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BASES IN SEQUENCE SPACES AND EXPANSION OF A FUNCTION IN A SERIES OF POWER SERIES

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Abstract. In this paper we establish a relation between the existence of a basis and the solution of an infinite linear system. Then we study the expansion of a function in a series of power series.

1. Introduction

Bases in infinite dimensional sequence spaces have been studied by many authors. Concerning Schauder bases, let us cite Wilansky [15] and more recently Malkowsky [7] and [8]. New Schauder bases have been found in the spaces $c_0(\Delta^\mu)$ and $c(\Delta^\mu)$ by the last author. We also have many results concerning AK spaces, for example in the Λ-strongly null and Λ-strongly convergent sequence spaces. In this work we establish a relation between the notion of a basis and the solution of an infinite linear system. Let us recall that some results concerning linear infinite systems have been studied in Cooke [1]. We can find results concerning Pólya systems in Petersen and Baker [12] and Petersen [13]. More recently, some results concerning summability have been put together in Maddox [4], we find a study on the Walsh functions in Mursaleen [9]. R. Labas and B. de Malafosse [2, 3] gave an application of the theory of the sum of operators to the theory of infinite matrices. Note that infinite linear systems have been used for the study of the spectrum of the Cesàro operator in certain spaces, see Reade [14], Okutoyi [11] and de Malafosse [6].

The plan of this paper is organized as follows. In section 2, we recall some results concerning AK spaces and Schauder basis [7, 8]. Next, we recall the spaces s_c and S_c (see [5, 6, 9]) and define bases of α, β type. In section 3, we study how a function can have an expansion in a given series of power series. In the first subsection, we prove that this problem is equivalent to the existence of a solution of an infinite linear system and we give an application where the Cesàro operator is used. In the subsection 2 are given some properties of such expansions. In the third subsection, we study a particular expansion and we study when a function can have a unique expansion, or infinitely many expansions, or no expansion at all.

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2. Bases in certain sequence spaces

2.1. Some known results

A Banach space E of complex sequences, with the norm $\|\cdot\|_E$ is called a BK space if every projection $P_n : E \to C$, defined by $P_n(X) = x_n$ is continuous. Denote by e_n, $n \geqslant 0$, the vector $(0, \dots, 1, 0, \dots)$, where 1 is in the $(n+1)$-st position and by e the vector $(1, 1, \dots)$. A BK space E is said to have AK (see [15]), if for every $B = (b_n)_{n\geqslant 0} \in E$, $B = \sum_{m=0}^{\infty} b_m e_m$, i.e.

$$\Big\| B - \sum_{m=0}^{n} b_m e_m \Big\|_E \to 0 \qquad (n \to \infty).$$

It is well-known [15], that the space l^p, $p > 0$, of the sequences $X = (x_n)_{n\geqslant 0}$ such that the series $\sum_n |x_n|^p$ are convergent have AK with respect to the norm $\|X\|_{l^p} = (\sum_n |x_n|^p)^{1/p}$. It is the same for the space c of all convergent sequences, with the norm $\|X\|_{l^\infty} = \sup_{n\geqslant 0} |x_n|$. In this last case, if $B = (b_n)_{n\geqslant 0} \in c$, and $(b_n)_{n\geqslant 0} \to b$, we have a unique representation $B = be + \sum_{m=0}^{\infty} (b_m - b) e_m$, i.e.

$$\Big\| B - \Big(be + \sum_{m=0}^{N} (b_m - b) e_m \Big) \Big\|_{l^\infty} = \sup_{m \geqslant N+1} |b_m - b| \to 0, \quad (N \to \infty).$$

Elsewhere, $(\hat{a}_m)_{m\geqslant 0}$ is a Schauder basis of the normed vector space E, if for all $B \in E$ there is a unique sequence $X = (x_n)_{n\geqslant 0}$, such that $B = \sum_{m=0}^{\infty} x_m \hat{a}_m$, in the space E. That is

$$\lim_{N\to\infty} \Big\| B - \sum_{m=0}^{N} x_m \hat{a}_m \Big\|_E = 0.$$

Note that l^∞ has no Schauder basis.

Here, we shall consider sequences as column vectors and give other bases in new spaces. Then we must recall definitions of useful spaces permitting to do calculations on infinite linear systems.

2.2. Spaces S_α and s_α

In the following, we shall use infinite linear systems defined by

$$\sum_{m=0}^{\infty} a_{nm} x_m = b_n, \qquad n = 0, 1, 2, \dots$$

Such a system can be written as a matrix equation

$$AX = B,$$

where $A = (a_{nm})_{n,m\geqslant 0}$ and X, B are the one column matrices defined respectively by $(x_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$. The following spaces have been defined, for instance, in [2] and [3]. For a sequence $\alpha = (\alpha_n)_{n\geqslant 0}$, where $\alpha_n > 0$, for every $n \geqslant 0$, we consider the Banach algebra

$$S_\alpha = \Big\{ A = (a_{nm})_{n,m\geqslant 0} \;\Big|\; \sup_{n\geqslant 0} \Big(\sum_{m=0}^{\infty} |a_{nm}| \frac{\alpha_m}{\alpha_n} \Big) < \infty \Big\}, \tag{1}$$

normed by

$$\|A\|_{S_\alpha} = \sup_{n\geqslant 0}\Big(\sum_{m=0}^{\infty} |a_{nm}|\frac{\alpha_m}{\alpha_n}\Big). \tag{2}$$

We also define the Banach space s_α of one-column matrices by

$$s_\alpha = \Big\{\ X = (x_n)\ \Big|\ \sup_{n\geqslant 0}\Big(\frac{|x_n|}{\alpha_n}\Big) < \infty\ \Big\}, \tag{3}$$

normed by

$$\|X\|_{s_\alpha} = \sup_{n\geqslant 0}\Big(\frac{|x_n|}{\alpha_n}\Big). \tag{4}$$

If $\alpha = (\alpha_n)_{n\geqslant 0}$ and $\beta = (\beta_n)_{n\geqslant 0}$ are two sequences such that $0 < \alpha_n \leqslant \beta_n$ for each n, then $s_\alpha \subset s_\beta$.

