^{2m-1}}\, ds \cdot r^{2m-1} \int_r^\infty \rho(x)\, dx \le 2C_m \Phi(r) \cdot [u, u].$$
$$(Tu, Tu)_{[r,\infty)} \le 2C_m (\Phi(r) + \sup_{s\in[r,\infty)} \Phi(s)) \cdot [u, u].$$

Now, we estimate $(Tu, Tu)_{[s,\infty)}$ from below. We need a supremum, so, we choose a test function

$$u(x) = C \cdot \begin{cases} x^m & \text{if } 0 \le x \le s, \\ s^m + P_{m-1}(x) & \text{if } x > s. \end{cases}$$

The polynomial $P_{m-1}$ of degree at most $m - 1$ will be chosen so as to ensure the continuity of the function $u(x)$ and its derivatives up to order $m - 1$, inclusive. We choose the parameter $C$ so that $[u, u] = 1$:

$$\int_0^s (u^{(m)})^2\, dx = C^2 (m!)^2 s = 1,$$

whence we obtain the estimate

$$(Tu, Tu)_{[s,\infty)} = \int_s^\infty u^2 \rho\, dx = C^2 \int_s^\infty (s^m + P_{m-1}(x))^2 \rho\, dx \ge \frac{1}{(m!)^2} s^{2m-1} \int_s^\infty \rho(x)\, dx.$$

The positivity of $P_{m-1}(x)$ follows from the positivity of all derivatives $u^{(i)}(s)$, $i = 1, \dots, m - 1$. □

Bilateral estimates (2.4), (2.5) and (1.10) immediately imply Birman's discreteness criterion (1.4).

**Remark 2.1.** It follows from inequality (2.5) that condition (1.8) is necessary for the operator $T$ to be bounded.

# 3 General case of bilinear form

Here,

$$[u,v] := \int_I p(x)u^{(m)}(x)v^{(m)}(x)\,dx \tag{3.1}$$

under the same boundary conditions (1.7).

## 3.1 Inequalities

Denote

$$B_k = \int_I \frac{u^{(k)}(s)^2}{\omega_k(s)}\,ds, \quad k = 0, \dots, m.$$

The functions $\omega_k(s)$, $\varphi_k(x)$ are defined recursively: $\omega_m(s) = 1/p(s)$,

$$\varphi_k(s) = \int_0^s \omega_{k+1}(x)\,dx, \quad \omega_k(x) = \frac{\varphi_k^2(x)}{\omega_{k+1}(x)}, \quad k = m-1, \dots, 0. \tag{3.2}$$

Then

$$\Big(\frac{1}{\varphi_k}\Big)' = -\frac{\varphi_k'}{\varphi_k^2} = -\frac{\varphi_k'}{\omega_k\omega_{k+1}} = -\frac{1}{\omega_k}\,.$$

From here,

$$\frac{1}{\varphi_k(s)} = \int_s^\infty \frac{dx}{\omega_k(x)}\,.$$

Indeed, this function is positive and decreasing. Therefore, this improper integral converges. However, in fairness, for $k = m-1$ in the case of convergence of the integral $\int_0^\infty dx/p(x)$ it is better to write the inequality

$$\frac{1}{\varphi_k(s)} \le \int_s^\infty \frac{dx}{\omega_k(x)}\,.$$

Denote

$$A_k = \int_I \frac{|u^{(k)}(s)u^{(k+1)}(s)|}{\varphi_k(s)}\,ds.$$

Then $A_k^2 \le B_k B_{k+1}$,

$$\begin{aligned} B_k &= \int_I \frac{ds}{\omega_k(s)} 2\int_0^s u^{(k)}(x)u^{(k+1)}(x)\,dx \\ &= 2\int_I u^{(k)}(x)u^{(k+1)}(x)\,dx \int_x^\infty \frac{ds}{\omega_k(s)} = 2\int_I \frac{u^{(k)}(x)u^{(k+1)}(x)}{\varphi_k(x)}\,dx \le 2A_k. \end{aligned}$$

