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A NOTE ON THE HURWITZ ZETA FUNCTION

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Abstract. We show that the Hurwitz zeta function and polylogarithm, $\zeta(\nu,a)$ and $\mathrm{Li}_\nu(z)$, form a discrete Fourier transform pair for $\operatorname{Re}\nu > 1$. Many formulae, the majority of them previously unknown, are obtained as a corollary to this result. In particular, the transformation relation allows the evaluation of $\zeta(\nu,a)$ at rational values of the parameter a. It is also shown that, by making use of the transform pair, various known results can be deduced easily and in a unified manner. For instance,

$$2\zeta(2n+1,1/3) = (3^{2n+1}-1)\zeta(2n+1) + (-1)^{n-1}3^{2n}\sqrt{3}\frac{(2\pi)^{2n+1}}{(2n+1)!}B_{2n+1}(1/3),$$

$n \geq 1$, where $B_n(\cdot)$ stands for the Bernoulli polynomial of degree n.

1. Introduction

As the various zeta functions (Riemann, Hurwitz, Epstein, Lerch, Selberg and their generalisations) constantly find new applications in different areas of mathematics (number theory, analysis, numerical methods etc.) and physics (quantum field theory, string theory, cosmology etc.), further development of their theory is needed. In this note we extend the procedure used in [1] to establish a new relation between the Hurwitz zeta function and polylogartithms.

2. Statement of the results

The Riemann and the Hurwitz zeta functions, $\zeta(\nu)$ and $\zeta(\nu,a)$ respectively, are both analytic over the whole ν-complex plane, except at $\nu = 1$, where they have a simple pole. $\zeta(\nu)$ and $\zeta(\nu,a)$ can be defined for $\operatorname{Re}\nu \leqslant 1$, $\nu \neq 1$, as analytic continuations of the following series [2, p. 19 and p. 22]

$$\zeta(\nu) = \sum_{k=1}^{\infty}\frac{1}{k^\nu} \quad \text{and} \quad \zeta(\nu,a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^\nu}, \quad 0 < a \leqslant 1,\ \operatorname{Re}\nu > 1. \tag{1}$$

Throughout the text $\mathrm{Li}_\nu(z)$, here referred to as the polylogarithm, denotes the Dirichlet power series defined by

$$\mathrm{Li}_\nu(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^\nu} \tag{2}$$

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which converges absolutely for all ν if $|z| < 1$, for $\operatorname{Re}\nu > 0$ if $|z| = 1$, $z \neq 1$, and for $\operatorname{Re}\nu > 1$ if $z = 1$. It is known that $\operatorname{Li}_\nu(z)$ admits an analytic continuation which makes it regular for every ν. It is evident that $\operatorname{Li}_\nu(1) = \zeta(\nu, 1) = \zeta(\nu)$.

The Bernoulli polynomial of degree n, denoted by $B_n(x)$, is defined by the power series [2, p. 25]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!}, \qquad |t| < 2\pi \tag{3}$$

and $B_n = B_n(0)$ is the n-th Bernoulli number. Note that the (Euler) relation exists between the even-indexed Bernoulli numbers B_{2n} and the values $\zeta(2n)$. For a more detailed discussion of $\zeta(\nu)$ and $\zeta(\nu, a)$ see Whittaker and Watson [3]. An extensive list of formulae involving $\operatorname{Li}_\nu(z)$ can be found in Prudnikov et al. [4, pp. 762–763]. The theory of this and related functions (for example the Legendre chi function, and the generalised and associated Clausen functions) is well covered in Lewin's standard text [5]. Formulae involving $B_n(x)$ and B_n can be found in Magnus et al. [2, pp. 25–32], Abramowitz and Stegun [6, pp. 803–806], Gradshteyn and Ryzhik [7, pp. 1076–1080] and Prudnikov et al. [4, pp. 765–766]. Our results are as follows.

