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RELATIONSHIPS BETWEEN USUAL AND APPROXIMATE INVERSE SYSTEMS

Ivan Lončar

Abstract. We shall prove that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an approximate inverse system of compact non-metric spaces with surjective bonding mappings p_{ab} such that each X_a is a limit of a usual τ-directed inverse system $X(a) = \{X_{(a,\gamma)},\ f_{(a,\gamma)(a,\delta)},\ \Gamma_a\}$ of metric compact spaces, then there exist: 1) a usual τ-directed inverse system $X_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$, 2) every X_d is a limit of an approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ of compact metric spaces $X_{(a,\gamma_a)}$, 3) if the mappings p_{ab} and $f_{(a,\gamma)(a,\delta)}$ are monotone, then $g_{(a,\gamma_a)(b,\gamma_b)}$ and F_{de} are monotone.

1. Introduction

In this paper we shall use the notion of *inverse systems* $\mathbf{X} = \{X_a, p_{ab}, A\}$ and their limits in the usual sense [1, p. 135].

The cardinality of a set X will be denoted by $\operatorname{card}(X)$. The cofinality of a cardinal number m will be denoted by $\operatorname{cf}(m)$. $Cov(X)$ is the set of all normal coverings of a topological space X. If $\mathcal{U}$, $\mathcal{V} \in Cov(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \leq \mathcal{U}$. For two mappings $f, g\colon Y \to X$ which are $\mathcal{U}$-near (for every $y \in Y$ there exists a $U \in \mathcal{U}$ with $f(y), g(y) \in U$), we write $(f, g) \leq \mathcal{U}$. A basis of (open) normal coverings of a space X is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U} \in Cov(X)$ admits a refinement $\mathcal{V} \in \mathcal{C}$. We denote by $cw(X)$ (*covering weight*) the minimal cardinal of a basis of normal coverings of X [9, p. 181].

LEMMA 1. [9, Example 2.2] *If X is a compact Hausdorff space, then $cw(X) = w(X)$.*

The notion of *approximate inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ will be used in the sense of S. Mardešić [11].

DEFINITION 1. An *approximate inverse system* is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, X_a, $a \in A$, is a topological space and $p_{ab}\colon X_b \to X_a$, $a \leq b$, are mappings such that $p_{aa} = \mathrm{id}$ and the following condition (A2) is satisfied:

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(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in Cov(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) \leq \mathcal{U}$, whenever $a \leq b \leq c \leq d$.

An approximate map $p = \{p_a : a \in A\} \colon X \to \mathbf{X}$ into an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a collection of maps $p_a \colon X \to X_a$, $a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $U \in Cov(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) \leq U$ for each $c \geq b$. (See [10]).

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system and let $\mathbf{p} = \{p_a : a \in A\} \colon X \to \mathbf{X}$ be an approximate map. We say that $\mathbf{p}$ is a *limit* of $\mathbf{X}$ provided it has the following universal property:

(UL) For any approximate map $\mathbf{q} = \{q_a : a \in A\} \colon Y \to \mathbf{X}$ of a space Y there exists a unique map $g \colon Y \to X$ such that $p_a g = q_a$.

Let $X = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called a *thread* of X provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall U \in Cov(X_a))(\exists b \geq a)(\forall c \geq b)\, p_{ac}(x_c) \in st(x_a, U)$.

If X_a is a $T_{3.5}$ space, then the sets $st(x_a, \mathcal{U})$, $\mathcal{U} \in Cov(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

$(L)^*$ $(\forall a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a$.

Let τ be an infinite cardinal. We say that a partially ordered set A is τ*-directed* if for each $B \subseteq A$ with $\operatorname{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. If A is $\aleph_0$-directed, then we will say that A is σ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be τ*-directed* if A is τ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if A is σ-directed.

The proof of the following theorem is similar to the proof of Theorem 1.1 of [4].

THEOREM 1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed approximate inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f \colon X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b \colon X_b \to Y$ such that $f = g_b p_b$.*

THEOREM 2. *Let X be a compact space. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof. It is well-known that there exists a usual inverse system $\mathbf{Y} = \{Y_\alpha, q_{\alpha\beta}, \Sigma\}$ of metric spaces Y_α and surjective bonding mappings such that X is homeomorphic to $\lim \mathbf{Y}$. By Theorem 9.5 of [12] there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $\lim \mathbf{X}$ is homeomorphic to $\lim \mathbf{Y}$ and each X_a is the limit of a countable inverse subsystem of $\mathbf{Y}$. This means that each X_a is a metric compact space. ■

THEOREM 3. [8, p. 163, Theorem 2.] *If X is a locally connected compact space, then there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a* *metric locally connected compact space, each* p_{ab} *is a monotone surjection and* X *is homeomorphic to* $\lim \mathbf{X}$. *Conversely, the inverse limit of such system is always a locally connected compact space.*

REMARK 1. We may assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ in Theorem 3 is σ-directed [12, Theorem 9.5].

THEOREM 4. [13, Corollary 2.9] *If* X *is a hereditarily locally connected continuum, then there exists a* σ*-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *such that each* X_a *is a metrizable hereditarily locally connected continuum, each* p_{ab} *is a monotone surjection and* X *is homeomorphic to* $\lim \mathbf{X}$.

THEOREM 5. [3, Corollary 3] *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-directed inverse system of hereditarily locally connected continua* X_a. *Then* $X = \lim \mathbf{X}$ *is hereditarily locally connected.*

The following theorem is Theorem 1.7 from [5].

THEOREM 6. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-directed inverse system of compact metrizable spaces and surjective bonding mappings. Then* $X = \lim \mathbf{X}$ *is metrizable if and only if there exists an* $a \in A$ *such that* $p_b \colon X \to X_b$ *is a homeomorphism for each* $b \geq a$.

2. Approximate subsystems

In this Section we investigate the approximate subsystem of an approximate system $\mathbf{X} = \{X_a, p_{ab}, \mathrm{A}\}$. We start with the following definition.

