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TO THE QUESTION OF SAVING THE UNIQUE SOLVABILITY OF THE GENERAL LINEAR BOUNDARY VALUE PROBLEM FOR A CLASS OF FUNCTIONAL DIFFERENTIAL SYSTEMS WITH DISCRETE MEMORY

Dedicated to the memory of Professor N.V. Azbelev
on the occasion of his 100th birthday anniversary

Abstract. A class of linear functional differential systems with continuous and discrete times and discrete memory is considered. The paper gives an explicit description of a family of uniquely solvable linear boundary value problems as a neighborhood of a fixed uniquely solvable boundary value problem. The description is based on an explicit representation of the principal components to the general solution representation such as the fundamental matrix and the Cauchy operator. In the study of the problems outside the class under consideration, the systems with discrete memory can be employed as a model or approximating ones. This can be useful as applied to systems with aftereffect under studying rough properties that hold under small disturbances of the parameters.

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რეზიუმე. განიხილება წრფივი ფუნქციონალურ-დიფერენციალური სისტემები უწყვეტი და დისკრეტული დროით და დისკრეტული მეხსიერებით. ნაშრომში მოცემულია ცალსახად ამოხსნადი წრფივი სასაზღვრო ამოცანების ოჯახის, როგორც ფიქსირებული ცალსახად ამოხსნადი სასაზღვრო ამოცანის მიდამოს ცხადი აღწერა. ეს აღწერა ეფუძნება ზოგადი ამონახსნის წარმოდგენის – მთავარი კომპონენტების, როგორებიცაა ფუნდამენტური მატრიცა და კოშის ოპერატორი, ცხად წარმოდგენას. ამოცანების შესწავლისას, რომლებიც არ შედის განსახილველ კლასში, შეიძლება გამოყენებულ იქნას სისტემები დისკრეტული მეხსიერებით როგორც მოდელი ან მისი აპროქსიმაციები. ეს მიდგომა შეიძლება გამოსადეგი იყოს უხეში თვისებების შესწავლისას სისტემებისთვის შემდგომი ეფექტით, რომლებიც ნარჩუნდება პარამეტრების მცირე დარღვევისას.

1 Introduction

Actual applied problems arising in various fields of applications constantly give rise to new types of mathematical models with ordinary derivatives. Here, we consider a class of systems containing simultaneously phase variables and equations with both continuous and discrete time; such models and corresponding systems are often called hybrid. An interest of researchers in various classes of hybrid models has been steadily increasing over the last 15 years. We just mention here the well-known works of Russian and foreign authors, see [1, 2, 6, 7, 9, 14, 15].

Here, we continue the study of linear continuous–discrete systems with aftereffect in the frame of an approach developed in the previous works [8, 11, 12]. For a class of linear systems with continuous and discrete times and discrete memory, we consider the general linear boundary value problem and propose an explicit description of a family of uniquely solvable linear boundary value problems as a neighborhood of a fixed uniquely solvable boundary value problem. The description is based on an explicit representation of the principal components to the general solution representation such as the fundamental matrix and the Cauchy operator. In the study of the problems outside the class under consideration, the systems with discrete memory can be employed as a model or approximating ones. This can be useful as applied to systems with aftereffect under studying rough properties that hold under small disturbances of the parameters.

The system under consideration includes simultaneously two types of variables, namely, the state variables depending on the continuous time, $t \in [0,T]$, and the variables with dependence on the discrete time, $t \in \{0, t_1, \ldots, t_\mu\}$. A special feature of the systems is that the memory of the system operators is discrete and located at the points t_j, strictly preceding the current instant t ($t_j < t$). Some applications of such systems in economic dynamics problems are presented in [16].

The proposed approach uses significantly the ideas and results of the theory of the Abstract Functional Differential Equation (AFDE) constructed by N. V. Azbelev and L. F. Rakhmatullina and systematically described in [3, 4]. AFDE is the equation $\mathcal{L}y = f$ with the operator $\mathcal{L}$ acting from the Banach space $\mathbf{D}$ isomorphic to the direct product $\mathbf{B} \times R^n$, where $\mathbf{B}$ is a Banach space. The main idea of the applications of the AFDE theory is the appropriate choice of the $\mathbf{D}$ space when considering specific new problems. This choice allows, while remaining within the framework of the general theory, to apply standard schemes and statements when considering the tasks that previously required an individual approach and special design. This approach has demonstrated its effectiveness in the study of wide classes of actual problems (see [4]).

