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A NOTE ON POSITIVE AND POSITIVE DEFINITE COLOMBEAU'S GENERALIZED FUNCTIONS

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Abstract. The positivity and the positive definitness of Colombeau's generalized functions are investigated.

1. Introduction

The paper is devoted to positive and positive definite Colombeau's generalized functions. We extend definitions of these notions given in [5] in order to have the coherence with the positivity and positive definitness of distributions. More precisely, with new definitions a distribution f is positive, resp. positive definite, if and only if the corresponding Colombeau generalized function $Cd(f)$ is positive, resp. positive definite. Our investigations are motivated by the investigations of generalized stochastic processes [5].

There exists the so called simplified version of Colombeau's theory of generalized functions ([2], [3], [7]). Our paper shows the preferences of Colombeau's standard theory in which families $\mathcal{A}_q$, $q \in \mathbb{N}_0$ appear.

2. Basic notions

For the general theory of Colombeau's geenralized functions, we refer to [2], [3] and [7]. Stochastic processes in the framework of $\mathcal{G}$ are considered in [1], [5] and [6].

Let T be an open interval of $\mathbb{R}$ and $C_0^\infty(T)$ the space of complex valued functions defined on $\mathbb{R}$ with compact supports contained in T. Denote $\mathcal{A}_0(\mathbb{R}) = \{\, \varphi \in C_0^\infty(\mathbb{R});\ \int \varphi(x)\,dx = 1 \,\}$, and for $q \in \mathbb{N}_0$, $\mathcal{A}_q(\mathbb{R}) = \{\, \varphi \in \mathcal{A}_0;\ \int x^j \varphi(x)\,dx = 0,\ 0 < j < q \,\}$; $A_q(\mathbb{R}^n) = \{\, \phi \in C_0^\infty(\mathbb{R}^n);\ \phi(x_1, \dots, x_n) = \prod_{i=1}^n \varphi(x_i),\ \varphi \in \mathcal{A}_0(\mathbb{R}) \,\}$. Put $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, $\check{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, $\varepsilon > 0$. Further on, ε will be positive (usually small) real number. $h_\varepsilon = \mathcal{O}(\varepsilon^a)$ means that $h_\varepsilon/\varepsilon^a \le C$ as $\varepsilon \to 0$. Here, we use spaces $A_q(\mathbb{R}^2)$ and denote $\phi(x, y) = \varphi(x)\varphi(y)$.

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The basic space $\mathcal{E}(T)$ consists of functions $g\colon \mathcal{A}_0(\mathbb{R}) \to C^\infty(T)$. It is an algebra with multiplication. More important is its subalgebra of moderate elements $\mathcal{E}_M(T)$, where

$$\mathcal{E}_M(T) = \{\, g(\varphi,x) \in \mathcal{E}(T) : \ (\forall K \Subset T)\,(\forall \alpha \in \mathbb{N}_0^n)\,(\exists N \in \mathbb{N}_0)\\ (\forall \varphi \in \mathcal{A}_N(\mathbb{R})) \quad (\sup\{|\partial^\alpha g(\varphi_\varepsilon,x)|, \ \ x \in K\} = \mathcal{O}(\varepsilon^{-N}))\,\}.$$

We use notation $g(\varphi,x)$ or $g(\varphi_\varepsilon,x)$ for elements of $\mathcal{E}_M(T)$. Denote by Γ the set of sequences $\{a_q\}$ with positive elements which strictly increase to infinity. Then the set of null elements $\mathcal{N}(T)$ in $\mathcal{E}(T)$ is defined as follows.

$$\mathcal{N}(T) = \{\, g(\varphi,x) \in \mathcal{E}(T) : \ (\forall K \Subset T)\,(\forall \alpha \in \mathbb{N}_0^n)\,(\exists N \in \mathbb{N}_0)\,(\exists \{a_q\} \in \Gamma)\\ (\forall q \geq N)\,(\forall \varphi \in \mathcal{A}_q(\mathbb{R})) \ \ (\sup\{|\partial^\alpha g(\varphi_\varepsilon,x)|, \ x \in K\} = \mathcal{O}(\varepsilon^{a_q - N}))\,\}.$$

The quotient space

$$\mathcal{G}(T) = \mathcal{E}_M(T)/\mathcal{N}(T)$$

is the space of Colombeau's generalized functions. An element $G = [g(\varphi_\varepsilon,x)] \in \mathcal{G}(T)$ is a class of equivalence and is represented by any of its elements. The pointwise product, the addition and the derivation in $\mathcal{G}(T)$ are naturally defined by using representatives.

