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DIFFERENCES OF DECREASING SLOWLY VARYING FUNCTIONS

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Abstract. The class of slowly varying functions is not closed under subtraction. In previous papers we found some subclasses of nondecreasing slowly varying functions, characterized by the good decomposition property, which (under additional conditions) are closed under subtraction. The main result of this paper is that the assumption that functions are nondecreasing is essential because only nondecreasing functions can have the good decomposition property.

A positive measurable function l, defined on some neighborhood of infinity, is said to be *slowly varying* (SV) if, for every $s > 0$,

$$\frac{l(st)}{l(t)} \to 1 \quad (t \to +\infty). \tag{1}$$

These functions were introduced by Karamata [4] (see also [1] and [5]).

We deal with differences of slowly varying functions. The class of all slowly varying functions is not closed under subtraction, though it is closed under the operations of addition, multiplication and division of functions. We consider additional assumptions under which it is possible to say something about subtraction of slowly varying functions. In [2] and [3] we studied the differences of slowly varying functions in the case when the functions are monotone nondecreasing. In this paper we show that, in contrast to the nondecreasing case, in the class of nonincreasing slowly varying functions it is not possible to find a subclass which behaves nicely with respect to the operation of subtraction.

The main object of our study in [2] was the following property of good decomposition in the class of nondecreasing functions, which is closely related with the operation of subtraction.

Definition 1. A nondecreasing slowly varying function l is said to have *the property of good decomposition in the class of nondecreasing functions* if whenever we decompose l into a sum $l = f + g$ of two nondecreasing functions f and g, then f and g are necessarily slowly varying.

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In this paper we consider other classes of monotone functions, such as: nonincreasing, convex decreasing, concave increasing and study the good decomposition property in these classes. We adapt Definition 1 to be appropriate to these cases, by assuming that when l is in one of these classes, then the summands f and g also belong to the same class.

Convex increasing functions appear in this context in the following way: we consider functions of the form $l(e^x)$, which are convex increasing and l is slowly varying (note that an increasing slowly varying function l cannot be convex, because its rate of growth is smaller than the rate of growth of any power function).

By making the change of variables $L(x) = l(e^x)$ we pass to the class of *additively slowly varying* functions. These are positive measurable functions L defined on some neighborhood of infinity and such that

$$\lim_{x\to+\infty} \frac{L(y+x)}{L(x)} = 1, \tag{2}$$

for every $y \in \mathbb{R}$. Obviously, L is additively slowly varying if and only if l is slowly varying.

In Section 1 we consider the nonincreasing functions and in Section 2 functions having monotone derivatives. The functions in Section 1 are slowly varying in the multiplicative sense (see (1)) and the result of Theorem 1 is complementary to those from [2], while in Section 2 the functions are slowly varying in the additive sense (see (2) and [3]).

1. Nonincreasing functions

In paper [2] we found a subclass, denoted by $O\Pi^+$ (see Definition 2 below), of nondecreasing slowly varying functions satisfying that the difference of two functions from that class, if nondecreasing, is slowly varying.

DEFINITION 2. A positive nondecreasing function f belongs to the class $O\Pi^+$ if

$$\limsup_{t\to+\infty}(f(st) - f(t)) = M(s)$$

is finite for every $s > 1$.

The characterization theorem for the class $O\Pi^+$ from [2] is the following.

THEOREM A. *Let l be a nondecreasing slowly varying function. Then l has the good decomposition property in the class of nondecreasing functions if and only if it belongs to the class $O\Pi^+$.*

In [2] we considered nondecreasing slowly varying functions only. This can be justified by the following theorem, which shows that nonincreasing slowly varying functions never have the good decomposition property in the class of nonincreasing functions.