DEFINITION 2. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let B be a directed subset of A such that $\{X_b, p_{bc}, B\}$ is an approximate inverse system. We say that $\{X_b, p_{bc}, B\}$ is an *approximate subsystem* of $\mathbf{X} = \{X_a, p_{ab}, A\}$ if there exists a mapping $q\colon \lim \mathbf{X} \to \lim\{X_b, p_{bc}, B\}$ such that

$$p_b q = P_b, \quad b \in B,$$

where $p_b\colon \lim\{X_b, p_{bc}, B\} \to X_b$ and $P_b\colon \lim \mathbf{X} \to X_b$, $b \in B$, are natural projections.

We say that an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *irreducible* if for each $B \subset A$ with $\mathrm{card}(B) < \mathrm{card}(A)$ it follows that B is not cofinal in A.

LEMMA 2. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system. There exists a cofinal subset* B *of* A *such* $\mathbf{X} = \{X_a, p_{ab}, B\}$ *is irreducible.*

Proof. Consider the family $\mathcal{B}$ of all cofinal subsets of B of A. The set $\{\mathrm{card}(B) : B \in \mathcal{B}\}$ has the minimal element b since each $\mathrm{card}(B)$ is some initial ordinal number. Let $B \in \mathcal{B}$ be such that $\mathrm{card}(B) = b$. It is clear that $\{X_a, p_{ab}, B\}$ is irreducible. ■

In the sequel we will assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is irreducible.

LEMMA 3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces such that* $\mathrm{card}(A) = \aleph_0$. *Then there exists a countable well-ordered subset* B

of A such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ *is homeomorphic to* $\lim\{X_b, p_{bc}, B\}$.

Proof. Let ν be any finite subset of A. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. Since A is infinite, there exists a sequence $\{\nu_n : n \in \mathbb{N}\}$ such that $\nu_1 \subseteq \ldots \subseteq \nu_n \subseteq \cdots$ and $A = \bigcup\{\nu_n : n \in \mathbb{N}\}$. Recursively, we define the sets $A_1, \ldots, A_n, \ldots$ by

$$A_1 = \nu_1 \cup \{\delta(\nu_1)\},$$

and

$$A_{n+1} = A_n \cup \nu_{n+1} \cup \{\delta(A_n \cup \nu_{n+1})\}.$$

It follows that there exists a sequence

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \cdots$$

of finite sets A_n such that $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Let $b_1 = \delta(A_1)$ and $b_n \geq \delta(A_n)$, b_{n-1} if $n \geq 2$. We obtain a sequence $B = \{b_n : n \in \mathbb{N}\}$ such that B is cofinal in A. By virtue of [10, Theorem (1.19)] it follows that $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. ■

Now we consider irreducible approximate inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ with $\mathrm{card}(A) \geq \aleph_1$.

LEMMA 4. *Let A be a directed set. For each subset B of A there exists a directed set $F_\infty(B)$ such that* $\mathrm{card}(F_\infty(B)) = \mathrm{card}(B)$.

Proof. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup\{\delta(\nu) : \nu \in B\}$, where ν is a finite subset of B and $\delta(\nu)$ is defined as in the proof of Lemma 3. Put

$$F_{n+1} = F_1(F_n(B),$$

and

$$F_\infty(B) = \bigcup\{F_n(B) : n \in \mathbb{N}\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \ldots \subseteq F_n(B) \subseteq \cdots$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$.

If B is finite, then $\mathrm{card}(F_\infty(B)) = \aleph_0$. If $\mathrm{card}(B) \geq \aleph_0$, then we have $\mathrm{card}(\{\delta(\nu) : \nu \in B\}) \leq \mathrm{card}(B)\aleph_0$. We infer that $\mathrm{card}(F_1(B)) \leq \mathrm{card}(B)\aleph_0$. Similarly, $\mathrm{card}(F_n(B)) \leq \mathrm{card}(B)\aleph_0$. This means that $\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0$. Thus

$$\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0, \qquad \text{if} \qquad \mathrm{card}(B) < \mathrm{card}(A).$$

The proof is completed. ■

LEMMA 5. *Let $\{X_a, p_{ab}, A\}$ be an approximate inverse system such that $cw(X_a) < \mathrm{card}(A)$, $a \in A$. For each subset B of A with $\mathrm{card}(B) < \mathrm{card}(A)$, there exists a directed set $G_\infty(B) \supseteq B$ such that the collection $\{X_a, p_{ab}, G_\infty(B)\}$ is an approximate system and $\mathrm{card}(G_\infty(B)) = \mathrm{card}(B)$.*

Proof. Let $\mathcal{B}_a$ be a base of normal coverings of X_a. Let $\mathcal{U}_a \in \mathcal{B}_a$. By virtue of (A2) there exists an $a(\mathcal{U}_a) \in$A such that $(p_{ad}, p_{ac}p_{cd}) \leq \mathcal{U}_a$, $a \leq a(\mathcal{U}_a) \leq c \leq d$. For each subset B of A we define $G_\infty(B)$ by induction as follows:

a) Let $G_1(B) = F_\infty(B)$. From Lemma 4 it follows that $\mathrm{card}(G_1(B)) = \mathrm{card}(F_\infty(B)) = \mathrm{card}(B)$.

b) For each $n > 1$ we define $G_n(B)$ as follows:

1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$,

2) If n is even, then $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in \mathcal{B}_a,\ a \in G_{n-1}(B)\}$. Since $\mathrm{card}(\mathcal{B}_a) < \mathrm{card}(A)$ the set $G_n(B)$ has the cardinality $< \mathrm{card}(A)$. Now we define $G_\infty(B) = \bigcup\{G_n(B) : n \in \mathbb{N}\}$. It is obvious that $\mathrm{card}(G_\infty(B)) < \mathrm{card}(A)$.