A very useful special case is the one where $\alpha_n = r^n$, $r > 0$. We denote, then, by S_r and s_r, the spaces S_α and s_α. When $r = 1$, we obtain the space of bounded sequences $l^\infty = s_1$. If

$$\|I - A\|_{S_\alpha} < 1, \tag{5}$$

then A is invertible in the space S_α.

S_α being a unit algebra, we have the useful result: if we suppose that (5) holds, then for every $B \in s_\alpha$, the equation $AX = B$ admits one and only one solution in s_α, given by

$$X = \sum_{k=0}^{\infty}(I - A)^k B. \tag{6}$$

We have seen [3] that a matrix A, which verifies (5), for a given sequence $\alpha = (\alpha_n)_{n\geqslant 0}$ is not necessarily of Pòlya type (see [1, 12, 13]). Recall that a matrix A satisfies the Pòlya condition if

$$\liminf_{m} C(n,m) = 0, \qquad n = 1, 2, \dots$$

where

$$C(n,m) = \frac{|a_{0m}| + |a_{1m}| + \cdots + |a_{n-1,m}|}{|a_{nm}|}.$$

We see that a matrix $A = (a^{|m-n|})_{n,m\geqslant 0}$, where $0 < a < 1/3$, verifies (5), when we replace S_α by S_1, but, for all $n \geqslant 1$, $\sum_{k=0}^{n-1} |\frac{a_{km}}{a_{nm}}| \geqslant a > 0$, which shows that it is not of Pòlya type.

2.3. Bases of α, β type

In the following $\alpha = (\alpha_n)_{n\geqslant 0}$ and $\beta = (\beta_n)_{n\geqslant 0}$ are two given sequences that satisfy $0 < \alpha_n \leqslant \beta_n$ for all n.

Definition 1. We shall say that $(\hat{a}_m)_{m\geqslant 0}$ is a basis of the α, β type if $\hat{a}_m \in s_\alpha$ for all $m \geqslant 0$ and for every $B \in s_\alpha$ there is a unique sequence $X = (x_n)_{n\geqslant 0}$, such that

$$B = \sum_{m=0}^{\infty} x_m \hat{a}_m$$

in the space s_β.

If there are two reals r_1, r_2 with $0 < r_1 \leqslant r_2$ such that $\alpha = (r_1^n)_{n\geqslant 0}$ and $\beta = (r_2^n)_{n\geqslant 0}$, we shall say that $(\hat{a}_m)_{m\geqslant 0}$ is a basis of r_1, r_2 type.

In order to establish a relation between the solution of an infinite linear system and the notion of basis, we need a lemma.

Lemma 2. *Let $\hat{a}_m = {}^t(a_{0m}, \dots, a_{nm}, \dots) \in s_\alpha$ for all $m \geqslant 0$ and assume that the series $\sum_{m=0}^{\infty} x_m \hat{a}_m$ is convergent in the space s_β. The following assertions are equivalent:*

i) $(\hat{a}_m)_{m\geqslant 0}$ is a basis of α, β type;

ii) equation $AX = B$ admits only one solution, for all $B \in s_\alpha$.

Proof. Suppose that i) is satisfied. For all $B \in s_\alpha$ there exists $X = (x_n)_{n\geqslant 0}$ such that $\|B - \sum_{m=0}^{N} x_m \hat{a}_m\|_{s_\beta}$ tends to 0 as N tends to infinity. We have

$$\sup_{n\geqslant 0} \left(\frac{|b_n - \sum_{m=0}^{N} a_{nm} x_m|}{\beta_n} \right) = o(1) \quad \text{as} \quad N \to \infty.$$

Then for every value of n, $|b_n - \sum_{m=0}^{N} a_{nm} x_m| = \beta_n o(1)$ as $N \to \infty$ and $b_n = \sum_{m=0}^{\infty} a_{nm} x_m$, which proves that $B = AX$. Hence we deduce ii).

Conversely, denote by $X = (x_n)_{n\geqslant 0}$ the solution of the equation $AX = B$, $B \in s_\alpha$. We see that $B = AX = (\sum_{m=0}^{\infty} a_{nm} x_n)_{n\geqslant 0}$, and $B \in s_\beta$ since $s_\alpha \subset s_\beta$. Then for all N

$$\Big\| B - \sum_{m=0}^{N} x_m \hat{a}_m \Big\|_{s_\beta} = \Big\| \Big(\sum_{m=N+1}^{\infty} a_{nm} x_m \Big)_{n\geqslant 0} \Big\|_{s_\beta}.$$

The last term of the identity is equal to $\|\sum_{m=N+1}^{\infty} x_m \hat{a}_m\|_{s_\beta}$, which converges to 0, as N tends to infinity, since the series $\sum_{m=0}^{\infty} x_m \hat{a}_m$ is convergent in the space s_β. This proves that $(\hat{a}_m)_{m\geqslant 0}$ is a basis of α, β type. ∎

Remark 1. Note first that we can have $\sum_{m=0}^{\infty} x_m \hat{a}_m = 0$ in a space s_{r_2}, for a sequence $X = (x_n)_{n\geqslant 0} \neq 0$, although the vectors $\hat{a}_0$, $\hat{a}_1$, … are linearly independent (that is, if for all $k \in \mathbf{N}$, $\sum_{m=0}^{k} x_m \hat{a}_m = 0$, then $x_m = 0$, for all $m = 0, 1, \dots, k$. In fact, consider the family $(\hat{a}_m)_{m\geqslant 0}$ defined by ${}^t\hat{a}_0 = e_0$ and ${}^t\hat{a}_n = e_{n-1} + e_n$ for all $n \geqslant 1$. It is easy to see that these vectors are linearly independent. Moreover, for all $m \geqslant 0$, $\hat{a}_m \in s_{r_2}$, for any $r_2 > 1$ and we have for every integer N: $\|\sum_{m=0}^{N} (-1)^m \hat{a}_m\|_{s_{r_2}} = 1/r_2^{N+1}$, which tends to 0 as N tends to infinity. Then we have $\sum_{m=0}^{\infty} (-1)^m \hat{a}_m = 0$ in the space s_{r_2}.