The space of generalized complex numbers is defined as $\bar{\mathbb{C}} = \mathcal{E}_c/\mathcal{N}_c$, where

$$\mathcal{E}_c = \{\, z : \mathcal{A}_0(\mathbf{R}) \to \mathbb{C} : \ (\exists N \in \mathbb{N})\ (\forall \varphi \in \mathcal{A}_N(\mathbb{R}))(|z(\varphi_\varepsilon)| = \mathcal{O}(\varepsilon^{-N}))\,\}.$$

$$\mathcal{N}_c = \{\, z \in \mathcal{E}_c : (\exists N \in \mathbb{N})(\exists \{a_q\} \in \Gamma)(\forall \varphi \in \mathcal{A}_q(\mathbb{R}))(q \geq N)\ (|z(\varphi_\varepsilon)| = \mathcal{O}(\varepsilon^{a_q - N}))\,\}.$$

The space of generalized real numbers $\bar{\mathbb{R}}$ is defined in appropriate way ($z\colon \mathcal{A}_0(\mathbb{R}) \to \mathbb{R}$). It is a subspace of $\bar{\mathbb{C}}$.

The mapping $Cd\colon \mathcal{D}'(T) \to \mathcal{G}(T)$, the canonical imbedding of Schwartz's distributions into $\mathcal{G}(T)$, is defined in the following way. Denote by κ_ε a sequence of smooth functions on $C^\infty(T)$ suct that $\kappa_\varepsilon \geq 0$, $\kappa_\varepsilon(x) = 0$ for $x \in \{x \in T : d(x, \mathbb{R} \setminus T) \leq \varepsilon\}$, $\kappa_\varepsilon(x) = 1$ for $x \in \{x \in T : \ d(x, \mathbb{R} \setminus T) \geq 2\varepsilon\}$. Let $f \in \mathcal{D}'(T)$. Then $Cd(f) = [g(\varphi_\varepsilon,x)]$, where

$$g(\varphi_\varepsilon,x) = (f\kappa_\varepsilon * \check{\varphi}_\varepsilon)(x) = \langle f(t)\kappa_\varepsilon(t), \check{\varphi}_\varepsilon(x-t)\rangle, \quad x \in T, \ \ \varphi_\varepsilon \in \mathcal{A}_0(\mathbb{R}).$$

This is an injective mapping. If $T = \mathbb{R}$ then a function κ_ε is not needed.

Let K be a compact set in $\mathbb{R}$ and $G = [g(\varphi_\varepsilon,x)] \in \mathcal{G}(T)$. The integral $\int_K G(x)\,dx$ is defined by its representative $\int_K \kappa(x) g(\varphi_\varepsilon,x)\,dx$, where $\kappa \in C^\infty(T)$, $\kappa(x) = 1$ on T.

It is said that $G \in \mathcal{G}(T)$ is equal to zero in the sense of generalized functions, $G = 0$ (g.d.), if for every $\varphi \in \mathcal{D}(T)$, $\int_T G(x)\varphi(x)\,dx = 0$ in $\bar{\mathbb{C}}$. $G_1 = G_2$ (g.d.) iff $G_1 - G_2 = 0$, (g.d.).

The space of Colombeau's generalized tempered functions is defined as follows.

Put

$$\mathcal{E}_{Mt}(\mathbb{R}) = \{g(\varphi, x) \in \mathcal{E}(\mathbb{R}) : (\forall \alpha \in \mathbb{N}_0^n)\,(\exists N \in \mathbb{N}_0)\,(\forall \varphi \in \mathcal{A}_N(\mathbb{R}))$$
$$(\sup\{\frac{|\partial^\alpha g(\varphi_\varepsilon, x)|}{(1+|x|)^N},\ x \in \mathbb{R}\} = \mathcal{O}(\varepsilon^{-N})\,\}.$$
$$\mathcal{N}_t(\mathbb{R}) = \{g(\varphi, x) \in \mathcal{E}(\mathbb{R}) : (\forall \alpha \in \mathbb{N}_0^n)\,(\exists N \in \mathbb{N}_0)\,(\exists \{a_q\} \in \Gamma)(\forall q \geq N)$$
$$(\forall \varphi \in \mathcal{A}_q(\mathbb{R}))\ (\sup\{\frac{|\partial^\alpha g(\varphi_\varepsilon, x)|}{(1+|x|)^N},\ x \in \mathbb{R}\} = \mathcal{O}(\varepsilon^{a_q - N}))\,\}.$$

Then

$$\mathcal{G}_t(\mathbb{R}) = \mathcal{E}_{Mt}(\mathbb{R})/\mathcal{N}_t(\mathbb{R})$$

is the space of Colombeau's tempered generalized functions.

