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CONVEXITY AND REFLEXIVITY

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Abstract. In recent years some papers have appeared containing generalizations of the concept of convexity with the help of the notion of measure of noncompactness. Furthermore, some authors have introduced the concept of near uniform convexity by means of the Hausdorff measure and of weak near uniform convexity by means of the De Blasi measure. In this work we present a generalization of these concepts by means of a general set quantity.

1. Introduction

The concept of convexity plays an important role in the classical geometry of normed spaces. It is frequently used in the metric fixed point theory and other branches of nonlinear analysis [6, 9, 10, 13, 15].

In recent years some papers have appeared containing generalizations of the concept of convexity with the help of the notion of measure of noncompactness. Namely, Huff [15] and Goebel and Sekowski [14] introduced independently the concept of near uniform convexity which seems to be a natural generalization of uniform convexity defined by Clarkson [9]. Recently Cabrera [8] has introduced the concept of weak near uniform convexity using the De Blasi measure of weak noncompactness.

In order to recall this concept let us introduce some notation. Assume $(E, \|\ \|)$ is an infinite-dimensional Banach space with the zero element θ. Denote by B_E and S_E the closed unit ball and the unit sphere in E, respectively. For a subset X of E we denote by $\operatorname{conv} X$ the convex hull and by $\operatorname{Conv} X$ the closed convex hull of X.

Moreover, if we assume that X is a nonempty and bounded set in E, then the quantity $\chi(X)$ defined in the following way:

$$\chi(X) = \inf\{\, \varepsilon > 0 \ :\ \text{there exists a compact set } Y \text{ such that } X \subset Y + \varepsilon B_E \,\}$$

is called the Hausdorff measure of noncompactness of the set X. For the properties of the function χ we refer to [2].

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We say that a Banach space E is nearly uniformly convex (NUC, for short) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if X is a convex closed subset of B_E with $\operatorname{dist}(\theta, X) \geqslant 1 - \delta$, then $\chi(X) \leqslant \varepsilon$ [15]. For further purposes we recall that this definition is equivalent to the following one [1]:

$$E \text{ is NUC if and only if } \lim_{\varepsilon \to 0} \sup\{\, \chi(F(f, \varepsilon)) \;:\; f \in S_{E^*} \,\} = 0$$

where E^* denotes the dual space of E and $F(f, \varepsilon)$ is defined as the slice of B_E, that is to say

$$F(f, \varepsilon) = \{\, x \in B_E \;:\; f(x) \geqslant 1 - \varepsilon \,\}.$$

Similarly we define the next concept. Namely, a space E is said to be locally nearly uniformly convex (LNUC) if

$$\lim_{\varepsilon \to 0} \chi(F(f, \varepsilon)) = 0 \quad \text{for any } f \in S_{E^*}.$$

Finally, recall that a space E is referred to as nearly strictly convex (NSC) if its unit sphere does not contain noncompact and convex sets. Equivalently, E is NSC if $\chi(F(f, 0)) = 0$ for each $f \in S_{E^*}$ such that $F(f, 0) \neq \emptyset$.

Notice that NUC $\Rightarrow$ LNUC $\Rightarrow$ NSC, but no converse implication is true. The concepts introduced above have a lot of applications in geometry of Banach spaces and in fixed point thoery [1, 13, 15, 18]. For example, they enable us to select some different classes of Banach spaces which are equivalent from the viewpoint of the classical geometry of Banach spaces.

Let us also mention that each NUC or LNUC Banach space is reflexive but in general the converse implication is not true [1].

Recently Cabrera [8] has generalized the above mentioned concepts with the help of the notion of the De Blasi measure of weak noncompactness. Namely, if we assume that X is nonempty and bounded in E and if we denote by $\mathcal{M}_E^{\omega}$ the family consisting of all relatively weakly compact subsets of E, the quantity $\beta(X)$ defined in the following way

$$\beta(X) = \inf\{\, \varepsilon > 0 \;:\; \text{there exists a set } Y \in \mathcal{M}_E^{\omega} \text{ such that } X \subset Y + \varepsilon B_E \,\}$$

is called the De Blasi measure of weak noncompactness of the set X.

Observe that in the case when E is reflexive we have $\beta(X) = 0$ for every $X \in \mathcal{M}_E$. We refer to [5, 11] for more details.

We give the following definitions formally analogous to the above mentioned which appear in [8].