In the following we shall use the increasing sequence $\Lambda = (\lambda_{n+1})_{n\geqslant 0}$ of strictly positive reals tending to infinity. Denote by $C = (a_{nm})_{n,m\geqslant 0}$ the well-known infinite triangle matrix defined from Λ by $a_{nm} = 1/\lambda_{n+1}$ for $0 \leqslant m \leqslant n$, the other coefficients being equal to 0 (see [7, 8, 15]). Recall that if $\lambda_n = n$ for all $n \geqslant 1$, C is called the Cesàro operator (see [6, 11, 14]). We shall assert the following reslusts in which

$$\hat{c}_m = {}^t(0, \dots, 1/\lambda_{m+1}, 1/\lambda_{m+2}, \dots), \quad m = 0, 1, \dots$$

the first non-zero coefficient being in the $(m + 1)$-st position.

PROPOSITION 3. *i) Let* $(\hat{a}_m)_{m\geqslant 0}$ *be the sequence defined by*

$$\hat{a}_m = {}^t(0,\dots,a_{m,m},a_{m+1,m},a_{m+2,m},\dots),$$

$m=0,1,2,\dots$, $a_{m,m}\neq 0$ *being in the* $(m+1)$*-st position. Assume that there is* $M>0$ *such that for all* $n,m\geqslant 0$*:*

$$|a_{nm}|\leqslant M \quad \text{and} \quad \left|\frac{a_{nm}}{a_{mm}}\right|\leqslant M. \tag{7}$$

For real $r_1>M+1$ *and* $r_2>r_1$, $(\hat{a}_m)_{m\geqslant 0}$ *is a basis of* r_1,r_2 *type.*

ii) For $2<r_1<r_2$, $(\hat{c}_m)_{m\geqslant 0}$ *is a basis of* r_1,r_2 *type.*

iii) Assume that $\alpha=(\alpha_n)_{n\geqslant 0}$ *and* $\beta=(\beta_n)_{n\geqslant 0}$ *are two sequences for which* $(\alpha_n)_{n\geqslant 0}$ *is increasing and*

$$\overline{\lim_{n\to\infty}}\left(\frac{\alpha_n}{\beta_n}\right)=0. \tag{8}$$

Then $(\hat{c}_m)_{m\geqslant 0}$ *is a basis of* α,β *type.*

Proof. Assertion i). Consider for all m the sequence defined by $\hat{a}'_m=a_{mm}^{-1}\hat{a}_m={}^t(0,\dots,1,\frac{a_{m+1,m}}{a_{mm}},\dots)$. As $r_1>1$, we deduce from (7) that

$$\sup_{n\geqslant m+1}(|a_{nm}|/|a_{mm}r^n|)<\infty.$$

Define the infinite matrix A', whose $(m+1)$-st column is $\hat{a}'_m$. We see that A' is a lower triangle, whose diagonal entries are equal to 1 and the coefficient in the $(n+1)$-st row and of the $(m+1)$-st column, with $0\leqslant m\leqslant n-1$ for $n\geqslant 1$, is equal to a_{nm}/a_{nn}. We have

$$\|I-A'\|_{S_{r_1}}=\sup_{n\geqslant 2}\left[\frac{1}{|a_{nn}|}\left(\frac{|a_{n0}|}{r_1^n}+\frac{|a_{n1}|}{r_1^{n-1}}+\dots+\frac{|a_{n,n-1}|}{r_1}\right)\right],$$

that is, using (7),

$$\|I-A'\|_{S_{r_1}}\leqslant\frac{M}{r_1-1}\sup_{n\geqslant 2}\left(1-\frac{1}{r_1^n}\right).$$

Hence if $r_1>M+1$ then $\|I-A'\|_{S_{r_1}}<1$. We deduce that for every $B\in s_{r_1}$, the equation $A'X=B$ admits only one solution $X=(A')^{-1}B\in s_{r_1}$. Furthermore, there exists a constant $K>0$ such that

$$|x_m|\|\hat{a}'_m\|_{s_{r_2}}\leqslant Kr_1^m\sup_{n\geqslant m+1}\left(\frac{|a_{nm}|}{|a_{mm}r_2^n|}\right)\leqslant\frac{KM}{r_2}\left(\frac{r_1}{r_2}\right)^m.$$

Then the series with general term $|x_m|\|\hat{a}'_m\|_{s_{r_2}}$ is convergent, since $r_2>r_1$. s_{r_2} being a Banach space, the series $\sum_{m=0}^\infty x_m\hat{a}'_m$ is convergent in s_{r_2}. Using Lemma 2 we deduce that for all $B\in s_{r_1}$ there exists a unique sequence $X=(x_n)_{n\geqslant 0}$ such that $B=\sum_{m=0}^\infty x_m\hat{a}'_m=\sum_{m=0}^\infty x_m a_{mm}^{-1}\hat{a}_m$. And since

$$\sup_{n\geqslant m}\left(\frac{|a_{nm}|}{r_1^n}\right)\leqslant\frac{M}{r_1^m}\leqslant M,$$

we see that $\hat{a}_m\in s_{r_1}$ for all $m\geqslant 0$. This completes the proof of i).

Assertion ii). We can apply i) to the sequence $(\hat{c}_m)_{m\geqslant 0}$. Indeed, Λ being increasing, we have here $a_{nm} = 1/\lambda_{n+1} \leqslant 1/\lambda_1$ and $\frac{|a_{nm}|}{|a_{mm}|} = \frac{\lambda_{m+1}}{\lambda_{n+1}} \leqslant 1$ for all $n \geqslant 0$ with $m \leqslant n$. It is enough to take $M = \sup(1, 1/\lambda_1)$.