Let $\psi \in \mathcal{S}$ and $G = [g(\varphi_\varepsilon, x)] \in \mathcal{G}_t$. Then $\langle G, \psi \rangle = \int G(x)\psi(x)\,dx$ is defined by its representative

$$\int_{\mathbb{R}} g(\varphi_\varepsilon, x)\psi(x)\,dx, \qquad \varphi \in \mathcal{A}_0(\mathbb{R}), \quad \varepsilon \in (0,1];$$

$G_1, G_2 \in \mathcal{G}_t(\mathbb{R})$ are equal in the sense of generalized tempered distributons, $G_1 = G_2$ (g.t.d.) if $\langle G_1, \psi \rangle = \langle G_2, \psi \rangle$ in $\bar{\mathbb{C}}$, for every $\psi \in \mathcal{S}$.

3. Positive and positive definite Colombeau's generalized functions

Recall, an element in $\mathcal{D}'(T)$ is positive if for every positive $\varphi \in \mathcal{D}(T)$, $\varphi \geq 0$, $\langle f, \varphi \rangle \geq 0$, [4].

Definition 1. Let $G \in \mathcal{G}(T)$. Then G is positive, $G \geq 0$, (resp. negative, $G \leq 0$) on T if G has a representative $g(\varphi_\varepsilon, x)$ such that

$$(\forall \rho \in \mathcal{D}(T), \rho \geq 0)(\forall a > 0)(\exists N \in \mathbb{N})(\forall \varphi \in \mathcal{A}_N)(\exists \varepsilon_0 \in (0,1))$$
$$\langle g(\varphi_e, x), \rho(x) \rangle > +\varepsilon^a \geq 0, \quad \text{for } \varepsilon < \varepsilon_0 \tag{1}$$
$$(\text{resp.}, \langle g(\varphi_e, x), \rho(x) \rangle > -\varepsilon^a \leq 0, \text{ for } \varepsilon < \varepsilon_0)$$

Proposition 1. *(i) If $G = [g(\varphi_\varepsilon, x)] \in \mathcal{G}(T)$ is such that $G \geq 0$ and $G \leq 0$, then for every $\rho \in \mathcal{D}(T)$, $\rho \geq 0$,*

$$\langle g(\varphi_\varepsilon, x), \rho(x) \rangle \in \mathcal{N}_c.$$

(ii) Let $f \in \mathcal{D}'(T)$. Then $Cd(f)$ is positive if and only if f is positive.

Proof. (i) By assumption, there exists representatives $g_1(\varphi_e, x)$ and $g_2(\varphi_\varepsilon, x)$ such that

$$(\forall \rho \in \mathcal{D}(T), \rho \geq 0)(\forall a > 0)(\exists N \in \mathbb{N})(\forall \varphi \in \mathcal{A}_N)(\exists \varepsilon_0 \in (0,1))$$
$$\langle g_1(\varphi_e, x), \rho(x) \rangle > +\varepsilon^a \geq 0, \quad \varepsilon < \varepsilon_0,$$
$$\langle g_2(\varphi_e, x), \rho(x) \rangle > -\varepsilon^a \leq 0, \quad \varepsilon < \varepsilon_0,$$
$$-\varepsilon^a \leq \langle g_1(\varphi_\varepsilon, x) - g_2(\varphi_e, x), \rho(x) \rangle \leq \varepsilon^a, \quad \varepsilon < \varepsilon_0.$$

Thus, for every $a > 0$,

$$0 \leqslant \langle g_1(\varphi_\varepsilon, x) + \varepsilon^a, \rho(x)\rangle \leq \langle g_2(\varphi_\varepsilon, x) + 2\varepsilon^a, \rho(x)\rangle \leq (1+2C)\varepsilon^a, \quad \varepsilon < \varepsilon_0,$$

where $C = \int \rho(x)\,dx > 0$. This implies that for every $\rho \in \mathcal{D}(T)$, $\rho \geq 0$, $\langle g(\varphi_\varepsilon, \rho(x))\rangle \in \mathcal{N}_c$.

(ii) Let $f \in \mathcal{D}'(T)$ be positive. Then for $\varphi \in \mathcal{A}_q$, and every $\rho \in \mathcal{D}(T)$, $\rho \geq 0$,

$$\begin{aligned}\langle f * \varphi_\varepsilon, \rho\rangle &= \langle f, \rho * \check{\varphi}_\varepsilon\rangle = \langle f(x), \int \rho(x-\varepsilon t)\varphi(-t)\,dt\rangle \\ &= \langle f(x), \int (\rho(x) + (-\varepsilon t)\rho'(x) + \cdots + \frac{(-\varepsilon t)^q}{q!}\rho^{(q)}(x-\xi(t)))\varphi(-t)\,dt\rangle \\ &= \langle f(x), \rho(x)\rangle + C(f,\rho)\varepsilon^q,\end{aligned}$$

where $C(f,\rho)$ is a suitable constant. This implies that $f * \varphi_\varepsilon$ satisfies (1), i.e. $Cd(f)$ is positive.