Theorem 1. *Every nonincreasing slowly varying function l is a sum of two nonincreasing functions such that at least one of them is not slowly varying.*

Proof. In the proof we shall decompose the slowly varying function l into a sum of two nonincreasing functions f and g such that, for an increasing sequence t_n, f is constant on every even interval $[t_{2n}, t_{2n+1}]$ and g is constant on every odd interval $[t_{2n+1}, t_{2n+2}]$. Note that such a construction is possible for every nonincreasing, not necessarily slowly varying, function, but if l is not slowly varying, the assertion of the theorem is trivial.

Since l is positive nonincreasing it has a finite limit $\lim_{t\to\infty} l(t) = c \geqslant 0$. Suppose first $c = 0$.

Because l is slowly varying, it cannot be identically equal to zero starting from some t. Let t''_n be an increasing sequence of continuity points of l, $t''_n \to \infty$, $n \to \infty$, and such that

$$l(t''_n - 1) > l(t''_n). \tag{3}$$

For a fixed s, $0 < s < 1$, choose a subsequence t'_n of t''_n such that

$$t'_n < st'_{n+1}. \tag{4}$$

For n large enough, by (3) we have that

$$l(st'_n) - l(t'_n) > l(t'_n - 1) - l(t'_n) > 0.$$

Thus condition (3) ensures that, if the function l has infinitely many intervals on which it is constant, then the points st'_n and t'_n do not belong to the same interval.

Then, since $l\,(t) \to 0$, $t \to \infty$, we can chose again a subsequence t_n of t'_n such that

$$l(st_n) - l(t_n) > l(t_{n+1}). \tag{5}$$

Define

$$f(t) = \begin{cases} \sum_{i=2n+1}^{\infty} (-1)^{i+1} l(t_i), & t \in [t_{2n}, t_{2n+1}] \\ l(t) - \sum_{i=2n}^{\infty} (-1)^i l(t_i), & t \in [t_{2n+1}, t_{2n+2}] \end{cases}$$

and $g(t) = l(t) - f(t)$, which has a similar form. Note that f and g are uniquely determined by the decomposition above. From the definition of f it follows that $f(t_{2n}) = f(t_{2n+1}) < l(t_{2n+1})$ and also that $f(st_{2n}) - f(t_{2n}) = l(st_{2n}) - l(t_{2n})$, since by (4) we have $st_{2n} \in [t_{2n-1}, t_{2n}]$. Then we have

$$\frac{f(st_{2n}) - f(t_{2n})}{f(t_{2n})} > \frac{l(st_{2n}) - l(t_{2n})}{l(t_{2n+1})} > 1$$

where the last inequality follows from (5). This proves that f is not slowly varying. In a similar way by considering the points t_{2n+1}, it can be shown that g is not slowly varying either. So in the case when l tends to zero it can always be decomposed into a sum of two nonincreasing functions neither of which is slowly varying.

If $\lim_{t\to\infty} l(t) = c > 0$, since f and g are nonincreasing, one of them must have a positive limit too, and hence is slowly varying. Thus by applying the same construction as above to the function $l(t) - c$ we find a decomposition $l(t) - c = f(t) + g(t)$, such that $l(t) = f(t) + g_1(t)$, where $f \notin SV$ and $g_1(t) = g(t) + c \in SV$. ■

2. Decreasing convex functions

Functions from the class $\mathrm{O\Pi}^+$ cannot grow faster than $\log t$. In order to be able to say something about differences of functions with faster rate of growth we imposed [3] some second order monotonicity conditions, i.e. monotonicity conditions on derivatives. As already mentioned, an increasing slowly varying function l cannot be convex, it can only be concave. But even with the additional assumption of concavity in Definition 1 all functions with the good decomposition property are those from $\mathrm{O\Pi}^+$ (see [2]).

In order to be able to consider the good decomposition property for functions such as $l(t) = \log^\alpha t$, $\alpha > 1$, we studied in [3] slowly varying functions l, such that the function $L(x) = l(e^x)$ is convex increasing and we introduced the class $\mathrm{O\Pi}_2^+$ which is characterized by the good decomposition property in the class of convex increasing functions.