The set $G_\infty(B)$ is directed. Let a, b be a pair of elements of $G_\infty(B)$. There exists an $n \in \mathbb{N}$ such that $a, b \in G_n(B)$. We may assume that n is odd. Then $a, b \in F_\infty(G_{n-1}(B))$. Thus there exists a $c \in F_\infty(G_{n-1}(B))$ such that $c \geq a, b$. It is clear that $c \in G_\infty(B)$. The proof of directedness of $G_\infty(B)$ is completed.

The collection $\{X_a, p_{ab}, G_\infty(B)\}$ is an approximate system. It suffices to prove that the condition (A2) is satisfied. Let a be any member of $G_\infty(B)$. There exists an $n \in \mathbb{N}$ such that $a \in G_n(B)$. We have two cases.

1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$. This means that $a \in F_\infty(G_{n-1}(B))$. By definition of $F_\infty(G_{n-1}(B))$ we infer that $a(\mathcal{U}_a) \in F_\infty(G_{n-1}(B))$. Thus (A2) is satisfied.

2) If n is even, then $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in Cov(X_a),\ a \in G_{n-1}(B)\}$. In this case $a \in G_{n+1}(B) \subseteq G_\infty(B)$. Arguing as in the case 1, we infer that (A2) is satisfied. ∎

THEOREM 7. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces. If $\lambda \leq w(X_a) \leq \tau < \mathrm{card}(A)$ for each $a \in A$, then $\lim \mathbf{X}$ is homeomorphic to a limit of a λ-directed usual inverse system $\{X_\alpha, q_{\alpha\beta}, T\}$, where each X_α is a limit of an approximate inverse subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$.*

Proof. The proof consists of several steps.

Step 1. Let $\mathcal{B} = \{B_\mu : \mu \in M\}$ be a family of all subsets of A with $\mathrm{card}(B_\alpha) = \lambda$. Put $A_\mu = G_\infty(B_\mu)$ (Lemma 5) and let $\Delta = \{A_\mu : \mu \in M\}$ be ordered by inclusion $\subseteq$.

Step 2. *If Φ and Ψ are in Δ such that $\Phi \subset \Psi$, then there exists a mapping $q_{\Phi\Psi}\colon \lim\{X_\alpha, p_{\alpha\beta}, \Psi\} \to \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$.*

Namely, if $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then by definition of the threads of $\{X_\alpha, p_{\alpha\beta}, \Psi\}$ the condition (L) is satisfied. If (L) is satisfied for $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then it is satisfied for $(x_\gamma, \gamma \in \Phi)$ since the required a' in (L) lies—by definition of the set Φ—in the set Φ. This means that $(x_\gamma, \gamma \in \Phi) \in \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. Now we define $q_{\Phi\Psi}(x) = (x_\gamma, \gamma \in \Phi)$.

Step 3. *The collection $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ is a usual inverse system.* It suffices to prove transitivity, i.e., if $\Phi \subseteq \Psi \subseteq \Omega$, then $q_{\Phi\Psi}q_{\Psi\Omega} = q_{\Phi\Omega}$. This easily follows from the definition of $q_{\Phi\Psi}$.

Step 4. *The space* $\lim \mathbf{X}$ *is homeomorphic to* $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$, *where* $X_\Phi = \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. We shall define a homeomorphism $H\colon \lim \mathbf{X} \to \lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Let $x = (x_a : a \in A)$ be any point of $\lim \mathbf{X}$. Each collection $\{x_a : a \in \Phi \in \Delta\}$ is a point x_Φ of X_Φ since $X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Moreover, from the definition of $q_{\Phi\Psi}$ (Step 2) it follows that $q_{\Phi\Psi}(x_\Psi) = x_\Phi$, $\Psi \supseteq \Phi$. Thus, the collection $\{x_\Phi : \Phi \in \Delta\}$ is a point of $\lim\{X_\Phi, q_{\Phi\Psi}, \Delta\}$. Let $H(x) = \{x_\Phi, \Phi \in \Delta\}$. Thus, H is a continuous mapping of $\lim \mathbf{X}$ to $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. In order to complete the proof it suffices to prove that H is 1–1 and onto. Let us prove that H is 1–1. Let $x = (x_a : a \in A)$ and $y = (y_a : a \in A)$ be a pair of points of $\lim \mathbf{X}$. This means that there exists an $a \in$A such that $y_a \neq x_a$. There exists a $\Phi \in \Delta$ such that $a \in \Phi$. Thus, the collections $\{x_a : a \in \Phi\}$ and $\{y_a : a \in \Phi\}$ are different. From this we conclude that $x_\Phi \neq y_\Phi$, $x_\Phi, y_\Phi \in X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Hence H is 1–1. Let us prove that H is onto. Let $y = (y_\Phi : \Phi \in \Delta)$ be any point of $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Each y_Φ is a collection $\{x_a : a \in \Phi\}$ and if $\Psi \supseteq \Phi$, then the collection $\{x_a : a \in \Phi\}$ is the restriction of the collection $\{x_a : a \in \Psi\}$ on Φ. Let x be the collection which is the union of all collections $\{x_a : a \in \Phi\}$, $\Phi \in \Delta$. Hence x is a collection $(x_a : a \in A)$ which is a point of $\lim \mathbf{X}$ and $H(x) = y$.

Step 5. *Inverse system* $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ *is a* λ*-directed inverse system.* Let $\{\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\} : \kappa \leq \lambda\}$ be a collection of approximate subsystems $\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\}$. The set $\Phi = \bigcup\{\Phi_\kappa : \kappa \leq \lambda\}$ has the cardinality $\leq \lambda$ since $\operatorname{card}(\Phi_\kappa) \leq \lambda$. By virtue of Steps 1–4 there exists an approximate subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \lambda$. This means that $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ is a λ-directed inverse system. ∎

COROLLARY 1. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact metric spaces. Then* $\lim \mathbf{X}$ *is homeomorphic to the limit of a* σ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \Delta\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \aleph_0$.