2 The system description

Let us introduce the Banach spaces where the operators and the equations are considered and describe the main subject. Fix a segment $[0,T] \subset R$. We denote by $L^n = L^n[0,T]$ the space of summable functions $v : [0,T] \to R^n$ with the norm $\|v\|_{L^n} = \int\limits_0^T |v(s)|_n \, ds$, where $|\cdot|_n$ (or $|\cdot|$ for short if the dimension value is clear) stands for the norm in R^n; $AC^n = AC^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ with the norm $\|x\|_{AC^n} = |x(0)|_n + \|\dot{x}\|_{L^n}$. Next, we fix the set $J = \{t_0, t_1, \ldots, t_\mu\}$, $0 = t_0 < t_1 < \cdots < t_\mu = T$. Let $FD^\nu = FD^\nu\{t_0, t_1, \ldots, t_\mu\}$ be the space of functions $z : J \to R^\nu$ under the norm

$$\|z\|_{FD^\nu} = \sum_{i=0}^{\mu} |z(t_i)|_\nu.$$

In the sequel, for any pair of Banach spaces X and Y equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, we define the norm in the product $X \times Y$ by the equality $\|\cdot\|_{X\times Y} = \|\cdot\|_X + \|\cdot\|_Y$.

We consider the system

$$\dot{x}(t) = \sum_{j:\, t_j<t} A_j(t)x(t_j) + \sum_{j:\, t_j<t} B_j(t)z(t_j) + f(t), \quad t \in [0,T], \tag{2.1}$$

$$z(t_i) = \sum_{j<i} D_{ij}x(t_j) + \sum_{j<i} H_{ij}z(t_j) + g(t_i), \quad i = 1, \ldots, \mu. \tag{2.2}$$

Here, the columns of $(n \times n)$-matrices A_j and $(n \times \nu)$-matrices B_j belong to the space L^n, $f \in L^n$; $(\nu \times n)$-matrices D_{ij} and $(\nu \times \nu)$-matrices H_{ij} have constant elements, $g : J \to R^\nu$.

System (2.1), (2.2) is a special case of the general continuous-discrete system considered in detail in [11]. Theorem 1 [11] gives the representation of the general solution in the form

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{Y} \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} + \mathcal{C} \begin{pmatrix} f \\ g \end{pmatrix},$$

where $z = \operatorname{col}(z(t_1), \dots, z(t_\mu))$, $g = \operatorname{col}(g(t_1), \dots, g(t_\mu))$,

$$\mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{pmatrix}$$

is the fundamental matrix,

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix}$$

is the Cauchy operator. Here, the block components $\mathcal{Y}_{ij}$, $\mathcal{C}_{ij}$, $i, j = 1, 2$, are the operators acting as follows:

$$\mathcal{Y}_{11} : R^n \to AC^n,\ \ \mathcal{Y}_{12} : R^\nu \to AC^n,\ \ \mathcal{Y}_{21} : R^n \to R^{\nu\mu},\ \ \mathcal{Y}_{22} : R^\nu \to R^{\nu\mu},$$
$$\mathcal{C}_{11} : L^n \to AC^n,\ \ \mathcal{C}_{12} : R^{\nu\mu} \to AC^n,\ \ \mathcal{C}_{21} : L^n \to R^{\nu\mu},\ \ \mathcal{C}_{22} : R^{\nu\mu} \to R^{\nu\mu}.$$

In the sequel, we will use the explicit representation of $\mathcal{Y}$ and $\mathcal{C}$ obtained in [12] in terms of the system matrix parameters. To give the representation, we recall the following notation from [12]:

$$\mathcal{A}_{ij} = \int_0^{t_i} A_j(s)\chi_j(s)\,ds, \quad \mathcal{B}_{ij} = \int_0^{t_i} B_j(s)\chi_j(s)\,ds,$$

where $\chi_j(s)$ is the characteristic function of $(t_j, T]$;

$$\mathcal{A}_j(t) = \int_0^{t} A_j(s)\chi_j(s)\,ds, \quad \mathcal{B}_j(t) = \int_0^{t} B_j(s)\chi_j(s)\,ds,$$