We say that a Banach space E is weakly nearly uniformly convex (WNUC) if

$$\lim_{\varepsilon \to 0} \sup\{\, \beta(F(f, \varepsilon)) \;:\; f \in S_{E^*} \,\} = 0.$$

A Banach space is said to be weakly locally nearly uniformly convex (WLNUC) if

$$\lim_{\varepsilon \to 0} \beta(F(f, \varepsilon)) = 0 \quad \text{for any } f \in S_{E^*}.$$

Finally, a Banach space E is referred to as weakly strictly convex (WNSC) if

$$\beta(F(f,0)) = 0 \quad \text{for each } f \in S_{E^*} \text{ such that } F(f,0) \neq \emptyset.$$

It is proved in [8] that the concepts of WNUC and WLNUC do not select new classes of Banach spaces but they create a characterization of reflexivity. Moreover, an example of nonreflexive Banach space is given in [8] which is WNSC.

In the above mentioned generalizations of the concept of convexity the main tools are the Hausdorff measure of noncompactness and the De Blasi measure of weak noncompactness. Notice that these measures are the Hausdorff distance between a nonempty and bounded subset C of E and a certain family of nonempty and bounded subsets of this space (relatively compact subsets of E and relatively weakly compact subsets of E, respectively).

In this paper we study some facts about the reflexivity and the convexity obtained by using the Hausdorff distance to a certain family of nonempty and bounded subsets of E. This generalization of measures of noncompactness has been treated in [4].

2. Hausdorff distance

For $x \in E$, $C \subset E$ and for $\varepsilon > 0$ we write

$$d(x,C) = \inf\{\, \|x-y\| \;:\; y \in C \,\}, \quad B(C,\varepsilon) = \{\, y \in E \;:\; d(y,C) \leqslant \varepsilon \,\}.$$

Observe that $B(C,\varepsilon) = C + \varepsilon B_E$. The family of all nonempty and bounded susbets of E will be denoted by $\mathcal{M}_E$.

For $C, D \in \mathcal{M}_E$ we consider the Hausdorff nonsymmetric distance defined in the following way:

$$H'(C,D) = \inf\{\, \varepsilon > 0 \;:\; C \subset B(D,\varepsilon) \,\} = \inf\{\, \varepsilon > 0 \;:\; C \subset D + \varepsilon B \,\}.$$

The Hausdorff distance between C and D is defined as

$$H(C,D) = \max\{H'(C,D), H'(D,C)\}.$$

Observe that the function H is a metric on the family $\mathcal{M}_E^{bc}$ where $\mathcal{M}_E^{bc} = \{\, C \in \mathcal{M}_E \;:\; C = \overline{C} \,\}$.

In what follows let $\mathcal{N}$ be a nonempty subfamily of $\mathcal{M}_E$. Consider two real functions defined on the family $\mathcal{M}_E$ in the following way:

$$H'_{\mathcal{N}}(C) = \inf\{\, H'(C,P) \;:\; P \in \mathcal{N} \,\}, \quad H_{\mathcal{N}}(C) = \inf\{\, H(C,P) \;:\; P \in \mathcal{N} \,\}.$$

For further goals let us recall that the function $H_{\mathcal{N}}$ was introduced in [3] on a complete metric space S. The main result obtained in [3] is formulated in the following proposition.

PROPOSITION 1. *Let $\mathcal{N}$ be a nonempty subfamily of $\mathcal{M}_E$ satisfying the condition $\mathcal{M} \in \mathcal{N}$ and $\emptyset \neq P \subset \mathcal{M} \implies P \in \mathcal{N}$. Then the equality $H'_{\mathcal{N}}(C) = H_{\mathcal{N}}(C)$ holds for every $C \in \mathcal{M}_E$.*

In what follows we recall the concept of set quantity which appears in [4].

Definition 1. A mapping $\mu\colon \mathcal{M}_E \to [0,\infty)$ is said to be a set quantity if it satisfies the following conditions for $C, D \in \mathcal{M}_E$ and $\lambda \in K$:

1. $\mu(C \cup D) = \max\{\mu(C), \mu(D)\}$,
2. $\mu(\lambda C) = |\lambda|\mu(C)$,
3. $\mu(C + D) \leqslant \mu(C) + \mu(D)$,
4. $\mu(\operatorname{conv} C) = \mu(C)$.

From this definition it can be easily seen that μ satisfies:

(a) $\mu(\{0\}) = 0$,

(b) $C \subset D \implies \mu(C) \leqslant \mu(D)$,

(c) $\mu(C) = \mu(\overline{C})$.

The family $\ker\mu$ defined in the usual way, $\ker\mu = \{\, C \in \mathcal{M}_E \,:\, \mu(C) = 0 \,\}$, will be called the kernel of the quantity μ.

If $\ker\mu$ is the collection of all nonempty and relatively compact subsets of E then μ is referred to as a measure of noncompactness. In the case when $\ker\mu$ is the family of all nonempty and relatively weakly compact subsets of E we will say that μ is a measure of weak noncompactness.