Assertion iii). First we see that $\hat{a}_m \in s_\alpha$ for all $m \geqslant 0$. In fact, $\lim_{n\to\infty} \lambda_n = \infty$ and $\alpha_n \geqslant \alpha_0$ imply that $\sup_{n\geqslant m}(\frac{1}{\lambda_{n+1}\alpha_n}) < \infty$ for every m. Further, the matrix whose columns are successively $\hat{c}_0$, $\hat{c}_1$, ... is equal to C. The equation $CX = B$, where $B \in s_\alpha$, admits only one solution. To calculate this solution we need the infinite lower triangle matrix $\Delta = (\eta_{nm})_{n,m\geqslant 0}$ defined by $\eta_{n,n-1} = -1$ for $n \geqslant 1$, $\eta_{nn} = 1$ for $n \geqslant 0$, the other entries being equal to 0. We get then $C\Delta D = I$, where $D = (\lambda_{n+1}\delta_{nm})_{n,m\geqslant 0}$, with $\delta_{nn} = 1$ and $\delta_{nm} = 0$ for $m \neq n$ (see [6]). Hence $C^{-1} = \Delta D$. If we denote $C^{-1} = (c'_{nm})_{n,m\geqslant 0}$, then $c'_{nn} = \lambda_{n+1}$ for all $n \geqslant 0$ and $c'_{n,n-1} = -\lambda_n$ for $n \geqslant 1$. Calculating the product $C^{-1}B$ we deduce that:

$$\begin{cases} x_0 = b_0, \\ x_n = -\lambda_n b_{n-1} + \lambda_{n+1} b_n, & \text{if } n \geqslant 1. \end{cases} \tag{9}$$

The series $R_n = \sum_{m=N+1}^{\infty} x_m \hat{c}_m$, for a given integer N, is equal to the product

$$R_N = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\lambda_{N+2}} & 0 & 0 & \dots & 0 \\ \frac{1}{\lambda_{N+3}} & \frac{1}{\lambda_{N+3}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} x_{N+1} \\ x_{N+2} \\ \dots \\ \dots \\ \dots \end{pmatrix}.$$

If we set $R_N = (y_n)_{n\geqslant 0}$, then $y_n = 0$ for all $n = 0, 1, \dots, N$, $y_{N+1} = x_{N+1}/\lambda_{N+2}$ and $y_n = \frac{1}{\lambda_{n+1}}(\sum_{m=N+1}^{n} x_m)$ for $n \geqslant N+2$. We have

$$\Big\| \sum_{m=N+1}^{\infty} x_m \hat{c}_m \Big\|_{s_\beta} = \sup_{n\geqslant N+1} \left(\frac{|y_n|}{\beta_n} \right).$$

For $n = N+1$, there is a constant $K_1 > 0$ such that

$$\frac{|y_{N+1}|}{\beta_{N+1}} \leqslant \frac{K}{\lambda_{N+2}} \frac{\alpha_{N+1}}{\beta_{N+1}}. \tag{10}$$

Using (9), we obtain for $n \geqslant N+2$

$$\frac{|y_n|}{\beta_n} = \frac{|\lambda_{n+1} b_n - \lambda_{N+1} b_N|}{\lambda_{n+1}\beta_n}.$$

There is a constant $K_2 > 0$ such that

$$\frac{|y_n|}{\beta_n} \leqslant K_2 \left[\frac{\alpha_n}{\beta_n} + \frac{\lambda_{N+1}}{\lambda_{n+1}} \frac{\alpha_N}{\beta_n} \right] \leqslant 2K_2 \frac{\alpha_n}{\beta_n}, \tag{11}$$

since the sequences $\Lambda = (\lambda_{n+1})_{n\geqslant 0}$ and $(\alpha_n)_{n\geqslant 0}$ are increasing. Now put $\tau_1(N) = \frac{K_1}{\lambda_{N+2}} \frac{\alpha_{N+1}}{\beta_{N+1}}$ and $\tau_2(N) = 2K_2 \sup_{n\geqslant N+2}(\frac{\alpha_n}{\beta_n})$; (10) and (11) imply

$$\sup_{n\geqslant N+1} \left(\frac{|y_n|}{\beta_n} \right) \leqslant \sup(\tau_1(N), \tau_2(N)).$$

$\tau_1(N) \to 0$ as $N \to \infty$, since Λ tends to infinity and $0 < \alpha_n \leqslant \beta_n$ for all n. Using (8), we see that $\tau_2(N) \to 0$ as $N \to \infty$ and the conclusion follows from Lemma 2. ∎

REMARK 2. For any $X = (x_n)_{n\geqslant 0}$ let us denote $|X| = (|x_n|)_{n\geqslant 0}$. Consider the space of Λ-strongly null sequences $w(\Lambda)$. Recall that $B = (b_n)_{n\geqslant 0} \in w(\Lambda)$ if and only if $C(|B|) \in c_0$, that is $\lim_{n\to\infty}(\frac{1}{\lambda_n}\sum_{k=0}^{n}|b_k|) = 0$. It is well-known (see [7]) that $w(\Lambda)$ has AK with respect to the norm

$$\|B\| = \sup_{n\geqslant 0}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}|b_k|\Big),$$

i.e. $\|B - \sum_{m=0}^{N} b_m e_m\| \to 0$ as $N \to \infty$.

We shall consider now a new kind of basis using power series.

3. Expansion of a function in a series of power series

Until here we have studied the representation of a vector $B = (b_n)_{n\geqslant 0} \in s_\alpha$ as $B = \sum_{m=0}^{\infty} x_m(\sum_{n=0}^{\infty} a_{nm}e_n)$ in a space s_β for a given sequence $(a_{nm})_{n,m\geqslant 0}$. Analogously, we are going to see how a function f can be written as $f(z) = \sum_{m=0}^{\infty} x_m(\sum_{n=0}^{\infty} a_{nm}z^n)$ in a given disk $\overline{D}(0,R)$, the set of complex numbers z that satisfy $|z| \leqslant R$.

3.1. Definition and first properties

Let $A = (a_{nm})_{n,m\geqslant 0}$ be an infinite matrix and suppose that there are $r, R > 0$ such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|a_{nm}|R^n r^m < \infty. \tag{12}$$

Define by $\hat{A} = (\hat{A}_m)_{m\geqslant 0}$ the sequence whose general term is

$$\hat{A}_m(z) = \sum_{n=0}^{\infty} a_{nm}z^n, \qquad m = 0, 1, \dots$$

defined for $|z| \leqslant R$. We shall say that the infinite matrix A is associated to the sequence $\hat{A}$. We deduct from (12) that for every $m \geqslant 0$ the series $\sum_{n=0}^{\infty}|a_{nm}|R^n$ is convergent. Hence, if r_m denotes the radius of convergence of $\hat{A}_m(z)$, $m \geqslant 0$, we must have $r_m > 0$ for all m, and we conclude that $\inf_{m\geqslant 0}(r_m) \geqslant R$.

