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INCLUSION THEOREMS FOR THE SPACES $\mathcal{F}_\alpha$

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Abstract. We give a new proof of one known inclusion theorem for the space $\mathcal{F}_0$ that enables us to extend this theorem from the unit disc in $\mathbf{C}$ to the unit ball in $\mathbf{C}^n$, $n > 1$. We also improve an inclusion relation between Bergman spaces and the spaces $\mathcal{F}_\alpha$, $\alpha > 0$.

1. Introduction

Let Δ denote the unit disc in $\mathbf{C}$ with boundary T. We denote by m the Lebesgue measure on Δ and by σ the normalized Lebesgue measure on T.

Let $\mathcal{M}$ denote the set of complex valued Borel measures on T. For each $\alpha \geqslant 0$ let $\mathcal{F}_\alpha = \mathcal{F}_\alpha(\Delta)$ denote the family of functions on Δ having the property that there exists a measure $\mu \in \mathcal{M}$ such that

$$f(z) = \int_T K_\alpha(z\bar{\xi})\,d\mu(\xi), \qquad z \in \Delta, \tag{1.1}$$

where for $\alpha > 0$, $K_\alpha(w) = (1-w)^{-\alpha}$, $w \in \Delta$, and $K_0(w) = 1 + \log\frac{1}{1-w}$, $w \in \Delta$. In (1.1) and throughout this paper each logarithm means the principal branch. The family $\mathcal{F}_\alpha$ is a Banach space with respect to the norm defined by $\|f\|_{\mathcal{F}_\alpha} = \inf\|\mu\|$, where μ varies over all measures in $\mathcal{M}$ for which (1.1) holds and where $\|\mu\|$ denotes the total variation norm of μ.

A function f holomorphic in Δ (abbreviated $f \in H(\Delta)$) is said to belong to the Hardy space H^p, $0 < p < \infty$, if $\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty$ and to the weighted Bergman space $A^{p,\alpha}$, $0 < p < \infty$, $\alpha > -1$, if

$$\|f\|^p_{A^{p,\alpha}} = \int_0^1 (1-r)^\alpha M_p^p(r,f)\,dr < \infty.$$

Here $M_p(r,f)$ is the $L^p(\{z \in \Delta : |z| = r\})$ "norm" of f.

Let $g(z) = \sum_{k=0}^\infty a_k z^k$ be holomorphic in Δ. We define the multiplier transformation $D^\beta g$ of g, where β is a real number, by

$$D^\beta g(z) = \sum_{k=0}^\infty (k+1)^\beta a_k z^k.$$

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A function $f \in H(\Delta)$ is said to belong to the space H^p_s if $\|D^s f\|_{H^p} < \infty$ and to the space $A^{p,\alpha}_s$ if $\|D^s f\|_{A^{p,\alpha}} < \infty$.

For an arc $I \subset T$ and a function $g \in H^1$ let $MO(g,I)$ denote the mean oscilation of g over I, i.e.

$$MO(g,I) = \frac{1}{\sigma(I)} \int_I \left| g(\xi) - \frac{1}{\sigma(I)} \int_I g(\eta)\, d\sigma(\eta) \right| d\sigma(\xi),$$

and let, as usual, $BMOA = BMOA(\Delta)$ be the space of functions $g \in H^1$ such that $\|g\|_{BMOA} = \sup MO(g,I) < \infty$, where the supremum runs over all arcs $I \subset T$.

We are now ready to state our first result

Theorem 1. $\mathcal{F}_0(\Delta) \subset BMOA(\Delta)$.

Theorem is known (see [5]). The argument given in [5] is limited to the one dimensional case. Our proof is different and may be easily extended to the corresponding spaces $\mathcal{F}_0(B^n)$ and $BMOA(B^n)$ of holomorphic functions on the unit ball B^n in $\mathbf{C}^n$, $n > 1$. Thus we have

Theorem 2. $\mathcal{F}_0(B^n) \subset BMOA(B^n)$.