LEMMA 6. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system such that* X_a, $a \in A$, *are compact locally connected spaces and* p_{ab} *are monotone surjections. If* $\mathbf{Y} = \{X_b, p_{cd}, B\}$ *is an approximate subsystem of* $\mathbf{X}$, *then the mapping* $q_{AB}\colon \lim \mathbf{X} \to \lim \mathbf{Y}$ *(defined in Step 2 of the proof of Theorem 7) is a monotone surjection.*

Proof. Let $P_a\colon \lim \mathbf{X} \to X_a$, $a \in A$, be the natural projection. Similarly, let $p_a\colon \lim \mathbf{Y} \to X_a$, $a \in B$, be the natural projection. From the definition of q_{AB} (Step 2 of the proof of Theorem 7) it follows that $p_a q_{AB} = P_a$ for each $a \in B$. By virtue of [10, Corollary 4.5] and [7, Corollary 5.6] it follows that P_a and p_a are monotone surjections. Let us prove that q_{AB} is a surjection. Let $y = (y_a : a \in B) \in \lim \mathbf{Y}$. The sets $P_a^{-1}(y_a)$, $a \in B$, are non-empty since P_a is surjective for each $a \in A$. From the compactness of $\lim \mathbf{X}$ it follows that a limit superior $Z = \operatorname{Ls}\{P_a^{-1}(y_a), a \in B\}$ is a non-empty subset of $\lim \mathbf{X}$. We shall prove that for each $z = (z_a : a \in A) \in Z$, $P_a(z) = y_a$. Suppose that $P_a(z) \neq y_a$. There exists a pair U, V of open disjoint subsets of X_a such that $y_a \in U$ and $P_a(z) \in V$. For sufficiently large $b \in B$, $P_a(P_b^{-1}(_b))$ is in U because of (AS). This means that $P_a^{-1}(V) \cap P_b^{-1}(y_b) = \emptyset$ for sufficiently large $b \in B$. This contradicts the assumption $z \in \operatorname{Ls}\{P_a^{-1}(y_a), a \in B\}$. Hence q_{AB} is a surjection. In order to complete the proof it suffices to prove that

q_{AB} is monotone. Take a point $y \in \lim \mathbf{Y}$ and suppose that $q_{AB}^{-1}(y)$ is disconnected. There exists a pair U, V of disjoint open sets in $\lim \mathbf{X}$ such that $q_{AB}^{-1}(y) \subseteq U \cup V$. From the compactness of $\lim \mathbf{X}$ it follows that q_{AB} is closed. This means that there exists an open neighborhood W of y such that $q_{AB}^{-1}(y) \subseteq q_{AB}^{-1}(W) \subseteq U \cup V$. From the definition of the basis in $\lim \mathbf{Y}$ it follows that there exists an open set W_a in some X_a, $a \in$B such that $y \in p_a^{-1}(W_a) \subseteq W$. Moreover, we may assume that W_a is connected since X_a is locally connected. Then $P_a^{-1}(W_a)$ is connected since P_a is monotone [7, Corollary 5.6]. Moreover, $q_{AB}^{-1}(y) \subseteq P_a^{-1}(W_a)$ and $P_a^{-1}(W_a) \subseteq U \cup V$ since $P_a = p_a q_{AB}$. This is impossible since U and V are disjoint open sets and $P_a^{-1}(W_a)$ is connected. ■

THEOREM 8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces. If* $\lambda \leq w(X_a) < \mathrm{card}(A)$ *for each* $a \in A$ *and* $\mathrm{cf}(\mathrm{card}(A)) \neq \lambda$, *then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* λ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$. *Moreover, if* $\mathrm{card}(A)$ *is a regular cardinal, then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* λ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$.

A directed preordered set $(A, \leq)$ is said to be *cofinite* provided each $a \in A$ has only finitely many predecessors. If $a \in A$ has exactly n predecessors, we shall write $p(a) = n + 1$. Hence, $a \in A$ is the first element of $(A, \leq)$ if and only if $p(a) = 1$.

LEMMA 7. *If* $(A, \leq)$ *is cofinite, then it satisfies the following principle of induction:*

Let $B \subset A$ *be a set such that:*

(i) *B contains all the first elements of A,*

(ii) *if B contains all the predecessors of $a \in A$, then $a \in B$.*

Then $B = A$.

LEMMA 8. [15, Lemma 1] *Let* $q = (q_a)\colon Y \to \mathcal{Y} = \{Y_b, \mathcal{V}_b, q_{ab'}, B\}$ *be an approximate map (approximate resolution) of a space* Y. *Then there exists an approximate map (approximate resolution)* $q = (q_a)\colon Y \to \mathcal{Y} = \{Y_c^{'}, \mathcal{V}_c^{'}, q_{cc'}, C\}$ *of the space* Y *and an increasing surjection* $t\colon C \to B$ *satisfying the following conditions:*

(i) *C is directed, unbounded, antisymmetric and cofinite set;*

(ii) $(\forall c \in C)(\forall b \in B)(\exists c' > c)\, t(c') > b$;

(iii) $(\forall c \in C)\, Y_c' = Y_{t(c)}$, $\mathcal{V}_c' = \mathcal{V}_{t(c)}$, $q_c' = q_{t(c)}$ *and* $q_{cc'}' = q_{t(c)t(c')}$, *whenever* $c < c'$.

COROLLARY 2. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces. Then there exists a cofinite approximate inverse system* $\mathbf{Y} = \{Y_c, p_{cc'}, C\}$ *such that each* Y_c *is some* X_a, *each* $p_{cc'}$ *is some* p_{ab} *and* $\lim \mathbf{X}$ *is homeomorphic to* $\lim \mathbf{Y}$.