Theorem. *Assume that t is a positive integer and set $\omega = \exp(i\,2\pi/t)$. Let $\zeta(\nu, a)$ and $\operatorname{Li}_\nu(z)$ be the Hurwitz zeta function and the polylogarithm defined as in (1) and (2). Then:*

$$\zeta(\nu, s/t) = \frac{1}{t}\sum_{r=1}^{t} t^\nu \operatorname{Li}_\nu(\omega^r)\omega^{-rs}, \qquad s = 1, 2, \ldots, t, \tag{4a}$$

$$\operatorname{Li}_\nu(\omega^r) = \frac{1}{t^\nu}\sum_{s=1}^{t} \zeta(\nu, s/t)\omega^{rs}, \qquad r = 1, 2, \ldots, t. \tag{4b}$$

Corollary 1. *Consider*

$$S_\nu(x) = \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^\nu} \quad \text{and} \quad C_\nu(x) = \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^\nu}, \quad \operatorname{Re}\nu > 1 \tag{5}$$

and assume that p and q, with $p \geqslant 0$ and $q > 0$, are integers. Then:

$$\text{(i)}\ \ S_\nu(p/q) = \frac{1}{q^\nu}\sum_{s=1}^{q} \zeta(\nu, s/q)\sin(s\,2\pi p/q), \quad 0 \leqslant p \leqslant q; \tag{6a}$$

$$\text{(ii)}\ \ C_\nu(p/q) = \frac{1}{q^\nu}\sum_{s=1}^{q} \zeta(\nu, s/q)\cos(s\,2\pi p/q), \quad 0 \leqslant p \leqslant q; \tag{6b}$$

$$\text{(iii)}\ \ \zeta(\nu, p/q) = q^{\nu-1}\sum_{s=1}^{q} [C_\nu(s/q)\cos(s\,2\pi p/q) + S_\nu(s/q)\sin(s\,2\pi p/q)], \ \ 1 \leqslant p \leqslant q; \tag{6c}$$

$$\text{(iv)}\ \ \zeta(\nu, p/q) - \zeta(\nu, 1 - p/q) = 2q^{\nu-1}\sum_{s=1}^{q} S_\nu(s/q)\sin(s\,2\pi p/q), \ \ 1 \leqslant p \leqslant q - 1; \tag{6d}$$

$$\text{(v)}\ \ \zeta(\nu, p/q) + \zeta(\nu, 1 - p/q) = 2q^{\nu-1}\sum_{s=1}^{q} C_\nu(s/q)\cos(s\,2\pi p/q), \ \ 1 \leqslant p \leqslant q - 1. \tag{6e}$$

Corollary 2. *Assume that p, q and n, with $p \geqslant 0$ and $q, n > 0$, are integers. Let $B_n(x)$ be the Bernoulli polynomial of degree n defined in (3). If $n \geqslant 1$, the following holds:*

(i) $$B_{2n+1}(p/q) = (-1)^{n-1} \frac{2(2n+1)!}{(2\pi q)^{2n+1}} \sum_{s=1}^{q} \zeta(2n+1, s/q) \sin(s\, 2\pi p/q), \quad 0 \leqslant p \leqslant q; \tag{7a}$$

(ii) $$B_{2n}(p/q) = (-1)^{n-1} \frac{2(2n)!}{(2\pi q)^{2n}} \sum_{s=1}^{q} \zeta(2n, s/q) \cos(s\, 2\pi p/q), \quad 0 \leqslant p \leqslant q; \tag{7b}$$

(iii) $$\zeta(2n+1, p/q) - \zeta(2n+1, 1-p/q) = \\ = (-1)^{n-1} q^{2n} \frac{(2\pi)^{2n+1}}{(2n+1)!} \sum_{s=1}^{q} B_{2n+1}(s/q) \sin(s\, 2\pi p/q), \quad 1 \leqslant p \leqslant q-1; \tag{7c}$$

(iv) $$\zeta(2n, p/q) - \zeta(2n, 1-p/q) = \\ = (-1)^{n-1} q^{2n-1} \frac{(2\pi)^{2n}}{(2n)!} \sum_{s=1}^{q} B_{2n}(s/q) \cos(s\, 2\pi p/q), \quad 1 \leqslant p \leqslant q-1. \tag{7d}$$

Note. The formula (4b) in the Theorem and the parts (i) and (ii) of Corollary 1 and Corollary 2 have been recently deduced [1]. The formula in (7a) was first given by Almkvist and Meurman [8], but it appears that the remaining results and the Theorem in the form given above are unknown. For further discussion of the results and illustrative examples see Section 4.