In [6] it is shown that if $\alpha > 0$, then $A^{1,\alpha-1}_1 \subset \mathcal{F}_\alpha$ (Lemma 1 and Lemma 2, pp. 159–160). Since $A^{1,\alpha-1}_1 \subset H^1_{1-\alpha}$ and $H^1_{1-\alpha} \setminus A^{1,\alpha-1}_1$ is a non-empty set, the following theorem is an improvement of this inclusion.

Theorem 3. *Let* $\alpha > 0$. *Then* $H^1_{1-\alpha} \subset \mathcal{F}_\alpha$.

2. Proof of Theorem 1.

Let $f \in \mathcal{F}_0$. Then there exists $\mu \in \mathcal{M}$ such that

$$f(z) = f(0) + \int_T \log \frac{1}{1 - z\bar{\xi}}\, d\mu(\xi). \tag{2.1}$$

To show that $f \in BMOA$ it suffices to show that

$$\sup_{z\in\Delta} \int_\Delta \frac{1-|z|^2}{|1-z\bar{w}|^2} |f'(w)|^2 (1-|w|^2)\, dm(w) < \infty,$$

(see [4], p. 240). Using (2.1) we find that

$$\int_\Delta \frac{1-|z|^2}{|1-z\bar{w}|^2} |f'(w)|^2 (1-|w|^2)\, dm(w) \leqslant C \int_T d|\mu|(\xi) \int_\Delta \frac{(1-|z|^2)(1-|w|^2)\, dm(w)}{|1-z\bar{w}|^2 |1-\bar{w}\xi|^2}.$$

Since $|\mu|(T) < \infty$, it is sufficient to show that

$$\int_\Delta \frac{(1-|w|^2)\, dm(w)}{|1-z\bar{w}|^2 |1-\bar{w}\xi|^2} \leqslant \frac{C}{1-|z|^2}, \quad \text{for all } z \in \Delta \text{ and } \xi \in T. \tag{2.2}$$

In this note we follow the custom of using letters C, C_1, C_2, ..., to stand for positive constants which change their values from one occurence to another while remaining independent of the important variables.

Given $z \in \Delta$ and $\xi \in T$ we consider the following partition of Δ:

$$\begin{aligned}
\Omega_1 &= \{\, w \in \Delta : |1 - z\bar{w}| \leqslant \tfrac{1}{2}|1 - z\bar{\xi}|\,\},\\
\Omega_2 &= \{\, w \in \Delta : |1 - w\bar{\xi}| \leqslant \tfrac{1}{2}|1 - z\bar{\xi}|\,\},\\
\Omega_3 &= \{\, w \in \Delta : \tfrac{1}{2}|1 - z\bar{\xi}| < |1 - z\bar{w}| \leqslant |1 - w\bar{\xi}|\,\},\\
\Omega_4 &= \{\, w \in \Delta : \tfrac{1}{2}|1 - z\bar{\xi}| < |1 - w\bar{\xi}| \leqslant |1 - w\bar{z}|\,\}.
\end{aligned}$$

With this notation we have

$$\begin{aligned}
|1 - \bar{w}z| &\leqslant C_1|1 - z\bar{\xi}| \leqslant C_2|1 - \bar{w}\xi|, && w \in \Omega_1,\\
|1 - \bar{w}\xi| &\leqslant C_1|1 - z\bar{\xi}| \leqslant C_2|1 - z\bar{w}|, && w \in \Omega_2,\\
|1 - z\bar{\xi}| &\leqslant C_1|1 - z\bar{w}| \leqslant C_2|1 - \bar{w}\xi|, && w \in \Omega_3,\\
|1 - z\bar{\xi}| &\leqslant C_1|1 - \bar{w}\xi| \leqslant C_2|1 - \bar{w}z|, && w \in \Omega_4.
\end{aligned}$$