$$\mathcal{D}_{ij} = D_{ij} \ \text{ if } \ j < i, \quad \mathcal{D}_{ij} = 0 \ \text{ otherwise}, \quad \mathcal{H}_{ij} = H_{ij} \ \text{ if } \ j < i, \quad \mathcal{H}_{ij} = 0 \ \text{ otherwise},$$
$$\mathcal{A}_0 = (\mathcal{A}_{10}, \dots, \mathcal{A}_{\mu 0})', \quad \mathcal{B}^0 = (\mathcal{B}_{10}, \dots, \mathcal{B}_{\mu 0})', \quad \mathcal{D}_0 = (\mathcal{D}_{10}, \dots, \mathcal{D}_{\mu 0})', \quad \mathcal{H}_0 = (\mathcal{H}_{10}, \dots, \mathcal{H}_{\mu 0})'$$

(here and below, $(\,\cdot\,)'$ stands for transposition);

$$\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{B} = (\mathcal{B}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{D} = (\mathcal{D}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{H} = (\mathcal{H}_{ij})_{i,j=1,\dots,\mu},$$
$$\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{B} = (\mathcal{B}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{D} = (\mathcal{D}_{ij})_{i,j=1,\dots,\mu}, \quad \mathcal{H} = (\mathcal{H}_{ij})_{i,j=1,\dots,\mu},$$
$$P = \begin{pmatrix} \mathcal{E} + \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{D}_0 & \mathcal{H}_0 \end{pmatrix},$$

where $(n\mu \times n)$-matrix $\mathcal{E}$ is defined by the equality $\mathcal{E} = (E_n, \dots, E_n)'$;

$$\mathbf{A} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{D} & \mathcal{H} \end{pmatrix}, \quad \mathbf{Q} = (\mathbf{E} - \mathbf{A})^{-1}. \tag{2.3}$$

Let us denote by $\mathbf{Y}$ the product $\mathbf{QP}$ and employ for $\mathbf{Y}$ and $\mathbf{Q}$ the following block forms:

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

with $(n\mu \times n)$-matrix Y_{11}, $(n\mu \times \nu)$-matrix Y_{12}, $(\nu\mu \times n)$-matrix Y_{21}, $(\nu\mu \times \nu)$-matrix Y_{22}.

The blocks of the fundamental matrices $\mathcal{Y}$ are defined by the equalities

$$\mathcal{Y}_{11} = E_n + \mathcal{A}_0(t) + \sum_{j=1}^{\mu} \mathcal{A}_j(t) Y_{11}^j + \sum_{j=1}^{\mu} \mathcal{B}_j(t) Y_{21}^j, \tag{2.4}$$

$$\mathcal{Y}_{12} = \mathcal{B}_0(t) + \sum_{j=1}^{\mu} \mathcal{A}_j(t) Y_{12}^j + \sum_{j=1}^{\mu} \mathcal{B}_j(t) Y_{22}^j, \tag{2.5}$$

$$\mathcal{Y}_{21} = Y_{21}, \quad \mathcal{Y}_{22} = Y_{22}, \tag{2.6}$$

where Y_{k1}^j is the j-th group of n-rows to Y_{k1}, $k = 1,2$, and Y_{k2}^j is the j-th group of ν-rows to Y_{k2}, $k = 1,2$. For more definiteness, we note that the following relationships take place for the solutions to the homogeneous system (2.1), (2.2) ($f = 0$, $g = 0$):

$$x(t_j) = Y_{11}^j x(0) + Y_{12}^j z(0), \;\; j = 1, \dots, \mu,$$
$$z(t_j) = Y_{21}^j x(0) + Y_{22}^j z(0), \;\; j = 1, \dots, \mu.$$

As for the Cauchy operator $\mathcal{C}$, its blocks are defined by the following equalities:

$$(\mathcal{C}_{11} f)(t) = \int_0^t \bigg\{ E_n + \sum_{k=1}^{\mu} \bigg[\int_s^t \sum_{j=1}^{\mu} (A_j(\tau) Q_{11}^{jk} + B_j(\tau) Q_{21}^{jk}) \chi_{(t_j,T]}(\tau)\, d\tau \bigg] \chi_{[0,t_k]}(s) \bigg\} f(s)\, ds,$$

where Q_{11}^{jk} is the k-th group of n-columns to Q_{11}^j, Q_{21}^{jk} is the k-th group of n-columns to Q_{21}^j (the expression inside of $\{\cdots\}$ is the Cauchy matrix $C_{11}(t,s)$);