Therefore, $A_k \le 2B_{k+1}$, $B_k \le 4B_{k+1}$, $B_0 \le 4^m B_m$,

$$A_0 \le 4^{m-1/2}B_m = 2^{2m-1}B_m = 2^{2m-1}[u,u]. \tag{3.3}$$

## 3.2 Upper and lower estimates

**Lemma 3.1.** *The following estimates hold:*

$$\sup_{[u,u]=1}(Tu,Tu)_{[s,\infty)} \geq \varphi(s)\int\limits_s^\infty \rho(x)\,dx, \tag{3.4}$$

*where*

$$\varphi(s) := u(s)^2 = \Bigg(\int\limits_0^s \frac{dx}{p(x)}\Bigg)^{-1}\Bigg(\int\limits_0^s \frac{(s-t)^{m-1}}{(m-1)!\,p(t)}\,dt\Bigg)^2,$$

*and*

$$(Tu,Tu)_{[r,\infty)} \leq 2^{2m+1}\sup_{s\in[r,\infty)}\Phi(s)[u,u], \tag{3.5}$$

*where*

$$\Phi(s) := \varphi_0(s)\int\limits_s^\infty \rho(x)\,dx. \tag{3.6}$$

*Proof.* The test function for lower estimate $p(x)u^{(m)} = C$ on $[0,s]$, whence

$$u(x) = C\int\limits_0^x \frac{(x-t)^{m-1}}{(m-1)!p(t)}\,dt, \quad x\in[0,s],$$

$u(x) = P_{m-1}(x)$ on $[s,\infty)$. We find the constant $C$ from the condition $[u,u]=1$:

$$\int\limits_0^s p(x)\,\frac{C^2}{p(x)^2}\,dx = 1, \quad C^2 = \Bigg(\int\limits_0^s \frac{dx}{p(x)}\Bigg)^{-1}.$$

Now, we get the lower bound. Since $P_{m-1}(x)$ does not decrease,

$$(Tu,Tu) = \int\limits_{[s,\infty)} u^2\rho\,dx \geq u(s)^2\int\limits_s^\infty \rho(x)\,dx.$$

Substituting the value of $u(s)^2$, we get $\varphi(s)$.

Now, we get the upper estimate:

$$\begin{aligned}(Tu,Tu)_{[r,\infty)} &= \int\limits_r^\infty u^2\rho\,dx = \int\limits_r^\infty \rho(x)\,dx\Bigg(u(r)^2 + 2\int\limits_r^x u(s)u'(s)\,ds\Bigg)\\ &= u(r)^2\int\limits_r^\infty \rho(x)\,dx + 2\int\limits_r^\infty u(s)u'(s)\,ds\int\limits_s^\infty \rho(x)\,dx = S_1 + 2S_2, \end{aligned} \tag{3.7}$$

$$S_2 = \int\limits_r^\infty \frac{u(s)u'(s)}{\varphi_0(s)}\,\Phi(s)\,ds \leq 2^{2m-1}\sup_{s\in[r,\infty)}\Phi(s)\cdot[u,u].$$

The value of $u(r)^2$ can be estimated in two ways:

$$u(r)^2 = \Bigg(\int\limits_0^r \frac{(r-s)^{m-1}}{(m-1)!}\,u^{(m)}(s)\,ds\Bigg)^2 \leq \frac{1}{((m-1)!)^2}\int\limits_0^r \frac{(r-s)^{2m-2}}{p(s)}\,ds\int\limits_0^r p(s)(u^{(m)}(s))^2\,ds$$

or

$$u(r)^2 = 2\int\limits_0^r u(s)u'(s)\,ds = 2\int\limits_0^r \frac{u(s)u'(s)}{\varphi_0(s)}\,\varphi_0(s)\,ds \leq 2^{2m}\varphi_0(r)[u,u].$$