Using this we find

$$\begin{aligned}
\int_{\Omega_1} \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^2|1 - \xi\bar{w}|^2} &\leqslant C \int_\Delta \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^4}\\
&\leqslant \int_0^1 (1-\rho)\,d\rho \int_T \frac{|d\eta|}{|1 - z\rho\bar{\eta}|^4} \leqslant C\int_0^1 \frac{(1-\rho)\,d\rho}{(1-\rho|z|)^3} \leqslant \frac{C}{1-|z|}, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega_2} \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^2|1 - \xi\bar{w}|^2} &\leqslant C \int_{\Omega_2} \frac{(1-|w|^2)\,dm(w)}{(|1 - z\bar{\xi}| + |1 - w\bar{\xi}|)^2|1 - \bar{w}\xi|^2}\\
&\leqslant C \int_\Delta \frac{(1-|w|^2)\,dm(w)}{(|1 - z\bar{\xi}| + (1-|w|))^2|1 - \bar{w}\xi|^2} \leqslant C\int_0^1 \frac{d\rho}{(|1 - z\bar{\xi}| + 1 - \rho)^2}\\
&\leqslant \frac{C}{|1 - z\bar{\xi}|}\int_0^\infty \frac{dx}{(1+x)^2} \leqslant \frac{C}{|1 - z\bar{\xi}|} \leqslant \frac{C}{1-|z|^2}, \qquad (2.4)
\end{aligned}$$

$$\int_{\Omega_3} \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^2|1 - \xi\bar{w}|^2} \leqslant C \int_{\Omega_3} \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^4} \leqslant \frac{C}{1-|z|^2}, \qquad (2.5)$$

$$\begin{aligned}
\int_{\Omega_4} \frac{(1-|w|^2)\,dm(w)}{|1 - z\bar{w}|^2|1 - \xi\bar{w}|^2} &\leqslant C \int_{\Omega_4} \frac{(1-|w|^2)\,dm(w)}{(|1 - z\bar{\xi}| + |1 - \bar{w}\xi|)^2|1 - \bar{w}\xi|^2}\\
&\leqslant C \int_\Delta \frac{(1-|w|^2)\,dm(w)}{(|1 - z\bar{\xi}| + (1-|w|))^2|1 - \bar{w}\xi|^2} \leqslant \frac{C}{|1 - z\bar{\xi}|} \leqslant \frac{C}{1-|z|^2}. \qquad (2.6)
\end{aligned}$$

Now (2.2) follows from (2.3), (2.4), (2.5) and (2.6). This finishes the proof of Theorem 1. ∎

3. Proof of Theorem 3.

Let α be a positive real numbers. Define the function H_α by

$$H_\alpha(z) = \sum_{n=0}^{\infty} (n+1)^{\alpha-1} z^n, \qquad \text{for } z \in \Delta.$$

Let $\mathcal{H}_\alpha$ denote the family of functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \Delta$, having the property that there exists $\mu \in \mathcal{M}$ such that

$$a_n = (n+1)^{\alpha-1} \int_T \bar{\xi}^n \, d\mu(\xi), \qquad n = 0, 1, 2, \ldots \tag{3.1}$$

Let $\|f\|_{\mathcal{H}_\alpha} = \inf \|\mu\|$, where μ varies over all members of $\mathcal{M}$ for which (3.1) holds. Then $\mathcal{H}_\alpha$ is a Banach space.

For the proof of Theorem 3 the following lemma is needed.

LEMMA 3.1. *If $\alpha > 0$ then $f \in \mathcal{F}_\alpha$ if and only if $f \in \mathcal{H}_\alpha$. There is a positive constant C depending only on α such that if $f \in \mathcal{F}_\alpha$ then*

$$C^{-1}\|f\|_{\mathcal{F}_\alpha} \leqslant \|f\|_{\mathcal{H}_\alpha} \leqslant C\|f\|_{\mathcal{F}_\alpha}.$$

Proof. Suppose that $f \in \mathcal{F}_\alpha$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \Delta$. Then there exists $\mu \in \mathcal{M}$ such that

$$a_n = A_n(\alpha) \int_T \bar{\xi}^n \, d\mu(\xi), \qquad n = 0, 1, 2, \ldots, \tag{3.2}$$

where $A_n(\alpha) = \dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$, $n = 0, 1, 2, \ldots$. From this it follows that

$$A_n(\alpha) = (n+1)^{\alpha-1} \left(\frac{1}{\Gamma(\alpha)} + B_n(\alpha) \right), \qquad \text{for } n = 1, 2, \ldots, \tag{3.3}$$