$$(\mathcal{C}_{12} g)(t) = \int_0^t \bigg\{ \sum_{k=1}^{\mu} \Big[\sum_{j=1}^{\mu} (A_j(s) Q_{12}^{jk} + B_j(s) Q_{22}^{jk}) \chi_{(t_j,T]}(s) \Big] g(t_k) \bigg\}\, ds,$$

where Q_{12}^{jk} is the k-th group of ν-columns to Q_{12}^j, Q_{22}^{jk} is the k-th group of ν-columns to Q_{22}^j ($Q_{i\,\ell}^j$ are defined in a perfect analogy with $Y_{i\ell}^j$, $i,\ell = 1,2$);

$$\mathcal{C}_{21} f = Q_{21} \bigg(\int_0^{t_1} f(s)\, ds, \dots, \int_0^{t_\mu} f(s)\, ds \bigg)', \;\; \mathcal{C}_{22} g = Q_{22} \big(g(t_1), \dots, g(t_\mu) \big)'.$$

It should be noted that all components of $\mathcal{Y}$ and $\mathcal{C}$ are expressed explicitly in the terms of coefficients of the system under consideration with the use of the matrix $\mathbf{Q} = (\mathbf{E} - \mathbf{A})^{-1}$.

3 Formulation of the problem

We consider the general linear boundary value problem (BVP) for system (2.1), (2.2) with the general linear boundary conditions

$$\ell \begin{pmatrix} x \\ z \end{pmatrix} = \alpha, \tag{3.1}$$

where $\ell : AC^n \times FD^\nu \to R^{n+\nu}$ is a linear bounded vector-functional, $\alpha \in R^{n+\nu}$. Let us recall the general representation of such a vector-functional

$$\ell \begin{pmatrix} x \\ z \end{pmatrix} = \Psi x(0) + \int_0^T \Phi(s) \dot{x}(s)\, ds + \Gamma_0 z(0) + \sum_{j=1}^{\mu} \Gamma_j z(t_j),$$

where Ψ is a constant $((n+\nu) \times n)$-matrix, Γ_j, $j = 0, \dots, \mu$, are constant $((n+\nu) \times \nu)$-matrices, the elements of a $((n+\nu) \times n)$-matrix Φ are measurable and bounded in essence on $[0,T]$.

To formulate the necessary and sufficient condition of the unique solvability to BVP (2.1), (2.2), (3.1), we introduce the extended fundamental matrix $\widetilde{\mathcal{Y}}$:

$$\widetilde{\mathcal{Y}} = \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ 0 & E_\nu \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{pmatrix}.$$

Next, define the $((n+\nu)\times(n+\nu))$-matrix

$$M = \ell\widetilde{\mathcal{Y}} = (\ell\widetilde{y}_1, \dots, \ell\widetilde{y}_{n+\nu}),$$

where $\widetilde{y}_k$, $k = 1, \dots, n+\nu$, is the k-th column of $\widetilde{\mathcal{Y}}$.

Note that due to (2.4), (2.5) and (2.6), we have the following explicit representation of M:

$$M = \biggl(\Psi + \sum_{j=0}^{\mu} \int_{t_j}^{T} \Phi(t)\bigl[A_j(t)Y_{11}^j + B_j(t)Y_{21}^j\bigr]\,dt + \sum_{i=0}^{\mu} \Gamma_i Y_{21}^i,$$
$$\sum_{j=0}^{\mu} \int_{t_j}^{T} \Phi(t)\bigl[A_j(t)Y_{12}^j + B_j(t)Y_{22}^j\bigr]\,dt + \sum_{i=0}^{\mu} \Gamma_i Y_{22}^i\biggr),$$

BVP (2.1), (2.2), (3.1) is uniquely solvable for any $f \in L^n$, $g \in FD^\nu$ if and only if the matrix M is invertible [8, Theorem 1].