From (3.7), we now get an estimate from above (3.5). $\square$

# 4 Discreteness conditions

From (3.4), for the discreteness of the spectrum of the operator (1.1) we obtain the necessary condition

$$\lim_{s\to\infty} \varphi(s) \int\limits_s^\infty \rho(x)\,dx = 0. \tag{4.1}$$

Note also the necessary condition for the operator $T$ to be bounded:

$$\sup_s \varphi(s) \int\limits_s^\infty \rho(x)\,dx < \infty. \tag{4.2}$$

**Example.** Let us find the necessary condition if $p(x) = x^\nu$, $\nu \in [0, 1)$. We have

$$\int\limits_0^s \frac{dx}{p(x)} = \frac{s^{1-\nu}}{1-\nu}\,, \quad \int\limits_0^s \frac{(s-t)^{m-1}}{(m-1)!p(t)}\,dt = \frac{s^{m-\nu}}{(m-1)!(1-\nu)(2-\nu)\cdots(m-\nu)}\,.$$

Therefore, we get the estimate

$$\sup_{[u,u]=1} (Tu, Tu)_{[s,\infty)} \ge \frac{(1-\nu)s^{2m-1-\nu}}{((m-1)!)^2(1-\nu)^2(2-\nu)^2\cdots(m-\nu)^2} \int\limits_s^\infty \rho(x)\,dx$$

and a necessary condition for the discreteness

$$\lim_{s\to\infty} s^{2m-1-\nu} \int\limits_s^\infty \rho(x)\,dx = 0. \tag{4.3}$$

The sufficient condition for the compactness of the operator $T$, which is also the discreteness condition, has the form

$$\lim_{s\to\infty} \varphi_0(s) \int\limits_s^\infty \rho(x)\,dx = 0. \tag{4.4}$$

**Example.** Find the function $\varphi_0$ in the case $p(x) = x^\nu$, $\nu \in [0, 1)$.

$$\omega_m(x) = x^{-\nu},\ \varphi_{m-1} = \int\limits_0^x s^{-\nu}\,ds = \frac{x^{-\nu+1}}{-\nu+1}\,,\ \omega_{m-1}(x) = \frac{x^{-2\nu+2}}{(1-\nu)^2x^{-\nu}} = \frac{x^{2-\nu}}{(1-\nu)^2}\,,$$

$$\varphi_{m-2}(x) = \frac{1}{(1-\nu)^2} \int\limits_0^x s^{2-\nu}\,ds = \frac{x^{3-\nu}}{(1-\nu)^2(3-\nu)}\,,$$

$$\omega_{m-2}(x) = \frac{x^{6-2\nu}}{(1-\nu)^4(3-\nu)^2} : \frac{x^{2-\nu}}{(1-\nu)^2} = \frac{x^{4-\nu}}{(1-\nu)^2(3-\nu)^2}\,,$$

$$\varphi_{m-3}(x) = \frac{1}{(1-\nu)^2(3-\nu)^2} \int\limits_0^x s^{4-\nu}\,ds = \frac{(5-\nu)x^{5-\nu}}{(1-\nu)^2(3-\nu)^2(5-\nu)^2}\,,\ \ldots\,.$$

Continuing, we get

$$\varphi_0(x) = \frac{(2m-1-\nu)x^{2m-1-\nu}}{(1-\nu)^2(3-\nu)^2\cdots(2m-1-\nu)^2}\,.$$

From (4.4) we obtain a sufficient discreteness condition

$$\lim_{s\to\infty} s^{2m-1-\nu} \int\limits_s^\infty \rho(x)\,dx = 0.$$

This coincides with the necessary condition (4.3), and therefore is a criterion for the discreteness of the spectrum in the case $p(x) = x^\nu$, $\nu \in [0, 1)$.

# 5 Lemmas

Let $L_2(I,p)$ be the space of functions measurable and with Lebesgue integrable square on $I$ and weight $p$.