and there is a positive constant $B(\alpha)$ such that $|B_n(\alpha)| \leqslant B(\alpha)/n$, for $n = 1, 2, \ldots$. For $n = 1, 2, \ldots$ let

$$c_n(\alpha) = B_n(\alpha) \int_T \bar{\xi}^n \, d\mu(\xi)$$

and define the function g by $g(z) = \sum_{n=1}^{\infty} c_n(\alpha) z^n$, for $z \in \Delta$. Since $|c_n(\alpha)| \leqslant |B_n(\alpha)| \, \|\mu\| \leqslant \frac{B(\alpha)}{n} \|\mu\|$, $g \in H^2$. Therefore $g \in \mathcal{F}_1$ (see [2]) and hence there exists $\nu \in \mathcal{M}$ such that

$$g(z) = \int_T \frac{d\nu(\xi)}{1 - z\bar{\xi}}.$$

This implies that

$$c_n(\alpha) = \int_T \bar{\xi}^n \, d\nu(\xi), \quad \text{for n=1,2, } \ldots$$

Thus

$$a_n = (n+1)^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}\int_T \bar{\xi}^n\,d\mu(\xi) + \int_T \bar{\xi}^n\,d\nu(\xi)\right), \quad \text{for } n = 1, 2, \ldots$$

Let $\lambda = \frac{1}{\Gamma(\alpha)}\mu + \nu + b\sigma$, where $b = \mu(T)(\frac{-1}{\Gamma(\alpha)} + 1) - \nu(T)$. Then

$$a_n = (n+1)^{\alpha-1}\int_T \bar{\xi}^n\,d\lambda(\xi), \qquad n = 0, 1, 2, \ldots$$

Since $\lambda \in \mathcal{M}$, $f \in \mathcal{H}_\alpha$.

The argument given above shows that

$$\|f\|_{\mathcal{H}_\alpha} \leqslant \frac{1}{\Gamma(\alpha)}\|\mu\| + \|\nu\| + |b| \leqslant C(\|\mu\| + \|g\|_{H^2}) \leqslant C\|\mu\|.$$

Here we have used again that $H^2 \subset H^1 \subset \mathcal{F}_1$. This inequality holds for every $\mu \in \mathcal{M}$ for which (3.2) holds. Hence $\|f\|_{\mathcal{H}_\alpha} \leqslant C\|f\|_{\mathcal{F}_\alpha}$.

The same argument shows that if $f \in \mathcal{H}_\alpha$ then $f \in \mathcal{F}_\alpha$ and $\|f\|_{\mathcal{F}_\alpha} \leqslant C\|f\|_{\mathcal{H}_\alpha}$. Instead of (3.3) it should use the relation

$$(n+1)^{\alpha-1} = A_n(\alpha)[\Gamma(\alpha) + D_n(\alpha)], \quad \text{for } n = 1, 2, \ldots, \tag{3.4}$$

and $|D_n(\alpha)| \leqslant \frac{D(\alpha)}{n}$, for some positive constant $D(\alpha)$. For (3.3) and (3.4), see [3] and [7]. ∎

Proof of Theorem 3. Let $f \in H^1_{1-\alpha}$ and $f(z) = \sum_{k=0}^n a_k z^k$, $z \in \Delta$. Then $D^{1-\alpha} f \in H^1$. Since $H^1 \in \mathcal{F}_1$ we have $D^{1-\alpha} \in \mathcal{F}_1$. Hence there exists a measure $\mu \in \mathcal{M}$ such that

$$(n+1)^{1-\alpha} a_n = \int_T \bar{\xi}^n\,d\mu(\xi), \qquad n = 0, 1, 2, \ldots,$$

or equivalently

$$a_n = (n+1)^{\alpha-1}\int_T \bar{\xi}^n\,d\mu(\xi), \qquad n = 0, 1, 2, \ldots$$

Therefore, $f \in \mathcal{H}_\alpha$ and by Lemma 3.1 we have $f \in \mathcal{F}_\alpha$. Thus, $H^1_{1-\alpha} \subset \mathcal{F}_\alpha$. ∎

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