3. Proof of the results

Before proving the Theorem, we recall the definition of the discrete Fourier transform [9, Chapter 8]. Let (a_r) $(r = 0, 1, \ldots, t-1, t \geqslant 1)$ be a sequence of real or complex numbers with period t $(a_{r+t} = a_r$ for all $r \in \mathbf{N}_0)$ and let $\omega = \exp(i\, 2\pi/t)$. Then, the discrete Fourier transform pair of the sequences (a_r) and (a_s^*) is defined as

$$a_s^* = \frac{1}{t} \sum_{r=0}^{t-1} a_r \omega^{-rs}, \qquad s = 0, 1, \ldots, t-1,$$
$$a_r = \sum_{s=0}^{t-1} a_s^* \omega^{rs}, \qquad r = 0, 1, \ldots, t-1.$$

The first relation is known as the *direct* discrete Fourier transform, and the second as the *inverse* discrete Fourier transform. We note that although it is usually asserted that $0 \leqslant r \leqslant t-1$ and $0 \leqslant s \leqslant t-1$, r and s can be arbitrary integers (or residues modulo t).

Proof of Theorem. We shall first derive the formula in (4b), i.e. show that the sequence $t^\nu \operatorname{Li}_\nu(\omega^r)$ is the inverse Fourier transform of $\zeta(\nu, s/t)$.

The sequence of the numbers $\mathrm{Li}_\nu(\omega^r)$ (and thus $t^\nu \,\mathrm{Li}_\nu(\omega^r)$) given by

$$\mathrm{Li}_\nu(\omega^r) = \sum_{k=1}^{\infty} \frac{\exp(i\,2k\pi r/t)}{k^\nu} = \sum_{k=0}^{\infty} \frac{\exp(i\,2(k+1)\pi r/t)}{(k+1)^\nu}, \quad r = 1, 2, \ldots, t$$

is clearly periodic with period t, and the absolute convergence of the series involved is assured when $\mathrm{Re}\,\nu > 1$ (see (2)).

Next, we recall the division law in $\mathbf{Z}$: for any $a \in \mathbf{Z}$, $b \in \mathbf{N}$ there exist unique $c, d \in \mathbf{Z}$ such that $a = bc + d$ and $0 \leqslant d < b$. Here, in the case of the series for $\mathrm{Li}_\nu(\omega^r)$, this means that any (k, t) $(k \in \mathbf{N}_0,\, t \in \mathbf{N})$ uniquely determine the integers m and s such that $k = tm + s$ where $m = 0, 1, 2, \ldots$ and $s = 0, 1, \ldots, t-1$. Hence, it follows by absolute convergence that

$$\begin{aligned}\mathrm{Li}_\nu(\omega^r) &= \sum_{m=0}^{\infty} \sum_{s=0}^{t-1} \frac{\exp[i\,2(tm+s+1)\pi r/t]}{(tm+s+1)^\nu} = \sum_{m=0}^{\infty} \sum_{s=1}^{t} \frac{\exp[i\,2(tm+s)\pi r/t]}{(tm+s)^\nu} \\ &= \frac{1}{t^\nu} \sum_{s=1}^{t} \sum_{m=0}^{\infty} \frac{\exp(i\,2m\pi r)\exp(i\,2s\pi r/t)}{(m+s/t)^\nu}\end{aligned}$$

which can be further simplified to

$$\mathrm{Li}_\nu(\omega^r) = \frac{1}{t^\nu} \sum_{s=1}^{t} \sum_{m=0}^{\infty} \frac{\exp(i\,2s\pi r/t)}{(m+s/t)^\nu} = \frac{1}{t^\nu} \sum_{s=1}^{t} \exp(i\,2s\pi r/t) \sum_{m=0}^{\infty} \frac{1}{(m+s/t)^\nu}$$

since $\exp(i\,2m\pi r) = 1$ (m and r are integers). In view of the definition of the Hurwitz zeta function in (1), the last double sum results in the required formula in (4b).