Proof. By virtue of Theorem (4.2) of [10] an approximate map $p\colon X \to \mathbf{X}$ is an approximate resolution if and only if it is a limit of $\mathbf{X} = \{X_a, p_{ab}, A\}$. Apply Lemma 8. ■

THEOREM 9. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact non-metric spaces with surjective bonding mappings p_{ab}. If each X_a is a limit of a usual σ-directed inverse system $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ of metric compact spaces, then:*

1. *there exists a usual σ-directed inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$,*
2. *every X_d is a limit of an approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ of compact metric spaces $X_{(a,\gamma_a)}$,*
3. *if the mappings p_{ab} and $f_{(a,\gamma)(a,\delta)}$ are monotone, then $g_{(a,\gamma_a)(b,\gamma_b)}$ and F_{de} are monotone.*

Proof. The proof consists of several steps. In the Steps 0.–11. we shall define a usual inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$.

Step 0. From Corollary 2 it follows that we may assume that A is cofinite.

Step 1. For each X_a there exists a σ-directed inverse system

$$\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\} \tag{1}$$

such that each $X_{(a,\gamma)}$ is a metric compact space, each $f_{(a,\gamma)(a,\delta)}$ is monotone and surjective and X_a is homeomorphic to $\lim \mathbf{X}(a)$. Now we have the following diagram

$$\begin{array}{ccccccc}
X_a & \xleftarrow{p_{ab}} & X_b & \xleftarrow{p_{bc}} & X_c & \xleftarrow{p_d} & X \\
\downarrow{\scriptstyle f_{(a,\gamma_a)}} & & \downarrow{\scriptstyle f_{(b,\gamma_b)}} & & \downarrow{\scriptstyle f_{(c,\gamma_c)}} & & \\
X_{(a,\gamma_a)} & & X_{(b,\gamma_b)} & & X_{(c,\gamma_c)} & & \\
\downarrow{\scriptstyle f_{(a,\gamma_a)(a,\delta_a)}} & & \downarrow{\scriptstyle f_{(b,\gamma_b)(b,\delta_b)}} & & \downarrow{\scriptstyle f_{(c,\gamma_c)(c,\delta_c)}} & & \\
X_{(a,\delta_a)} & & X_{(b,\delta_b)} & & X_{(c,\delta_c)} & & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{2}$$

Step 2. Put $B = \{(a,\gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$ and put C to be the set of all subsets c of B of the form

$$c = \{(a,\gamma_a) : a \in A\}, \tag{3}$$

where every γ_a is the fixed element of Γ_a.

Step 3. Let D be a subset of C containing all $c \in C$ for which there exist the mappings

$$g_{(a,\gamma_a)(b,\gamma_b)}\colon X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}, \qquad b \geq a, \tag{4}$$

such that

$$\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\} \tag{5}$$

is an approximate inverse system and each diagram

$$\begin{tikzcd} X_a & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\ X_{(a,\gamma_a)} \arrow[from=u, "f_{(a,\gamma_a)}"'] & X_{(b,g_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"] \end{tikzcd} \tag{6}$$

commutes, where $f_{(a,\gamma_a)}\colon X_a \to X_{(a,\gamma_a)}$ is the canonical projection. It is clear that the mapping $g_{(a,\gamma_a)(b,\gamma_b)}$ is unique since $f_{(b,\gamma_b)}$ is a surjection.

Step 4. *The set D is non-empty. Moreover, for each subset $S_a \subset \Gamma_a$, $a \in A$, $\mathrm{card}(S_a) \leq \aleph_0$, there exists a $d \in D$ such that $d = \{(a,\gamma_a) : a \in A\}$, $\gamma_a \geq \gamma$ for every $\gamma \in S_a$.* Let $a \in A$ be some first element of A and let $\gamma_a \in \Gamma_a$ such that $\gamma_a \geq \gamma$ for every $\gamma \in S_a$. The space $X_{(a,\gamma_a)}$ is a metric compact space and there exist mappings $f_{(a,\gamma_a)}p_{ab}\colon X_b \to X_{(a,\gamma_a)}$, $b \geq a$. By virtue of Theorem 1 for each $b \geq a$ there exists a $\gamma_b^1 \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^1, \gamma$, where $\gamma \in S_b$, there exists a monotone surjective mapping $g_{(a,\gamma_a)(b,\gamma_b)}\colon X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}$ with $f_{(a,\gamma_a)}p_{ab} = g_{(a,\gamma_a)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{tikzcd} X_a & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\ X_{(a,\gamma_a)} \arrow[from=u, "f_{(a,\gamma_a)}"'] & X_{(b,g_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"] \end{tikzcd} \tag{7}$$

commutes. Suppose that (a,γ_b^1), (a,γ_b^2), ... , (a,γ_b^{n-1}) are defined for each $a \in A$ with $p(a) \leq n-1$ such that the each diagram (6) commutes. Let $a \in A$ be a member of A with $p(a) = n$. This means that (a,γ_b^1), (a,γ_b^2), ... , (a,γ_b^{n-1}) are defined. From the cofinitness of A it follows that the set of γ_a^j which are defined in Γ_a is finite. Hence there exists $\gamma_a^n \geq \gamma_a^{n-1}, \ldots, \gamma_a^1$. We define $\gamma_b^n \in \Gamma_b$ considering the space $X_{(a,\gamma_a^n)}$ and the mappings $f_{(a,\gamma_a^n)}p_{ab}\colon X_b \to X_{(a,\gamma_a^n)}$. Again, by Theorem 1 for each $b \geq a$ there exists a $\gamma_b^n \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^n, \gamma_b^{n-1}, \ldots, \gamma_b^1$ and there is a mapping $g_{(a,\gamma_b)(b,\gamma_b)}\colon X_{(b,\gamma_b)} \to X_{(a,\gamma_a^n)}$ with $f_{(a,\gamma_a^n)}p_{ab} = g_{(a,\gamma_b)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{tikzcd} X_a & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\ X_{(a,\gamma_a^n)} \arrow[from=u, "f_{(a,\gamma_a^n)}"'] & X_{(b,g_b)} \arrow[l, "g_{(a,\gamma_a^n)(b,\gamma_b)}"] \end{tikzcd} \tag{8}$$