Denote by $E_{r,R}(\hat{A})$ the set of functions f, defined by the power series $f(z) = \sum_{n=0}^{\infty} b_n z^n$, such that there exists a sequence $X = (x_n)_{n\geqslant 0} \in s_r$ for which

$$f(z) = \sum_{m=0}^{\infty} x_m \hat{A}_m(z), \tag{13}$$

for all $z \in \overline{D}(0,R)$.

Notice that if ρ is the radius of convergence of f, we have $R < \rho$, since (13) is satisfied for all $z \in \overline{D}(0,R)$.

We shall give now a condition for which the vectors of the seqence $\hat{A} = (\hat{A})_{m\geqslant 0}$ are linearly independent, that is: if for any integer k and $z \in \overline{D}(0,R)$ the sum $\sum_{m=0}^{k} x_m \hat{A}_m(z) = 0$, then $x_m = 0$ for all $m = 0, 1, \dots, k$.

PROPOSITION 4. *For every integer k let $(n_i(k))_{0\leqslant i\leqslant k}$ be a sequence of integers with $n_0(k) < n_1(k) < \dots < n_k(k)$. Assume that for every k the determinant of the matrix $(a_{n_i(k),m})_{0\leqslant i,m<k}$ is different from 0. Then the vectors $\hat{A}_m$, $m = 0, 1, \dots$ are linearly independent.*

Proof. From the identities $\sum_{m=0}^{k} x_m \hat{A}_m(z) = \sum_{m=0}^{k} x_m (\sum_{n=0}^{\infty} a_{nm} z^n) = 0$, where $|z| \leqslant R$, we obtain that $\sum_{n=0}^{\infty}(\sum_{m=0}^{k} a_{nm} x_m) z^n = 0$. We obtain a linear system with infinitely many equations and a finite number of unknowns

$$\sum_{m=0}^{k} a_{nm} x_m = 0, \qquad n = 0, 1, \dots \tag{14}$$

Hence we have

$$\sum_{m=0}^{k} a_{n_i(k),m} x_m = 0, \qquad i = 0, 1, \dots, k,$$

and, since the determinant of the coefficients of the variables of this last system is different from zero, we deduct that $x_0 = x_1 = \dots = x_k = 0$. ∎

Now we can state the following results.

PROPOSITION 5. *Assume that (12) holds.*

i) Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $f \in E_{r,R}(\hat{A})$ if and only if the infinite linear system

$$\sum_{m=0}^{\infty} a_{nm} x_m = b_n, \qquad n = 0, 1, \dots \tag{15}$$

admits a solution in the space s_r.

ii) If we suppose that the matrix $A = (a_{nm})_{n,m\geqslant 0}$ satisfies the following conditions:

$$\sup_{n\geqslant 0}\Big(\sum_{m\neq n}^{\infty}\Big|\frac{a_{nm}}{a_{nn}}\Big| r^{m-n}\Big) < 1, \qquad \frac{b_n}{a_{nn}} = O(r^n) \quad (n \to \infty). \tag{16}$$

Then $f \in E_{r,R}(\hat{A})$ and the expansion is unique.

Proof. Assertion i). Suppose that

$$f(z) = \sum_{m=0}^{\infty} x_m \hat{A}_m(z) = \sum_{m=0}^{\infty} x_m \Big(\sum_{n=0}^{\infty} a_{nm} z^n\Big),$$

for all $z \in \overline{D}(0,R)$. Since we have (10), the order of symbols $\sum_n$ and $\sum_m$ can be interchanged. Then $f(z) = \sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} a_{nm} x_m) z^n$, and we deduce that (15) admits a solution in s_r. Conversely, assume that equation (15) admits a solution in s_r. The double series with general term $|a_{nm}||x_m||z|^n$ being convergent for all $z \in \overline{D}(0,R)$ and for all $X = (x_m)_n \in s_r$, we can write

$$f(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} a_{nm} x_m\Big) z^n$$

for $|z| \leqslant R$. And (12) implies $f(z) = \sum_{m=0}^{\infty} x_m (\sum_{n=0}^{\infty} a_{nm} z^n)$ for all $z \in \overline{D}(0,R)$, so $f \in E_{r,R}(\hat{A})$.

Assertion ii). Denote by D the infinite diagonal matrix $(\delta_{nm}/a_{nn})_{n,m\geqslant 0}$. From (16) we conclude $\|I - DA\|_{S_r} < 1$ and $DB \in s_r$. Them (13) is equivalent to $D(AX) = DB$. It is easy to verify that $D(AX) = (DA)X$. Hence $X = (DA)^{-1}DB$ is the unique solution of (15) in s_r. Finally, applying i) we conclude ii). ■

Definition 6. In the case when each $f \in E_{r,R}(\hat{A})$ admits a unique expansion as in (13) and satisfies Porposition 4, $\hat{A} = (\hat{A}_m)_{m\geqslant 0}$ is called an r, R basis.

We are going to give some expansions in nontrivial cases. Let us consider the first example, where the infinite matrix associated with $\hat{A}$ is of Pòlya type (see Subsection 2.2).

Example 7. Let a be a real with $0 < a < 1/3$ and set

$$\hat{A}_m(z) = \sum_{n=0}^{\infty} a^{|m-n|} z^n.$$

We are going to see that the function f, defined by $f(z) = \sum_{n=0}^{\infty} b_n z^n$, where $(b_n)_{n\geqslant 0}$ is bounded, belongs to $E_{1,R}(\hat{A})$ for all $R \in]0,1[$, is a $1, R$ basis. In fact, we have $a_{nn} = 1$ for every n and (16) is satisfied since $(b_n)_n \in s_1$ and $0 < a < 1/3$ implies

$$\sup_{n\geqslant 0} \Big(\sum_{m\neq n}^{\infty} a^{|m-n|} \Big) \leqslant 2 \sum_{k=1}^{\infty} a^k = \frac{2a}{1-a} < 1.$$

Furthermore,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a^{|m-n|} R^n = \sum_{n=0}^{\infty} \Big[R^n \Big(\sum_{m=0}^{n} a^m \Big) \Big] + \sum_{n=0}^{\infty} R^n \Big(\sum_{m=1}^{\infty} a^m \Big),$$

that is

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a^{|m-n|} R^n = \sum_{n=0}^{\infty} \Big[R^n \frac{1-a^{n+1}}{1-a} \Big] + \frac{a}{1-a} \sum_{n=0}^{\infty} R^n,$$

and this last term is equal to $\frac{1-a^2R}{(1-a)(1-R)(1-aR)}$. We see that condition (12) holds for all $R < 1$. Applying the previous proposition we conclude that $f \in E_{1,R}(\hat{A})$ and $\hat{A}$ is a $1, R$ basis.