In what follows we recall a theorem which appears in [4] and which gives us a way to construct set quantities.

Theorem 1. *Let $\mathcal{N}$ be a subfamily of $\mathcal{M}_E$ satisfying the following conditions: 1. $\mathcal{N} \neq \emptyset$. 2. $P, M \in \mathcal{N} \iff P \neq \emptyset$, $M \neq \emptyset$ and $P \cup M \in \mathcal{N}$. 3. $\lambda \in K$ and $P \in \mathcal{N} \implies \lambda P \in \mathcal{N}$. 4. $P, M \in \mathcal{N} \implies P + M \in \mathcal{N}$. 5. $P \in \mathcal{N} \implies \operatorname{conv} P \in \mathcal{N}$. 6. $\mathcal{N}$ is closed in the topology generated by the distance H on $\mathcal{M}_E$.*

Then $H_{\mathcal{N}}$ has the following properties: (a) $H_{\mathcal{N}}$ is a set quantity. (b) $\ker H_{\mathcal{N}} = \mathcal{N}$. (c) $\mathcal{N} = \mathcal{M}_E \iff H_{\mathcal{N}}(B_E) = 0$. (d) $\mathcal{N} \neq \mathcal{M}_E \iff H_{\mathcal{N}}(B_E) = 1$. (e) If μ is other set quantity in E with $\ker\mu = \mathcal{N}$ then $\mu \leqslant \mu(B_E)H_{\mathcal{N}}$.

3. Convexity induced by a set quantity

We start with the following definition.

Definition 2. We say that a Banach space E is μ-uniformly convex (μ-UC, for short) if

$$\lim_{\varepsilon\to 0} \sup\{\, \mu(F(f,\varepsilon)) \;:\; f \in S_{E^*} \,\} = 0,$$

where μ is a set quantity.

Similarly we define the next concept.

Definition 3. A space E is said to be μ-locally uniformly convex (μ-LUC, for short) if

$$\lim_{\varepsilon\to 0} \mu(F(f,\varepsilon)) = 0 \quad \text{for each } f \in S_{E^*}.$$

Finally, we give the following definition.

DEFINITION 4. A space E is referred to as μ-strictly convex (μ-SC, for short) if $\mu(F(f,0)) = 0$ for each $f \in S_{E^*}$ such that $F(f,0) \neq \emptyset$.

Notice that μ-UC $\Longrightarrow$ μ-LUC $\Longrightarrow$ μ-SC. A natural question is if the converse implications are true.

When $\mu = \chi$, the Hausdorff measure of noncompactness, there exist examples which prove the converse implications are not true [1].

When $\mu = \beta$, the De Blasi measure of noncompactness, in [8] it is proved that the concepts of β-UC and β-LUC coincide with the concept of reflexivity and, moreover, an example of nonreflexive Banach space is given which is β-SC.

Notice that the measures χ and β satisfy the Cantor condition [2]. We recall this definition.

DEFINITION 5. A set quantity μ satisfies the Cantor condition if for each decreasing sequence $(X_n)_{n=1,2,\ldots}$ of nonempty, closed and bounded subsets of E such that $\mu(X_n) \to 0$ when $n \to \infty$, the intersection of all X_n, $\bigcap X_n$ is nonempty.

In the following theorem we prove the main result.

THEOREM 2. *Let μ be a set quantity in E which satisfies the Cantor condition. If E is μ-LUC, then E is reflexive.*

Proof. Fix a functional $f \in S_{E^*}$ and consider the sequence of the slices $\{F(f,\frac{1}{n})\}$. Observe all the slices are nonempty, convex, closed and $F(f,\frac{1}{n}) \supset F(f,\frac{1}{n+1})$ for $n = 1,2,\ldots$. In view of our assumption we have $\lim \beta(F(f,\frac{1}{n})) = 0$ when $n \to \infty$. Taking into account the properties of the De Blasi measure of weak noncompactness β [3] we can deduce

$$\bigcap_{n=1}^{\infty} F\Big(f,\frac{1}{n}\Big) \neq \emptyset.$$

On the other hand we have

$$\bigcap_{n=1}^{\infty} F\Big(f,\frac{1}{n}\Big) = F(f,0)$$

which implies $F(f,0) \neq \emptyset$ for each $f \in S_{E^*}$.

In other words, we have that there exists $x \in S_E$ such that $f(x) = 1$. This means that every functional f of S_{E^*} attains its norm on the unit sphere S_E. The same is also true for an arbitrary functional $f \in E^*$ $(f \neq \theta)$ since the functional $f/\|f\|$ belongs to S_{E^*}.