commutes. By induction on A (Lemma 7) the set D is defined. It remains to prove that $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system. Let $\mathcal{U}$ be a normal cover of $X_{(a,\gamma_a)}$. Then $\mathcal{V} = f_{(a,\gamma_a)}^{-1}(\mathcal{U})$ is a normal cover of X_a. By virtue of (A2) there exists a $b \geq a$ such that for each $c \geq d \geq b$ we have $(p_{ad}, p_{ca}p_{cd} \leq \mathcal{V}$. By virtue of the commutativity of the diagrams of the form (8) it follows that

$$(g_{(a,\gamma_a)(d,\gamma_d)}, g_{(a,\gamma_a)(c,\gamma_c)}g_{(c,\gamma_c)(d,\gamma_d)}) \leq \mathcal{V}.$$

Thus, $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system.

Step 5. We define a partial order on D as follows. Let d_1, d_2 be a pair of members of D such that $d_1 = \{(a,\gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$ and $d_2 = \{(a,\delta_a) : a \in A, \delta_a \in \Gamma_a\}$. We write $d_2 \le d_1$ if and only if $\delta_a \le \gamma_a$ for each $a \in A$. From Step 4. it follows that $(D, \le)$ is τ-directed. Moreover, X_D is a usual inverse system.

Step 6. *For each $d \in D$ the limit space X_d of the inverse system (5) is a compact space. Moreover, there exists a mapping $F_d \colon X \to X_d$.* The existence of F_d follows from the commutativity of the diagram (6). The following diagram illustrates the construction of $d \in D$ and the space X_d.

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\delta_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\delta_b)}"] & X_c \arrow[l, "p_{bc}"'] \arrow[d, "f_{(c,\delta_c)}"] & X \arrow[l, "p_d"'] \\
X_{(a,\delta_a)} \arrow[d, "f_{(a,\gamma_a)(a,\delta_a)}"] & X_{(b,\delta_b)} \arrow[d, "f_{(b,\gamma_b)(b,\delta_b)}"] & X_{(c,\delta_c)} \arrow[d, "f_{(c,\gamma_c)(c,\delta_c)}"] & \\
X_{(a,\gamma_a)} \arrow[d] & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"] \arrow[d] & X_{(c,\gamma_c)} \arrow[l, "g_{(b,\gamma_b)(c,\gamma_c)}"] \arrow[d] & X_d \arrow[l, "g_{(c,\gamma_c)}"] \\
{} & {} & {} &
\end{tikzcd} \tag{9}$$

Step 7. If d_1, d_2 is a pair of members of D such that $d_1 = \{(a,\gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$, $d_2 = \{(a,\delta_a) : a \in A, \delta_a \in \Gamma_a\}$ and $d_2 \ge d_1$, then for each $a \in A$ commutes the diagram

$$\begin{tikzcd}
X_{(a,\delta_a)} \arrow[d, "f_{(a,\gamma_a)(a,\delta_a)}"'] & X_{(b,\delta_b)} \arrow[l, "g_{(a,\delta_a)(b,\delta_b)}"'] \arrow[d, "f_{(b,\gamma_v)(b,\delta_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"]
\end{tikzcd} \tag{10}$$

This follows from the surjectivity of the mappings $f_{(b,\gamma_b)}$ and from the commutativity of the diagrams of the form (6) for d_1 and d_2, i.e., from the commutativity of the diagrams

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a)}"'] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,g_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"]
\end{tikzcd} \tag{11}$$

and

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\delta_a)}"'] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\delta_b)}"] \\
X_{(a,\delta_a)} & X_{(b,d_b)} \arrow[l, "g_{(a,\delta_a)(b,\delta_b)}"]
\end{tikzcd} \tag{12}$$

Step 8. From Step 7. it follows that for $d_1, d_2 \in D$ with $d_2 \ge d_1$ there exists a mapping $F_{d_1 d_2} \colon X_{d_2} \to X_{d_1}$ (see [1, p. 138]) such that $F_{d_1} = F_{d_1 d_2} F_{d_2}$.

Proof of Step 8. Let $d_1, d_2, d_3 \in D$ and let $d_1 \le d_2 \le d_3$. Then $F_{d_1 d_3} = F_{d_1 d_2} F_{d_2 d_3}$. This follows from Step 7. and the commutativity condition in each inverse system $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ (see (1) of Step 1.).

Step 9. *The collection $\{X_d, F_{de}, D\}$ is a usual inverse system of compact spaces.*

Apply Steps 1.–8.

Step 10. *There is a mapping $F\colon X \to X_D$ which is 1–1.*

By Step 6. and Step 8. for each $d \in D$ there is a mapping $F_d\colon X \to X_d$ such that $F_{d_1} = F_{d_1 d_2} F_{d_2}$ for $d_2 \ge d_1$. This means that there exists a mapping $F\colon X \to X_D$ [1, p. 138]. Let us prove that F is 1–1. Take a pair x, y of distinct points of X. There exists an $a \in A$ such that $x_a = p_a(x)$ and $y_a = p_a(y)$ are distinct points of X_a. Now, there exists an (a, γ_a) such that $f_{(a,\gamma_a)}(x_a)$ and $f_{(a,\gamma_a)}(y_a)$ are distinct points of $X_{(a,\gamma_a)}$. From Step 4. it follows that there is a $d \in D$ such that $F_d(x)$ and $F_d(y)$ are distinct points of X_d. Thus, F is 1–1.