Now consider the sequence $\Lambda = (\lambda_{n+1})_{n\geqslant 0}$ defined in Subsection 2.3 and set $\hat{C} = (\hat{C}_m)_{m\geqslant 0}$, where $\hat{C}_m$ is defined by the power series $\hat{C}_m = \sum_{n=m}^{\infty} \frac{1}{\lambda_{n+1}} z^n$, $m = 0, 1, 2, \ldots$, defined for $|z| < 1$. We can give the following result.

Proposition 8. *Assume that $\Lambda \in s_r$ for $r > 1$. Let $f(z) = \sum_{n=0}^{\infty} b_n z^n$ be a power series such that $(b_n)_{n\geqslant 0}$ is bounded. Then $f \in E_{r,R}(\hat{C})$ when $R < 1/r$, that is*

$$f(z) = b_0 \hat{C}_0(z) + \sum_{m=1}^{\infty} [\lambda_{m+1} b_m - \lambda_m b_{m-1}] \hat{C}_m(z), \tag{17}$$

with $|z| \leqslant R$. Furthermore, $\hat{C}$ is an r, R basis.

Proof. We saw in Subsection 2.3 that the infinite matrix $A = (a_{nm})_{n,m\geqslant 0}$ associated to $\hat{C}$ is equal to C. Let us show that condition (12) is satisfied. We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{\lambda_{n+1}}R^n r^m = \sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}}R^n\left(\frac{r^{n+1}-1}{r-1}\right),$$

and since the sequence $(1/\lambda_{n+1})_{n\geqslant 0}$ is bounded, $R < 1$ and $rR < 1$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{\lambda_{n+1}}R^n r^m = \frac{1}{r-1}\Big(r\sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}}(rR)^n - \sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}}R^n\Big).$$

We deduce that the series $\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{\lambda_{n+1}}R^n r^m$ is convergent. Then (17) is equivalent to the linear system $CX = B$. As in Proposition 3, $X = (x_n)_{n\geqslant 0}$ is defined by (9), it can be verified that it belongs to s_r for $r > 1$. Finally, it is easy to see that $\hat{C} = (\hat{C}_m)_{m\geqslant 0}$ is an r, R basis, since the matrix C is a lower infinite triangle matrix with non-zero entries on the main diagonal.

Remark 3. The previous proposition can be applied to the case where C is the Cesàro operator, since the sequence $\Lambda = (n+1)_{n\geqslant 0} \in s_r$ for all $r > 1$.

3.2. Properties of the set $E_{r,R}(\hat{A})$

In this part it is useful to associate to every power series f, defined by $f(z) = \sum_{n=0}^{\infty} b_n z^n$, the upper triangular infinite matrix

$$[f] = \begin{pmatrix} b_0 & b_1 & b_2 & \dots \\ 0 & b_0 & b_1 & \dots \\ 0 & 0 & b_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

That is, setting $[f] = (\alpha_{nm})_{n,m\geqslant 0}$, $\alpha(nm) = b_{m-n}$ when $m \geqslant n$, and $\alpha_{nm} = 0$ when $m < n$ (see [2]). The function

$$f \mapsto [f]$$

maps the algebra of functions which can be expanded in a power series, into the algebra of the corresponding matrices. Recall that if $b_0 \neq 0$, $[f]$ is invertible and $[1/f] = [f]^{-1}$.

In the following we can suppose that $a_{nn} = 1$ for all values of n. When $a_{nn} \neq 0$ for all n, we can refer to this case by replacing $\hat{A}_m(z)$ by $\widehat{A'_m}(z) = a_{mm}^{-1}\hat{A}_m(z)$. We have the next result.

Proposition 9. *Assume that (12) holds for r and $R > 0$.*

i) If $f, g \in E_{r,R}(\hat{A})$, then $f + g \in E_{r,R}(\hat{A})$.

ii) Suppose that for $r \geqslant 1/R$,

$$\sup_{n\geqslant 0}\Big(\sum_{m\neq n}^{\infty}|a_{nm}|r^{m-n}\Big) < 1. \tag{18}$$

If $f, g \in E_{r,R}(\hat{A})$, then $f \cdot g \in E_{r,R}(\hat{A})$.

iii) Suppose that $f(z) = \sum_{n=0}^{\infty} b_n z^n$ *with* $b_0 \neq 0$ *satisfies the condition*

$$\sum_{n=2}^{\infty} |b_n b_0 - b_{n-1} b_1| R^n < |b_0^2|, \tag{19}$$

with $R \geqslant 1/r$, *and that (18) holds. Then* $f \in E_{r,R}(\hat{A})$ *implies* $1/f \in E_{r,R}(\hat{A})$.

Proof. If f and g belong to $E_{r,R}(\hat{A})$, there are two sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ belonging to s_r such that $(f+g)(z) = \sum_{m=0}^{\infty}(x_m + y_m)\hat{A}_m(z)$ for all $z \in \overline{D}(0,R)$. Since s_r is a Banach space, $(x_n + y_n)_{n\geqslant 0} \in s_r$, which proves i).