Finally we conclude, in virtue of James theorem [16], that E is reflexive. Thus the proof is complete. ∎

This result generalizes one obtained in [18]. Moreover, we can obtain the following characterization of reflexive spaces.

Theorem 3. *A Banach space E is reflexive if and only if there exists a set quantity μ satisfying the Cantor condition such that E is μ-LUC.*

Proof. If E is reflexive then we take the set quantity β and the result is obvious. The converse implication is proved in the preceding theorem. ■

Some interesting questions are the following.

Does there exist a set quantity such that it does not satisfy the Cantor condition?

Do there exist a Banach space E and a set quantity μ such that E is μ-LUC and is not reflexive?

In order to answer these questions we will study the convexity induced by the family of all weakly conditionally compact subsets. Recall that a subset A of E is called weakly conditionally compact (WCC) if every sequence in A has a weak Cauchy subsequence. It is obvious that WCC subsets are bounded. We denote by $P_{WCC}(E)$ the family of all nonempty and weakly conditionally compact subsets of E.

Proposition 2. *$P_{WCC}(E)$ satisfies the conditions of Theorem 1.*

Proof. The conditions 1–4 follow immediately. The condition 5 is proved in [7]. The condition 6 appears in [12, Exercise 2, p. 237]. ■

In virtue of Theorem 1, we can consider the set quantity $H_{\mathcal{N}}$ where $\mathcal{N} = P_{WCC}(E) = \mathcal{M}_E$. By Rosenthal characterization of Banach spaces containig no copies of l^1, if E contains no copy of l^1 then $\mathcal{N} = P_{WCC}(E) = \mathcal{M}_E$ and by (c) of Theorem 1, $H_{\mathcal{N}}$ is identically equal to zero in E. Consequently, E is an $H_{\mathcal{N}}$-LUC space.

Thus, every nonreflexive Banach space which contains no copy of l^1 (c_0, the James space $\mathcal{J}$, for example) solves the second question mentioned above.

On the other hand, in virtue of Theorem 1, the set quantity $H_{\mathcal{N}}$ does not sastisfy the Cantor condition in the spaces c_0 and $\mathcal{J}$ and this fact answers the first question mentioned above.

Proposition 3. *l^1 is not $H_{\mathcal{N}}$-SC.*

Proof. As l^1 is a Shur space, the weakly conditionally compact subsets are relatively norm compact and, as relatively norm compact subsets are, obviously, weakly conditionally compact, we deduce $H_{\mathcal{N}} = \chi$ in l^1. But l^1 is not χ-SC as it is proved in [14], and the proof is complete. ■

In what follows, we prove that under certain assumptions on the set quantity μ, there exist μ-SC Banach spaces which are not μ-LUC.

Recall that a Banach space E is said to be strictly convex (SC) if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$.

Proposition 4. *Let E be an SC Banach space and $f \in S_{E^*}$. If $F(f,0)$ is nonempty, then $F(f,0)$ contains only one element.*

Proof. Suppose $x_1, x_2 \in F(f,0)$ and $x_1 \neq x_2$. From $F(f,0) = \{ x \in S_{E^*} : f(x) = 1 \}$ we infer

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{1}{2}(f(x_1)+f(x_2)) = 1.$$

On the other hand, as $f \in S_{E^*}$ and $x_1, x_2 \in S_{E^*}$ we have

$$1 = f\left(\frac{x_1+x_2}{2}\right) \leqslant \|f\| \cdot \left\|\frac{x_1+x_2}{2}\right\| \leqslant \left\|\frac{x_1+x_2}{2}\right\| \leqslant 1$$

and this implies $\|\frac{x_1+x_2}{2}\| = 1$. This fact contradicts our hypothesis. ■

Notice that if μ is a set quantity such that the subsets containing only one element belong to its kernel, then every SC Banach space is μ-SC.

In what follows we recall a theorem due to Clarkson [9] which states that any separable Banach space may be given a new norm, equivalent to the original one, with respect to which the space is strictly convex. Finally, we are ready for another mentioned question.

THEOREM 4. *Let μ be a set quantity which satisfies the Cantor condition and its kernel contains the singletons. Under these assumptions there exists a μ-SC Banach space which is not μ-LUC.*

Proof. We consider a nonreflexive and separable Banach space E (l^1, for example). By Clarkson's theorem there exists an equivalent norm $|||\cdot|||$, such that $(E, |||\cdot|||)$ is SC Banach space. In virtue of the last proposition and Theorem 2, we deduce that $(E, |||\cdot|||)$ is not μ-LUC. ■

Finally, notice that the set quantity given by $\|A\| = \sup\{ \|x\| : x \in A \}$ has the kernel which does not contain the singletons.

An interesting question is to give other examples of set quantities such that the singletons do not belong to their kernels.

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