To give a description to a family of systems of form (2.1), (2.2), we introduce a more convenient notation. Denote by $\mathcal{T}_{11}$ the operator acting on x in the right-hand side of (2.1) and by $\mathcal{T}_{12}$ the operator acting on z in the right-hand side of (2.1). For subsystem (2.2), operators $\mathcal{T}_{21}$ and $\mathcal{T}_{22}$ are introduced in exactly the same way. Further, define the operator $\mathcal{L}$ by the equality

$$\mathcal{L}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} \dot{x} \\ z \end{pmatrix} - \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}.$$

Thus system (2.1),(2.2) can be written in a short form

$$\mathcal{L}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

In the sequel, we will write $\mathcal{L}^0$, referring to a system with all coefficients provided with the upper index 0 ($A_j^0(t)$, $B_j^0(t)$, and so on) and assume all such parameters to be fixed. As for the parameters to the operator $\mathcal{L}$, they are considered as arbitrary ones. Everything that has been said about indexing using the superscript, applies also to the fundamental matrices $\mathcal{Y}$, $\mathcal{Y}^0$, the Cauchy operators $\mathcal{C}$, $\mathcal{C}^0$ and to matrices M, M^0.

Assume that the BVP

$$\mathcal{L}^0\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \ell\begin{pmatrix} x \\ z \end{pmatrix} = \alpha$$

is uniquely solvable, i.e., M^0 is invertible. Our goal is to give a description of the set of operators $\mathcal{L}$ for which the property of the unique solvability is saved in a neighborhood of $\mathcal{L}^0$. Our consideration is based on the assertion that the inequality

$$\|\ell\widetilde{\mathcal{Y}} - \ell\widetilde{\mathcal{Y}}^0\| < \frac{1}{\|(M^0)^{-1}\|} \tag{3.2}$$

ensures the invertibility of $\ell\widetilde{\mathcal{Y}}$ (see, e.g., [10, Theorem 3.6.3]) and, therefore, the unique solvability of the BVP

$$\mathcal{L}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \ell\begin{pmatrix} x \\ z \end{pmatrix} = \alpha. \tag{3.3}$$

4 The main result

To give a description of the neighborhood mentioned above, we introduce some characteristics expressed in terms of parameters of the problem under consideration:

$$\theta_{11} = \sum_{j=0}^{\mu} \|A_j\|, \ \text{ where } \|A_j\| = \int_{t_j}^{T} \|A_j(t)\|_{R^n \to R^n}\, dt,$$

$$\theta_{12} = \max\big\{\|B_j\|,\ j = 0, \dots, \mu\big\}, \ \text{ where } \|B_j\| = \int_{t_j}^{T} \|B_j(t)\|_{R^\nu \to R^n}\, dt,$$

$$\theta_{21} = \sum_{j=0}^{\mu} \sum_{i=j+1}^{\mu} \|D_{ij}\|, \ \text{ where } \|D_{ij}\| = \|D_{ij}\|_{R^n \to R^\nu},$$

$$\theta_{22} = \max\Big\{\sum_{i=j+1}^{\mu} \|H_{ij}\|,\ j = 0, \dots, \mu\Big\}, \ \text{ where } \|H_{ij}\| = \|H_{ij}\|_{R^\nu \to R^\nu},$$

$$\Theta = \max\big\{\theta_{11} + \theta_{21}, \theta_{12} + \theta_{22}\big\}.$$

Next, introduce the characteristics for the difference $\mathcal{L} - \mathcal{L}^0$. Define

$$\theta_{11}^d = \sum_{j=0}^{\mu} \|A_j - A_j^0\|, \quad \theta_{12}^d = \max\big\{\|B_j - B_j^0\|,\ j = 0, \dots, \mu\big\},$$

$$\theta_{21}^d = \sum_{j=0}^{\mu} \sum_{i=j+1}^{\mu} \|D_{ij} - D_{ij}^0\|, \quad \theta_{22}^d = \max\Big\{\sum_{i=j+1}^{\mu} \|H_{ij} - H_{ij}^0\|,\ j = 0, \dots, \mu\Big\},$$

$$\Theta^d = \max\big\{\theta_{11}^d + \theta_{21}^d, \theta_{12}^d + \theta_{22}^d\big\}.$$