Step 11. *The mapping F is a homeomorphism onto X_D.* Let y be a point of X_D. Let us prove that there exists a point $x \in X$ such that $F(x) = y$. For each $d \in D$ we have a point $y_d = F_d(y)$. Now, we have the points $g_{(a,\gamma_a)} F_d(y)$ in $X_{(a,\gamma_a)}$ and the subsets $Y_a = f^{-1}_{(a,\gamma_a)}(g_{(a,\gamma_a)} F_d(y))$ of X_a. Let U be an open neighborhood Y_a. There exists an open neighborhood V of $g_{(a,\gamma_a)} F_d(y)$ such that $f^{-1}_{(a,\gamma_a)}(V) \subseteq U$. We infer that $\mathrm{Ls}\{g_{(b,\gamma_b)}(Y_b) : b \ge a\} \subseteq Y_a$ since $g_{(a,\gamma_a)} F_d(y) = \lim\{g_{(a,\gamma_a)(b,\gamma_b)} g_{(b,\gamma_b)} F_d(y) : b \ge a\}$ and the diagrams (6) commute. By virtue of [6, Lemma 2.1] it follows that there exists a non-empty closed subset C_d of $\lim \mathbf{X}$ such that $p_b(C_d) \subseteq Y_b$. The family $\{C_d : d \in D\}$ has the finite intersection property. This means that $X' = \bigcap\{C_d : d \in D\}$ is non-empty. For each $x \in X'$ we have $F_d(x) = F_d(y)$, $d \in D$. Thus, $F(y) = x$. The proof is completed. ■

By the similar method of proof we obtain the following theorem.

Theorem 10. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact non-metric spaces with surjective bonding mappings p_{ab}. If each X_a is a limit of a usual τ-directed inverse system $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ of compact spaces with $w(X_{(a,\gamma)}) \le \tau$, then:*

1. *there exists a usual τ-directed inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$,*
2. *every X_d is a limit of an approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ of compact spaces $X_{(a,\gamma_a)}$,*
3. *if the mappings p_{ab} and $f_{(a,\gamma)(a,\delta)}$ are monotone, then $g_{(a,\gamma_a)(b,\gamma_b)}$ and F_{de} are monotone.*

Corollary 3. *Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of compact non-metric spaces with surjective bonding mappings p_{nm}. If each X_n is a limit of a usual σ-directed inverse system $\mathbf{X}(n) = \{X_{(n,\gamma)}, f_{(n,\gamma)(n,\delta)}, \Gamma_n\}$ of metric compact spaces, then:*

1. *there exists a usual σ-directed inverse system* $\mathbf{X}_D = \{X_d, F_{de}, D\}$ *whose inverse limit* X_D *is homeomorphic to* $X = \lim \mathbf{X}$,
2. *every* X_d *is a limit of an approximate inverse sequence* $\{X_{(n,\gamma_n)}, g_{(n,\gamma_n)(m,\gamma_m)}, \mathbb{N}\}$ *of compact metric spaces* $X_{(n,\gamma_n)}$,
3. *if the mappings* p_{nm} *and* $f_{(n,\gamma)(n,\delta)}$ *are monotone, then* $g_{(n,\gamma_n)(m,\gamma_m)}$ *and* F_{de} *are monotone.*

Let $\mathcal{P}$ be a topological property of spaces.

THEOREM 11. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact non-metric spaces with surjective bonding mappings* p_{ab} *and let* $\mathcal{P}$ *be a topological property of spaces such that:*

1. *each* X_a *is a limit of a usual σ-directed inverse system* $\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ *of metric compact spaces with property* $\mathcal{P}$,
2. *if* X_d *is a limit of an approximate inverse system* $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ *of compact metric spaces* $X_{(a,\gamma_a)}$ *with property* $\mathcal{P}$, *then* X_d *has* $\mathcal{P}$,
3. *if* Y *is a limit of σ-directed usual inverse system of compact spaces with property* $\mathcal{P}$, *then* Y *has* $\mathcal{P}$.

Then $X = \lim \mathbf{X}$ *has the property* $\mathcal{P}$.

3. Applications

LEMMA 9. *Let* $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ *be an approximate inverse sequence of locally connected metric continua. If the bonding mappings are monotone and surjective, then* $X = \lim \mathbf{X}$ *is locally connected.*

Proof. There exists a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$, $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$ for each $i, j \in \mathbb{N}$ and a homeomorphism $H\colon \lim \mathbf{X} \to \lim \mathbf{Y}$ [2, Proposition 8]. Each mapping q_{ij} as a composition of the monotone mappings is monotone. This means that $\mathbf{Y}$ is a usual inverse sequence of locally connected continua with monotone bonding mappings q_{ij}. Hence $\lim \mathbf{Y}$ is locally connected. We infer that $X = \lim \mathbf{X}$ is locally connected since there exists a homeomorphism $H\colon \lim \mathbf{X} \to \lim \mathbf{Y}$. ■

LEMMA 10. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of locally connected continua such that* $\operatorname{card}(A) = \aleph_0$. *Then* $X = \lim \mathbf{X}$ *is locally connected.*

Proof. By virtue of Lemma 3 there exists a countable well-ordered subset B of A such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. From Lemma 9 it follows that $\lim\{X_b, p_{bc}, B\}$ is locally connected. Hence $X = \lim \mathbf{X}$ is locally connected. ■

LEMMA 11. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of locally connected metric continua and monotone bonding mappings. Then* $X = \lim \mathbf{X}$ *is locally connected.*

Proof. If $\operatorname{card}(A) = \aleph_0$ then we apply Lemma 10. If $\operatorname{card}(A) \geq \aleph_1$ then from Corollary 1 it follows that $X = \lim \mathbf{X}$ is homeomorphic to the limit of a σ-directed usual inverse system $\{X_\alpha, q_{\alpha\beta}, \Delta\}$, where each X_α is a limit of an approximate inverse subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \aleph_0$. From Lemma 10 it follows that each X_a is locally connected. By Theorem 3 we infer that the limit of $\{X_\alpha, q_{\alpha\beta}, \Delta\}$ is locally connected. Hence, X is locally connected since X is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, \Delta\}$. ■