Assertion ii). If $f, g \in E_{r,R}(\hat{A})$, then $[f]$ and $[g]$ belong to S_R since the expansion in (13) holds in the whole disk $\overline{D}(0,R)$. S_R being an algebra, we conclude that the infinite matrix $[f \cdot g] = [f][g] \in S_R$. Denote now $f(z)g(z) = \sum_{n=0}^{\infty} \gamma_n z^n$. The series $\sum_{n=0}^{\infty} |\gamma_n| R^n$ being convergent, we conclude that the sequence $\gamma = (\gamma_n)_{n\geqslant 0}$ belongs to $s_{1/R}$. Finally (18) is equivalent to $\|I - A\|_{S_r} < 1$, and since $\gamma \in s_{1/R}$ with $s_{1/R} \subset s_r$, the system

$$\sum_{m=0}^{\infty} a_{nm} x_m = \gamma_n, \quad n = 0, 1, \ldots$$

admits only one solution in s_r. Applying i) in Proposition 5, we conclude that $f \cdot g \in E_{r,R}(\hat{A})$.

Assertion iii). We shall modify the expression of $f(z)$ in order to know the behavior of ots inverse. So, let p_1 be the function defined by $p_1(z) = 1 - \frac{b_1}{b_0} z$ and consider the product

$$\frac{1}{b_0} f(z) p_1(z) = 1 + \sum_{n=2}^{\infty} \left(\frac{b_n}{b_0} - \frac{b_{n-1} b_1}{b_0^2} \right) z^n.$$

From (19) we conclude

$$\left\| I - \frac{1}{b_0}[f][p_1] \right\|_{S_R} = \sum_{n=2}^{\infty} \left| \frac{b_n}{b_0} - \frac{b_{n-1} b_1}{b_0^2} \right| R^n < 1. \tag{20}$$

Consider now the equation

$$[f]X = Y, \tag{21}$$

for any $Y \in s_R$. If we set $X = [p_1]X'$, (21) is equivalent to $([f][p_1])X' = Y$, since $[p_1] \in S_R$. Using (20), we see that this last equation admits a unique solution $X' = ([f][p_1])^{-1} Y$ in s_R. Then $X = [p_1]([f][p_1])^{-1} Y$ is the unique solution of (21) in the space s_R. Thus $[f]^{-1} = [p_1]([f][p_1])^{-1} \in S_R$, that is $[f]^{-1} = [1/f] \in S_R$. Now set $\frac{1}{f(z)} = \sum_{n=0}^{\infty} b'_n z^n$, the series $\sum_{n=0}^{\infty} |b'_n| R^n$ is convergent, and the sequence $B' = (b'_n)_{n\geqslant 0}$ belongs to $s_{1/R}$. So $B' \in s_r$, since $s_{1/R} \subset s_r$. From (18) we conclude that the system

$$\sum_{m=0}^{\infty} a_{nm} x_m = b'_n, \quad n = 0, 1, \ldots$$

admits only one solution in s_r. Using Proposition 5 we conclude that $1/f \in E_{r,R}(\hat{A})$. ∎

Remark 4. In the previous proposition, iii) holds when we replace the condition given by (19) by the following one: there exists $R_0 > 0$ such that $\sum_{n\geqslant 1}|b_n|R_0^n < |b_0|$. Indeed, we have then $\|I - \frac{1}{b_0}[f]\|_{S_{R_0}} < 1$. The choice of p_1 permits to take $R = R_1 > R_0$ as in (19) and we obtain a greater space $s_{R_1} \supset s_{R_0}$ in which the equation $AX = B$ admits only one solution (see the following application). p_1 has been chosen such that the coefficient of z in the polynomial $\frac{1}{b_0}f(z)p_1(z)$ is equal to 0 (see [9]). This method can be repeated, so p_2 can be chosen so that the coefficients of z and z^2 in $\frac{1}{b_0}f(z)p_1(z)p_2(z)$ are equal to 0, and $R_2 > R_1$ satisfy $\|I - \frac{1}{b_0}[fp_1p_2]\|_{S_{R_2}} < 1$, and so on.

3.2.1. An application

We shall adapt iii) in Proposition 9 in a case where we have not $a_{nn} = 1$ for all n. Consider the function $f(z) = \sum_{n=0}^{\infty}\frac{z^n}{(n!)^2}$. Here we choose the sequence $\Lambda = (\lambda_{n+1})_{n\geqslant 0}$ of Subsection 2.3 such that the series $\sum_{n=0}^{\infty} 1/\lambda_{n+1}$ is convergent. We are going to verify that the following expansion

$$\frac{1}{f(z)} = \sum_{m=0}^{\infty} x_m\Big(\sum_{n=m}^{\infty}\frac{1}{\lambda_{n+1}}z^n\Big) \tag{22}$$

holds for $|z| \leqslant R$ and $X \in s_r$. First we see that for $r = 1/R > 1$ the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\lambda_{n+1}}R^n r^m = \frac{1}{r-1}(r\sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}} - \sum_{n=0}^{\infty}\frac{1}{\lambda_{n+1}}R^n)$ is convergent (notice that $f \in E_{1/R,R}(\hat{A})$). Furthermore, (19) is satisfied, since $\sum_{n=2}^{\infty}[\frac{1}{[(n-1)!]^2} - \frac{1}{(n!)^2}] = 1$ implies that for all $R < 1$, $\sum_{n=2}^{\infty}[\frac{1}{[(n-1)!]^2} - \frac{1}{(n!)^2}]R^n < 1$ (remark here that the matrix $[f]$ does not satisfy $\|I - \frac{1}{b_0}[f]\|_{S_R} < 1$ for any $R > 0$). Finally, recall that C is the infinite matrix associated to the sequence with general terms $\hat{C}_m(z) = \sum_{n=m}^{\infty}\frac{1}{\lambda_{n+1}}z^n$, $m \geqslant 0$. Then $\frac{1}{f(z)} = \sum_{n=0}^{\infty} b'_n z^n$ is convergent for $|z| \leqslant R$ for all $R < 1$. So (22) is equivalent to $CX = B'$ with $B' = (b'_n)_{n\geqslant 0} \in s_{1/R}$, i.e. $B' \in s_r$ since $r = 1/R$. Thus the equation $CX = B'$ admits a unique solution $X = (x_n)_{n\geqslant 0}$ in $s_{1/R}$, given by (9). We see that $X \in s_{1/R}$. Using i) in Proposition 4, we conclude that (22) is satisfied for $|z| \leqslant R$ and $X \in s_{1/R}$ with $R < 1$. This expansion is unique and we obtain an r, R basis.