Finally, what remains is to show that the transform relations in (4a) and in (4b) form a discrete Fourier pair. Indeed, substitution of (4b) into (4a) yields

$$\begin{aligned}\zeta(\nu, s/t) &= \frac{1}{t} \sum_{r=1}^{t} t^\nu \,\mathrm{Li}_\nu(\omega^r)\omega^{-rs} = \frac{1}{t} \sum_{r=1}^{t} t^\nu \left(t^{-\nu} \sum_{s=1}^{t} \zeta(\nu, s/t)\omega^{rs} \right) \omega^{-rs} \\ &= \frac{1}{t} \sum_{s=1}^{t} \zeta(\nu, s/t) \sum_{r=1}^{t} \omega^{rs}\omega^{-rs} = \zeta(\nu, s/t), \quad s = 1, 2, \ldots, t\end{aligned}$$

because of the following orthogonality relationship

$$\sum_{r=1}^{t} \omega^{rs}\omega^{-rs} = \begin{cases} t, & \text{if } r = s, \\ 0, & \text{otherwise} \end{cases}$$

and thus the proposed transform relations in (4) are established for $\mathrm{Re}\,\nu > 1$. This completes the proof when $\mathrm{Re}\,\nu > 1$, and for other values of ν the relations hold by the principle of analytic continuation. ∎

Proof of Corollary 1. Clearly, the series in (5) are absolutely convergent for any real x and simple consideration of them shows that

$$S_\nu(1-x) = -S_\nu(x) \quad \text{and} \quad C_\nu(1-x) = C_\nu(x) \tag{8a}$$

while

$$S_\nu(1) = S_\nu(0) = 0 \quad \text{and} \quad C_\nu(1) = C_\nu(0) = \zeta(\nu). \tag{8b}$$

Moreover, the following

$$\mathrm{Li}_\nu[\exp(i\,2k\pi x)] = C_\nu(x) + iS_\nu(x) \tag{9}$$

holds by (2).

Now, in view of (9) the required formulae in (6a,b), valid for $1 \leqslant p \leqslant q$, follow trivially from (4b). It can be shown by direct verification that they remain valid for $p = 0$.

Next, by making use of (9) the expression in (4a) can be rewritten as follows

$$\begin{aligned}\zeta(\nu, s/t) &= t^{\nu-1} \sum_{r=1}^{t} (\Phi_1(r) + i\Phi_2(r)) = t^{\nu-1} \sum_{r=1}^{t} \Phi_1(r) \\ &= t^{\nu-1} \sum_{r=1}^{t} [C_\nu(r/t)\cos(r\,2\pi s/t) + S_\nu(r/t)\sin(r\,2\pi s/t)], \quad 1 \leqslant s \leqslant t\end{aligned} \tag{10a}$$

i.e. the imaginary part vanishes since for

$$\Phi_2(r) = S_\nu(r/t)\cos(r\,2\pi s/t) - C_\nu(r/t)\sin(r\,2\pi s/t)$$

we have that $\Phi_2(t) = 0$ and that $\Phi_2(t - r) = -\Phi_2(r)$ $(1 \leqslant r \leqslant t-1)$ (see (8a,b)) and therefore $\sum_{r=1}^{t-1} \Phi_2(r) = 0$.

In this way we have arrived at the proposed formula in (6c). Finally, the desired results in (6d) and (6e) are obtained from (10a) and

$$\zeta(\nu, 1 - s/t) = t^{\nu-1} \sum_{r=1}^{t} [C_\nu(r/t)\cos(r\,2\pi s/t) - S_\nu(r/t)\sin(r\,2\pi s/t)] \tag{10b}$$

where $1 \leqslant s \leqslant t - 1$. Here, the case $s = t$ should be excluded considering that for $\zeta(\nu, a)$ must be $a \neq 0$. ■

Proof of Corollary 2. It is easy to verify that parts (i)–(iv) are immediate consequences of (i), (ii), (iv) and (v) in Corollary 1 by recalling the following Fourier series representation of the Bernoulli polynomials $B_n(x)$ [6, p. 805, Entry 23.1.17 and 23.1.18]

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}}$$

where $0 \leqslant x \leqslant 1$ for $n = 2, 3, \dots$, $0 < x < 1$ for $n = 1$ and

$$B_{2n}(x) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}}$$

where $0 \leqslant x \leqslant 1$ for $n = 1, 2, 3, \dots$ ■

4. Concluding remarks

It has been shown that the Hurwitz zeta function and the polylogarithm form a discrete Fourier pair. We note that more non-obvious identities, some with useful physical applications, can be deduced from (4a,b) by using the theory of the discrete Fourier transform, particularly the theorems on convolution [9, Chapter 8]. As an illustration, we give

$$\sum_{r=1}^{t} t^{2\nu} C_\nu^2(r/t) + \sum_{r=1}^{t} t^{2\nu} S_\nu^2(r/t) = \sum_{r=1}^{t} \zeta^2(\nu, r/t)$$

whici is obtained by applying the Rayleigh-Parseval formula on the Fourier pair in (4a,b) where (4b) is combined by (9).