**Lemma 5.1.** *The space $W$ is Hilbert one.*

*Proof.* The justification is the isomorphism of the spaces $W$ and $L_2(I,p)$. If $g \in L_2(I,p)$, then, due to the inequalities

$$\left(\int_0^s |g(x)|\,dx\right)^2 \le \int_0^s p(x)g(x)^2\,dx \int_0^s \frac{dx}{p(x)} \le C\int_0^s \frac{dx}{p(x)},$$

the function $u(x)$ is uniquely determined from the equality $u^{(m)} = g$ and the boundary conditions (1.7). $\square$

**Lemma 5.2.** *Let $\sup\limits_{s\in I}\Phi(s) < \infty$ be satisfied. Then the operator $T$ is continuous.*

*Proof.* Follows from (3.5). $\square$

**Lemma 5.3.** *Let $f \in L_2(I,\rho)$. If $v \in W$, then*

$$\int_0^\infty f(x)v(x)\rho(x)\,dx = (-1)^m \int_0^\infty H(x)v^{(m)}(x)\,dx, \tag{5.1}$$

*where the function $H(x)$ is defined by the equalities $H^{(m)} = f\rho$, $H^{(m-k)}(\infty) = 0$, $k = 1,\dots,m$, and satisfies the relation*

$$(-1)^m H(x) = \int_x^\infty \frac{(t-x)^{m-1}}{(m-1)!} f(t)\rho(t)\,dt. \tag{5.2}$$

*Proof.* Integrating by parts, we get

$$\int_0^\infty f(x)v(x)\rho(x)\,dx = \sum_{k=1}^m (-1)^{k-1}H^{(m-k)}v^{(k-1)}\Big|_0^\infty + (-1)^m \int_0^\infty H(x)v^{(m)}(x)\,dx, \tag{5.3}$$

where the function $H(x)$ is defined by the equalities $H^{(m)} = f\rho$, $H^{(m-k)}(\infty) = 0$, $k = 1,\dots,m$, and therefore satisfies relation (5.2). $\square$

In order not to complicate the proofs, we make the following remark.

**Remark 5.1.** The proof is not complete due to the need to check the convergence of the improper integral. However, Lemma 5.3 is used below in Lemma 5.5. For the purposes of Lemma 5.5, the version of Lemma 5.3 for a finite interval suffices, so discussion of the convergence of the integral in this sense is not relevant.

Further, from Lemma 5.5, we can conclude that the integral on the right-hand side of (5.1) converges. Since $(-1)^m H(x) = p(x)u^{(m)}$, we have

$$\left(\int_I H(x)v^{(m)}\,dx\right)^2 \le \int_I p(x)(u^{(m)})^2\,dx \int_I p(x)(v^{(m)})^2\,dx = [u,u]\,[v,v].$$

**Lemma 5.4.** $\overline{T(W)} = L_2(I,\rho)$.

*Proof.* Suppose $\overline{T(W)} \neq L_2(I,\rho)$. Then there exists $h \in L_2(I,\rho)$, orthogonal to $\overline{T(W)}$, i.e.,

$$(\forall\, u \in W) \quad \int\limits_I u(x)h(x)\rho(x)\,dx = 0.$$

Due to (5.1),

$$\int\limits_0^\infty H(x)u^{(m)}(x)\,dx = 0 \ \text{ for any } \ u \in W,$$

where $H^{(m)} = h\rho$. Since $W$ maps isomorphically onto $L_2(I,p)$, this implies $H = 0$. Therefore, $h = 0$. □

**Lemma 5.5.** *The boundary value problem* (1.2), (1.7), (1.9) *is a representation of the operator* $\mathcal{L} = (T^*)^{-1}$.

Of course, in (1.2), the symbol $\mathcal{L}$ should only be understood as a definition for the expression on the right-hand side.

*Proof.* If $\mathcal{L}u = f$, then $u = T^*f$ and $[u,v] = (f, Tv)$ for any $v \in W$. From (5.1),

$$\int\limits_I p(x)u^{(m)}v^{(m)}\,dx = (-1)^m \int\limits_I Hv^{(m)}\,dx.$$

Because of the arbitrariness of $v^{(m)} \in L_2(I,p)$, $p(x)u^{(m)} = (-1)^m H(x)$ (almost everywhere in $I$, but we can assume that everywhere). That's why

$$(-1)^m (p(x)u^{(m)})^{(m)} = f\rho.$$

From this we get representation (1.1). Conditions (1.9) follow from the properties of the function $H$ (Lemma 5.3). □

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