Now, we get

$$\|\mathcal{L} - \mathcal{L}^0\| \le \Theta^d.$$

Next, put

$$c_{11}^0 = 1 + \sum_{k=1}^{\mu} \sum_{j=1}^{\mu} \int_{t_j}^{T} \big\|A_j^0(t)(Q_{11}^0)^{jk} + B_j^0(t)(Q_{21}^0)^{jk}\big\|\, dt,$$

$$c_{12}^0 = \max\Big\{\sum_{k=1}^{\mu} \int_{t_j}^{T} \big\|A_j^0(t)(Q_{12}^0)^{jk} + B_j^0(t)(Q_{22}^0)^{jk}\big\|\, dt,\ j = 1, \dots, \mu\Big\},$$

$$c_{21}^0 = \sum_{j=1}^{\mu} \sum_{k=1}^{\mu} \|(Q_{21}^0)^{jk}\|,$$

$$c_{22}^0 = \max\Big\{\sum_{j=1}^{\mu} \|(Q_{21}^0)^{jk}\|,\ k = 1, \dots, \mu\Big\}$$

and

$$C^0 = \max\big\{c_{11}^0 + c_{21}^0, c_{12}^0 + c_{22}^0\big\}.$$

Finally, let us define the constants λ and m by the inequalities

$$\lambda \ge \|\ell\|, \ \ m \ge \|(M^0)^{-1}\|.$$

Theorem 4.1. *Let the inequality*

$$\Theta^d < \frac{1}{C^0(\lambda m C^0(\Theta^0 + 1) + \lambda m + 1)} \tag{4.1}$$

be fulfilled. Then any BVP (3.3) *is uniquely solvable.*

Proof. First, we obtain an estimate of $\|\widetilde{\mathcal{Y}}-\widetilde{\mathcal{Y}}^0\|$ in terms of $\|\mathcal{L}-\mathcal{L}^0\|$. Here and in the sequel, we omit subscripts in the notation of the norms, the choice of spaces is usually clear. For instance, $\|\widetilde{\mathcal{Y}}\|$ means $\|\widetilde{\mathcal{Y}}\|_{AC^{n\times n}\times FD^{\nu\times\nu}}$.

Note that $\|\widetilde{\mathcal{Y}}-\widetilde{\mathcal{Y}}^0\|=\|\mathcal{Y}-\mathcal{Y}^0\|$, so we operate with $\mathcal{Y}$ and $\mathcal{Y}^0$. By definition, $\mathcal{Y}$ is the solution to the Cauchy problem

$$\mathcal{L}\mathcal{Y}=0,\ \ \mathcal{Y}(0)=\mathbb{E},$$

where $\mathbb{E}=E_{n+\nu}$.

Introducing $\mathcal{U}=\mathcal{Y}-\mathbb{E}$, we observe that

$$\mathcal{L}\mathcal{U}=-\mathcal{L}\mathbb{E},\ \ \mathcal{U}(0)=0. \tag{4.2}$$

Note that $\mathcal{U}-\mathcal{U}^0=\mathcal{Y}-\mathcal{Y}^0$. Let Λ be the narrowing of $\mathcal{L}$ onto $AC_0^n\times FD_0^\nu$, the subspace of $AC^n\times FD^\nu$ with zero element values at point 0. Hence (4.2) can be written as the equation

$$\Lambda\mathcal{U}=-\mathcal{L}\mathbb{E},$$

and $\Lambda^{-1}=\mathcal{C}$. Next, we denote $F=-\mathcal{L}\mathbb{E}$, $F^0=-\mathcal{L}^0\mathbb{E}$, and get $\mathcal{U}=\mathcal{C}F$ and $\mathcal{U}^0=\mathcal{C}^0F^0$. Further,

$$\mathcal{U}-\mathcal{U}^0=\mathcal{C}F-\mathcal{C}^0F^0=\mathcal{C}F-\mathcal{C}^0F^0+\mathcal{C}^0F-\mathcal{C}^0F=(\mathcal{C}-\mathcal{C}^0)F+\mathcal{C}^0(F-F^0).$$

By virtue of Theorem 3.6.3 from [10], we have

$$\|\mathcal{C}-\mathcal{C}^0\|\le\frac{\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|^2}{1-\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|}$$

under the condition

$$\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|<1.$$

Now, we have the estimation

$$\|\mathcal{U}-\mathcal{U}^0\|\le\frac{\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|}{1-\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|}\cdot\|\mathcal{C}^0\|\cdot\|F\|+\|\mathcal{C}^0\|\cdot\|F-F^0\|,$$

or, taking into account the inequalities

$$\|F\|\le\|F^0\|+\|F-F^0\|,\ \ \|F-F^0\|\le\|\mathcal{L}-\mathcal{L}^0\|\cdot\|\mathbb{E}\|,$$

we get

$$\|\mathcal{U}-\mathcal{U}^0\|\le\frac{\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|}{1-\|\Lambda-\Lambda^0\|\cdot\|\mathcal{C}^0\|}\cdot\|\mathcal{C}^0\|\cdot\big(\|F^0\|+\|F-F^0\|\big)+\|\mathcal{C}^0\|\cdot\|\mathcal{L}-\mathcal{L}^0\|\cdot\|\mathbb{E}\|.$$