The theorem allows evaluation of the Hurwitz zeta function $\zeta(\nu, a)$ at rational values of the parameter a. Recall that an analytic continuation from [10]

$$\zeta(1-\nu, s/t) = \frac{2\Gamma(\nu)}{(2\pi t)^\nu} \sum_{r=1}^{t} \zeta(\nu, r/t) \cos\pi(\nu/2 - 2rs/t), \quad 1 \leqslant s \leqslant t \tag{11}$$

has a similar property. However, equation (11) cannot in general be used directly to yield $\zeta(\nu, a)$ for ν a positive integer, unlike the following well known relations [6, p. 260, Eq. 6.4.7 in conjunction with Eq. 6.4.10]

$$\zeta(2n+1, x) - \zeta(2n+1, 1-x) = \frac{\pi}{(2n)!} \cot(\pi x)^{(2n)} \tag{12a}$$

$$\zeta(2n, x) + \zeta(2n, 1-x) = -\frac{\pi}{(2n-1)!} \cot(\pi x)^{(2n-1)} \tag{12b}$$

which are valid when $0 < x < 1$ and $n \geqslant 1$ and involve derivatives of $\cot(\pi x)$. Compare (12a,b) with our (7c,d) and observe that by combining them we arrive at the following interesting sums

$$\sum_{s=1}^{q} B_{2n+1}(s/q) \sin(s\, 2\pi p/q) = (-1)^{n-1} \frac{2n+1}{2(2\pi q)^{2n}} \cot(\pi x)^{(2n)} \Big|_{x=p/q}$$

$$\sum_{s=1}^{q} B_{2n}(s/q) \cos(s\, 2\pi p/q) = (-1)^{n} \frac{n}{(2\pi q)^{2n-1}} \cot(\pi x)^{(2n-1)} \Big|_{x=p/q}.$$

Further, in view of (6a,b) the trigonometric series $C_\nu(x)$ and $S_\nu(x)$ in (5) are in the general case, when x is a rational, summed in closed form in terms of the Hurwitz zeta function. For instance

$$S_\nu(1/3) = -S_\nu(2/3) = (1/2)3^{-\nu}\sqrt{3}(\zeta(\nu, 1/3) - \zeta(\nu, 2/3))$$

$$S_\nu(1/4) = -S_\nu(3/4) = 4^{-\nu}(\zeta(\nu, 1/4) - \zeta(\nu, 3/4))$$

$$S_\nu(1/6) = -S_\nu(5/6) = (1/2)6^{-\nu}\sqrt{3}[(\zeta(\nu, 1/6) - \zeta(\nu, 5/6)) + (\zeta(\nu, 1/3) - \zeta(\nu, 2/3))].$$

Note that (6a) can be rewritten as

$$\begin{aligned} S_\nu(p/q) &= \frac{1}{q^\nu} \sum_{s=1}^{q-1} \zeta(\nu, s/q) \sin(s\, 2\pi p/q) \\ &= \frac{1}{q^\nu} \sum_{s=1}^{[(q-1)/2]} (\zeta(\nu, s/q) - \zeta(\nu, 1-s/q)) \sin(s\, 2\pi p/q) \end{aligned}$$

and (6b) as

$$C_\nu(p/q) = \frac{1}{q^\nu} \sum_{s=1}^{[(q-1)/2]} (\zeta(\nu, s/q) + \zeta(\nu, 1-s/q)) \cos(s\, 2\pi p/q) + A\zeta(\nu) q^{-\nu}$$

where $A = 1$ and $A = 2 - 2^\nu$ when q is an odd and even integer respectively. Moreover, it is easy to verify that the other expressions given in Corollary 1 and Corollary 2 can be similarly rewritten.

In order to show that various results can be deduced easily and, in particular, in a unified manner, we give several examples of the application of the theorem and its corollaries. Almost all these results can be found in Hansen's text [11] while, for instance, Prudnikov et al. [4, pp. 765–766] list those given in Examples (i) and (ii), respectively.