THEOREM 12. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of locally connected continua and monotone bonding mappings. Then $X = \lim \mathbf{X}$ is a locally connected continuum.*

Proof. By virtue of Theorem 3 and Remark 1 every X_a is a limit of a usual σ-directed inverse system $X(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ of metric locally connected continua with monotone bonding mappings $f_{(a,\gamma)(a,\delta)}$. From Theorem 9 it follows that there exist : 1) a usual σ-directed inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$; 2) every X_d is a limit of an approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ of compact metric spaces $X_{(a,\gamma_a)}$ and 3) if the mappings p_{ab} and $f_{(a,\gamma)(a,\delta)}$ are monotone, then $g_{(a,\gamma_a)(b,\gamma_b)}$ and F_{de} are monotone. Now, every X_d as the limit of approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is locally connected because of Lemma 11. Finally, X is locally connected since X is homeomorphic to $X_D = \lim \mathbf{X}_D$ and X_D is locally connected (Theorem 3). ■

We shall say that a non-empty compact space is *perfect* if it has no isolated points.

A continuum is said to be *totally regular* [12, p. 47] if for each $x \neq y$ in X there is a positive integer n and perfect subsets $A_1, \dots, A_n$ of X such that $x_i \in A_i$ for $i = 1, \dots, n$ implies that $\{x_1, \dots, x_n\}$ separates x from y in X.

LEMMA 12. [12, Proposition 7.4] *Each totally regular continuum is hereditarily locally connected and rim-finite.*

The following theorem is a part of [12, Theorem 7.15].

THEOREM 13. *If X is a continuum then the following conditions are equivalent:*

1. *X is totally regular,*
2. *X is homeomorphic to $\lim\{X_a, f_{ab}, \Gamma\}$ such that each X_a is a totally regular continuum and each f_{ab} is a monotone surjection.*

THEOREM 14. [12, Theorem 7.7] *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of totally regular continua X_a and monotone surjective mappings p_{ab}. Then $X = \lim \mathbf{X}$ is totally regular.*

THEOREM 15. *Let X be a non-metric totally regular continuum. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is totally regular, each f_{ab} is a monotone surjection and X is homeomorphic to $\lim \mathbf{X}$.*

Proof. Apply [12, Theorem 9.4], Theorem 14 and Lemma 3.5 of [14]. ■

Now we shall prove the following theorem.

THEOREM 16. *Let* $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ *be an approximate inverse sequence of totally regular metric continua. If the bonding mappings are monotone and surjective, then* $X = \lim \mathbf{X}$ *is totally regular.*

Proof. There exists a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$, $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$ for each $i, j \in \mathbb{N}$ and a homeomorphism $H\colon \lim \mathbf{X} \to \lim \mathbf{Y}$ [2, Proposition 8]. Each mapping q_{ij} as a composition of the monotone mappings is monotone. This means that $\mathbf{Y}$ is a usual inverse sequence of totally regular continua with monotone bonding mappings q_{ij}. By virtue of Theorem 14 $\lim \mathbf{Y}$ is totally regular. We infer that $X = \lim \mathbf{X}$ is totally regular since there exists a homeomorphism $H\colon \lim \mathbf{X} \to \lim \mathbf{Y}$. ∎

THEOREM 17. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua such that* $\mathrm{card}(A) = \aleph_0$. *Then* $X = \lim \mathbf{X}$ *is totally regular.*

Proof. By virtue of Lemma 3 there exists a countable well-ordered subset B of A such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. From Theorem 16 it follows that $\lim\{X_b, p_{bc}, B\}$ is totally regular. Hence $X = \lim \mathbf{X}$ is totally regular. ∎

THEOREM 18. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua and monotone bonding mappings. If* $w(X_a) < \tau < \mathrm{card}(A)$ *for each* $a \in A$, *then* $X = \lim \mathbf{X}$ *is a totally regular continuum.*

Proof. By virtue of Theorem 7 (for $\lambda = \aleph_0$) there exists a σ-directed inverse system $\{X_\alpha, q_{\alpha\beta}, T\}$, where each X_α is a limit of an approximate inverse subsystem $\{X_\gamma, p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \aleph_0$. From Theorem 17 it follows that every X_α is totally regular. Theorem 14 completes the proof. ∎

THEOREM 19. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular metric continua and monotone bonding mappings. Then* $X = \lim \mathbf{X}$ *is totally regular continuum.*

Proof. If $\mathrm{card}(A) = \aleph_0$ then we apply Theorem 17. If $\mathrm{card}(A) \geq \aleph_1$ then from Theorem 18 it follows that X is totally regular. ∎

THEOREM 20. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular non-metric continua with surjective monotone bonding mappings* p_{ab}. *Then* $X = \lim \mathbf{X}$ *is totally regular.*

Proof. By virtue of Theorem 15 every X_a is a limit of a usual σ-directed inverse system $X(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ of metric totally regular continua with monotone bonding mappings $f_{(a,\gamma)(a,\delta)}$. From Theorem 9 it follows that there exist: 1) a usual σ-directed inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$, 2) every X_d is a limit of an approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ of compact metric spaces $X_{(a,\gamma_a)}$ and 3) if the mappings p_{ab} and $f_{(a,\gamma)(a,\delta)}$ are monotone, then $g_{(a,\gamma_a)(b,\gamma_b)}$ and F_{de} are monotone. Now, every X_d as the limit of approximate inverse system $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$

is totally regular because of Theorem 19. Finally, X is totally regular since X is homeomorphic to $X_D = \lim \mathbf{X}_D$ and X_D is totally regular (Theorem 14). ∎

THEOREM 21. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua with surjective monotone bonding mappings* p_{ab}. *Then* $X = \lim \mathbf{X}$ *is totally regular.*

Proof. Apply Theorems 19 and 20. ∎

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Faculty of Organization and Informatics Varaždin, Croatia

E-mail: ivan.loncar1@vz.tel.hr