From this it follows that under the condition

$$\|\mathcal{C}^0\|\cdot\max\big\{\|\Lambda-\Lambda^0\|,\|\mathcal{L}-\mathcal{L}^0\|\big\}\le\Delta<1$$

the estimate

$$\|\mathcal{U}-\mathcal{U}^0\|\le\frac{\Delta(\|\mathcal{C}^0\|\cdot\|F^0\|+\|\mathbb{E}\|)}{1-\Delta}$$

holds, or

$$\|\ell\widetilde{\mathcal{Y}}-\ell\widetilde{\mathcal{Y}}^0\|\le\frac{\lambda\Delta(\|\mathcal{C}^0\|\cdot\|F^0\|+\|\mathbb{E}\|)}{1-\Delta}\le\frac{\lambda\Delta(C^0\cdot(\Theta^0+1)+1)}{1-\Delta}\,.$$

Here, the latter inequality follows from the estimates $\|\mathcal{C}^0\|\le C^0$ and $\|F^0\|\le 1+\,\Theta^0$, since

$$\|\mathcal{T}_{11}\|\le\theta_{11},\quad\|\mathcal{T}_{12}\|\le\theta_{12},\quad\|\mathcal{T}_{21}\|\le\theta_{21},\quad\|\mathcal{T}_{22}\|\le\theta_{22}$$

for $\mathcal{T}=(\mathcal{T}_{ik})_{i,k=1,2}$ we have $\|\mathcal{T}\|\le\Theta$ and $\|\mathcal{L}\|\le 1+\Theta$. Let us recall the use of the upper index 0 to refer to a fixed system and its parameters. In doing so, we get

$$\|\mathcal{L}^0\|\le 1+\Theta^0,$$

where all θ_{ik}^0 correspond to the fixed system with the coefficients indexed by 0.

Solving the inequality

$$\frac{\Delta(C^0 \cdot (\Theta^0 + 1) + 1)}{1 - \Delta} < \frac{1}{\lambda m}$$

with respect to Δ and taking into account that $\Theta^d \geq \max\{\|\Lambda - \Lambda^0\|, \|\mathcal{L} - \mathcal{L}^0\|\}$, we find that inequality (4.1) ensures inequality (3.2). $\square$

5 An example

Consider the system

$$\begin{aligned}
\dot{x}(t) &= 0.5\, x(0) + 0.5 \sin(t)\chi_{(1,4]}(t)x(1) + 0.1 \exp(-0.1t)\chi_{(2,4]}(t)x(2) + 0.1t^2\chi_{(3,4]}(t)x(3) \\
&\quad + 0.3tz(0) + 0.2\chi_{(1,4]}(t)z(1) + 0.1t^2\chi_{(2,4]}(t)z(2) + 0.15\chi_{(3,4]}(t)z(3) + f(t), \quad t \in [0,4], \quad (5.1) \\
z(i) &= 0.4x(0) + 0.5\chi_{(1,4]}(i)x(1) + 0.4\chi_{(2,4]}(i)x(2) + 0.3\chi_{(3,4]}(i)x(3) \\
&\quad + 0.2iz(0) + 0.2\chi_{(1,4]}(i)z(1) + 0.3\chi_{(2,4]}(i)z(i) + 0.15\chi_{(3,4]}(i)z(3) + g(i), \quad i = 1, \dots, 4, \quad (5.2)
\end{aligned}$$

for which the fundamental matrix and the Cauchy operator are constructed in [12]. For the case of the BVP with the boundary conditions

$$x(0) = 0.2x(4), \quad z(0) = 0.2z(4),$$

by immediate calculations we obtain

$$\lambda = 1, \;\; m = 2, \Theta^0 = 6.3, \;\; C^0 = 14.$$

Thus the condition $\Theta^d \leq 0.00034$ ensures the unique solvability of the BVP for all systems in the 0.00034-neighborhood of system (5.1), (5.2).

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