EXAMPLES.

(i) Let C_ν be defined by (5). Then:

$$\begin{aligned}
&C_\nu(1/2) = (2^{1-\nu} - 1)\zeta(\nu)\\
&C_\nu(1/3) = C_\nu(2/3) = (1/2)(3^{1-\nu} - 1)\zeta(\nu)\\
&C_\nu(1/4) = C_\nu(3/4) = 2^{-\nu}(2^{1-\nu} - 1)\zeta(\nu)\\
&C_\nu(1/6) = C_\nu(5/6) = (1/2)(3^{1-\nu} - 1)(2^{1-\nu} - 1)\zeta(\nu).
\end{aligned}$$

(ii) Let $B_n(x)$ and B_n be the Bernoulli polynomials and numbers. If $n \geqslant 1$, then:

$$\begin{aligned}
B_{2n}(0) = B_{2n}(1) = B_{2n} &= (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n)\\
B_{2n}(1/2) &= (1/2)(2^{1-2n} - 1)B_{2n}\\
B_{2n}(1/3) = B_{2n}(2/3) &= (1/2)(3^{1-2n} - 1)B_{2n}\\
B_{2n}(1/4) = B_{2n}(3/4) &= 2^{-2n}(2^{1-2n} - 1)B_{2n}\\
B_{2n}(1/6) = B_{2n}(5/6) &= (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}.
\end{aligned}$$

(iii) If n is a positive integer and Q_n is given by $Q_n = (-1)^{n-1} \dfrac{(2\pi)^{2n+1}}{(2n+1)!}$ then:

$$\begin{aligned}
\zeta(2n+1, 1/2) &= (2^{2n+1} - 1)\zeta(2n+1)\\
2\zeta(2n+1, 1/3) &= (3^{2n+1} - 1)\zeta(2n+1) + \sqrt{3}\, B_{2n+1}(1/3) 3^{2n} Q_n\\
2\zeta(2n+1, 2/3) &= (3^{2n+1} - 1)\zeta(2n+1) - \sqrt{3}\, B_{2n+1}(1/3) 3^{2n} Q_n\\
2\zeta(2n+1, 1/4) &= 2^{2n+1}(2^{2n+1} - 1)\zeta(2n+1) + 2B_{2n+1}(1/4) 4^{2n} Q_n\\
2\zeta(2n+1, 3/4) &= 2^{2n+1}(2^{2n+1} - 1)\zeta(2n+1) - 2B_{2n+1}(1/4) 4^{2n} Q_n\\
2\zeta(2n+1, 1/6) &= (3^{2n+1} - 1)(2^{2n+1} - 1)\zeta(2n+1)\\
&\qquad + \sqrt{3}\,(B_{2n+1}(1/6) + B_{2n+1}(1/3)) 6^{2n} Q_n
\end{aligned}$$

$$
\begin{aligned}
2\zeta(2n+1,5/6) = (3^{2n+1}-1)(2^{2n+1}-1)\zeta(2n+1)& \\
-\sqrt{3}\,(B_{2n+1}(1/6)+B_{2n+1}(1/3))6^{2n}Q_n.&
\end{aligned}
$$

Proof. Note that in view of (8b) on putting $p=0$ in (6b) we have

$$\sum_{s=1}^{q}\zeta(\nu,s/q)=q^{\nu}\zeta(\nu)$$

and hence the following identities hold

$$
\begin{aligned}
\zeta(\nu,1/2) &= (2^{\nu}-1)\zeta(\nu)\\
\zeta(\nu,1/3)+\zeta(\nu,2/3) &= (3^{\nu}-1)\zeta(\nu)\\
\zeta(\nu,1/4)+\zeta(\nu,3/4) &= 2^{\nu}(2^{\nu}-1)\zeta(\nu)\\
\zeta(\nu,1/6)+\zeta(\nu,5/6) &= (2^{\nu}-1)(3^{\nu}-1)\zeta(\nu).
\end{aligned}
$$

Then, all the formulae for special values in Examples (i)–(iii) follow readily on making use of these identities in conjunction with parts (ii) of Corollary 1 and Corollary 2 and part (iii) of Corollary 2, respectively.

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