3.3. Study of a particular r, R basis

Consider now the case where $\hat{A}_0(z) = 1 + c'_0 z$, and for $m \geqslant 1$, $\hat{A}_m(z) = c_m z^{m-1} + z^m + z'_m z^{m+1}$. Assume that $rR < 1$ and the series with general terms $c_n(rR)^n$ and $c'_n(rR)^n$ are absolutely convergent. Then (12) is satisfied and equivalent to

$$\sum_{n=1}^{\infty}(|c'_{n-1}|r^{n-1} + r^n + |c_{n+1}|r^{n+1})R^n < \infty. \tag{23}$$

One has

PROPOSITION 10. *Suppose that (23) holds and $R < 1/r$. We have:*

i) If

$$\chi = \sup\Big\{\sup_{n\geqslant 0}\left(\frac{|c'_n|}{r} + |c_{n+2}|r\right), |c_1|r\Big\} < 1, \tag{24}$$

then $f \in E_{r,R}(\hat{A})$, and the expansion is unique.

ii) Assume that $B \in s_\alpha$, where $\alpha = (c_n r^n)_{n\geqslant 0}$ with:

$$|c_1| > 1/r \quad and \quad \sup_{n\geqslant 1}\left(\frac{|c'_{n-1}| + r}{|c_n|}\right) < r^2. \tag{25}$$

Then $f \in E_{r,R}(\hat{A})$ admits infinitely many expansions.

iii) Suppose that $B \in s_\beta$, where $\beta = (c'_{n-1}r^n)_{n\geqslant 1}$ and

$$\sup_{n\geqslant 0}\left(\frac{|c_{n+2}| + 1}{|c'_n|}\right) < \frac{1}{r}. \tag{26}$$

There is a real u_0 so that $b_0 \neq u_0$ implies that $f \notin E_{r,R}(\hat{A})$.

Proof. The infinite matrix $A = (a_{nm})_{n,m\geqslant 0}$ is definied here for all $n \geqslant 1$ by $a_{n,n-1} = c'_{n-1}$, $a_{nn} = 1$, $a_{n,n+1} = c_{n+1}$ and $a_{00} = 1$, $a_{01} = c_1$, the other elements being equal to 0. Relation (24) implies $\|I - A\|_{S_r} < 1$ (since $\chi = \|I - A\|_{S_r}$), which proves i), using Proposition 4.

Assertion ii). Denote by $A(e_0) = (a'_{nm})_{n,m\geqslant 0}$ the matrix obtained from A by addition of the supplementary row e_0. If $D_0 = (\delta_{mn}/a'_{nn})_{n,m\geqslant 0}$ is the diagonal matrix whose nonzero elements are the inverses of the diagonal entries of $A(e_0)$, then (25) expresses that $\|I - D_0A(e_0)\|_{S_r} < 1$. Hence the equation $AX = B$ with $D_0B(u) \in s_r$ admits infinitely many solutions in s_r defined for all scalars u by $X = [A(e_0)]^{-1}B(u)$ (see [5]). Since $D_0B(u) \in s_r$ is equivalent to $b_n = O(c_n r^n)$ as $n \to \infty$, we conclude that $f(z)$ has infinitely many expansions which can be written as

$$f(z) = \sum_{m=0}^{\infty} (x_m + uy_m)\hat{A}_m(z),$$

where $(x_n)_{n\geqslant 0}$, $(y_n)_{n\geqslant 0} \in s_r$ for all scalars u.

For iii), denote by A^* the matrix obtained from A by deleting the first row, and define B^* from B in the same way. D^* is the diagonal matrix whose nonzero entries are the inverses of the diagonal elements of A^*. (26) implies $\|I - D^*A^*\|_{S_r} < 1$, so the equation $A^*X = B^*$, with $D^*B^* = (b_{n+1}/c'_n)_{n\geqslant 0} \in s_r$, has a unique solution $X = (A^*)^{-1}B^*$ in s_r. Hence there is a real u_0 for which the equation $AX = B$ does not admit any solution in s_β, for $u_0 \neq b_0$. ■

EXAMPLE 11. Take a function $f(z) = \sum_{n=0}^{\infty} b_n z^n$. There is a real $r > 0$ such that $(b_n)_{n\geqslant 0} \in s_\alpha$, where $\alpha = (nr^n)_n$ and there exist infinitely many expansions of the form

$$f(z) = x_0 + x_1 + \sum_{m=1}^{\infty} (mx_{m-1} + m^2x_m + m^3x_{m+1})z^m. \tag{27}$$

Here we have $\hat{A}_0(z) = 1 + z$, $\hat{A}_1(z) = 1 + z + 2z^2$ and $\hat{A}_m(z) = (m-1)^3 z^{m-1} + m^2 z^m + (m+1) z^{m+1}$ for all $m \geqslant 2$. The matrix $A = (a_{nm})_{n,m \geqslant 0}$ is here defined by $a_{00} = a_{01} = a_{10} = a_{11} = a_{12} = 1$, and $a_{n,n-1} = n$, $a_{nn} = n^2$, $a_{n,n+1} = n^3$ for all $n \geqslant 2$. In order to refer to the previous case, where entries on the diagonal are equal to 1, consider the product $D_1 A$ with $D_1 = (d_n \delta_{nm})_{n,m \geqslant 0}$ and $d_0 = 1$, $d_n = n^{-2}$ for $n \geqslant 1$. Then $c'_n = 1/(n+1)$ for all $n \geqslant 0$; $c_1 = 1$ and $c_n = n - 1$ for $n \geqslant 2$. Set $\tau_1 = 1 + r$ and $\tau_2 = \sup_{n \geqslant 2} \frac{\frac{1}{n} - r}{n-1}$. Then condition ii) in the previous proposition yields

$$\sup\{\tau_1, \tau_2\} < r^2.$$

Hence $r^2 - r - 1 > 0$ that is $r > (1 + \sqrt{5})/2$. We conclude that for all $(b_n)_{n \geqslant 0} \in s_\alpha$ with $r > (1 + \sqrt{5})/2$ and for $|z| \leqslant R$, with $R < 2/(1 + \sqrt{5})$, we have the expansion in (27).

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