If $Cd(f) = f * \varphi_\varepsilon$ satisfies (1), by letting $\varepsilon \to 0$, it follows

$$\langle f, \rho\rangle \geq 0, \quad \text{for every } \rho \geq 0,\ \rho \in \mathcal{D}(T),$$

i.e. f is positive. ∎

Recall, an $f \in \mathcal{D}'(\mathbb{R})$ is positive definite if for every $\theta \in \mathcal{D}(\mathbb{R})$,

$$\langle f, \theta * \theta^*\rangle \geq 0,$$

where $\theta^*(x) = \overline{\theta(-x)}$.

Definition 2. Let $G \in \mathcal{G}(\mathbb{R})$. Then G is positive definite if it has a representative $g(\varphi_\varepsilon, x) \in \mathcal{E}_M(\mathbb{R})$ such that

$$\begin{gathered}(\forall\theta \in \mathcal{D}(\mathbb{R}))(\forall a > 0)(\exists N \in \mathbb{N})(\forall\varphi \in \mathcal{A}_N)(\exists\varepsilon_0 \in (0,1)) \\ \langle g(\varphi_\varepsilon, x), \theta(x) * \theta^*(x)\rangle > +\varepsilon^a \geq 0, \quad \text{for } \varepsilon < \varepsilon_0.\end{gathered} \tag{2}$$

Proposition 2. *Let $f \in \mathcal{D}'(\mathbb{R})$. Then f is a positive definite distribution if and only if $Cd(f)$ is positive definite.*

Proof. Let $Cd(f)$ be positive definite. Then, $(f * \varphi_\varepsilon)$ satisfies (2) and by letting $\varepsilon \to 0$ we obtain that f is positive definite.

Conversely, if $f \in \mathcal{D}'(\mathbb{R})$ is positive definite, by expanding

$$\int (\theta * \theta^*)(x - \varepsilon t)\varphi(t)\,dt$$

into a Taylor series, as in Proposition 1, we obtain the assertion. ∎

Colombeau space $\mathcal{G}_t(\mathbb{R})$ is not a subspace of $\mathcal{G}(\mathbb{R})$ since there exists $R(\varphi_\varepsilon, x) \in \mathcal{E}(\mathbb{R})$ with the property

$$R(\varphi_\varepsilon, x) \in (\mathcal{E}_{Mt}(\mathbb{R}) \setminus \mathcal{N}_t(\mathbb{R})) \cap \mathcal{N}(\mathbb{R}),$$

but every element of $\mathcal{E}_{Mt}(\mathbb{R})$ defines an element of $\mathcal{G}(\mathbb{R})$ which enables the definition of the canonical mapping $i\colon \mathcal{G}_t(\mathbb{R}) \to \mathcal{G}(\mathbb{R})$:

$$\mathcal{G}_t \ni G = [G_\varepsilon] \mapsto i(G) = [G_\varepsilon + \mathcal{N}(\mathbb{R})] \in \mathcal{G}(\mathbb{R}).$$

DEFINITION 3. An $F \in \mathcal{G}_t(\mathbb{R})$ is positive (respectively, positive definite) if $i(F) \in \mathcal{G}(\mathbb{R})$ is positive (respectively, positive definite).

REFERENCES

[**1**] S. Albeverio, Z. Haba, F. Russo, *Trivial solutions for a non-linear two space dimensional wave equation perturbed by space-time white noise*, Analyse, Probabilites, Topologie, Unitéde Recherche Associée au CNRS 0225.

[**2**] H. A. Biagioni, *A Nonlinear Theory of Generalized Functions*, Lecture Notes Math. **1421**, Springer-Verlag, Berlin, 1990.

[**3**] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions*, North Holand, 1982.

[**3**] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Academic Press, New York, 1964.

[**5**] Z. Lozanov Crvenković, S. Pilipović, *Some clases of Colombeau's generalized random processes*, Univ. Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., (in print)

[**6**] M. Oberguggenberger, *Generalized Functions and Stochastic Processes*, Progress in Probability, **36** (1995), 215–229.

[**7**] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Eqautions*, Pitman Research Notes Math, **259**, Longman, Harlow, 1992.

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