Definition 3. A positive convex increasing function F belongs to the class $\mathrm{O\Pi}_2^+$ if for every $y \in \mathbb{R}$ there is a constant $C = C(y)$ (not depending on x) such that

$$0 \leqslant F(x + 2y) - 2F(x + y) + F(x) \leqslant Cx, \tag{6}$$

for x large enough.

We proved that if a positive increasing convex function F belongs to $\mathrm{O\Pi}_2^+$, then F is additively slowly varying and the following theorem holds.

Theorem B. ([3]) *An increasing convex additively slowly varying function L has the good decomposition property in the class of increasing convex functions if and only if it belongs to the class $\mathrm{O\Pi}_2^+$.*

Except for the convex increasing case, in the class of positive functions with second order monotonicity conditions (i.e. having monotone left and right derivatives) there are two more cases to consider after the change of variables $L(x) = l(e^x)$: decreasing convex and increasing concave functions (since concave decreasing functions cannot be positive at infinity).

In Proposition 1 we consider the case of positive concave increasing functions, which always have the good decomposition property. In Theorem 2 we show that positive convex decreasing functions never have the good decomposition property.

Proposition 1. *Every positive concave additively slowly varying function L tending to $+\infty$ has the good decomposition property in the class of positive concave functions.*

Proof. If L tends to a constant the assertion is trivial. When $L(t) \to \infty$, the assertion follows from the fact that every positive concave increasing function

F is additively slowly varying. Indeed, since F' decreases, the difference $0 < F(x+y) - F(x)$ is bounded by a constant C, and since F tends to $+\infty$,

$$\frac{F(x+y)-F(x)}{F(x)} \leqslant \frac{C}{F(x)} \to 0.$$

Then for every decomposition $L = F+G$, where both F and G are positive concave increasing, it follows that both F and G are additively slowly varying. ∎

Now we deal with convex decreasing functions. Note that even if we decompose a convex decreasing function according to the construction from Theorem 1, the summands obtained are never convex.

The question is: if we impose in the decomposition the additional condition of the convexity of summands, does this yield their slow variation? Theorem 2 gives a negative answer to this question. We prove that a convex decreasing additively slowly varying function L never has the good decomposition property in the class of convex decreasing functions. We exclude the case when $L(x) = c > 0$ for all $x \geqslant x_0$, where in every decomposition of L into two convex decreasing summands, trivially both summands must be constant starting from x_0.

THEOREM 2. *Every positive convex decreasing additively slowly varying function L can be decomposed into a sum $L = F + G$ of two positive convex decreasing functions such that at least one of them is not additively slowly varying.*

Proof. In this proposition all derivatives are left derivatives. Note that L has a nonnegative limit and therefore $\lim_{x\to+\infty} L'(x) = 0$. Suppose first that $\lim_{x\to+\infty} L(x) = 0$. Consider

$$\mathcal{D}_{-y}L(x) = L(x-y) - L\,(x) + yL'(x),$$

for $y > 0$ fixed. Then obviously $\lim_{x\to+\infty} \mathcal{D}_{-y}L(x) = 0$. Also, since $-L'$ is positive nonincreasing, we have $-xL'(x) < 2L(x/2) \to 0$ and we can choose an increasing sequence x_n (of continuity points of L') tending to infinity and satisfying the following three conditions: $x_n - y > x_{n-1}$, $\mathcal{D}_{-y}L(x_n) > 0$, and

$$\mathcal{D}_{-y}L(x_n) > L(x_{n+1}) - x_{n+1}L'(x_{n+1}). \tag{7}$$

((7) is possible since on both sides of the preceding inequality we have positive functions tending to zero.)

Define F' and $G' = L' - F'$ in the following way

$$F'(x) = \begin{cases} \sum_{i=2n+1}^{\infty}(-1)^{i+1}L'(x_i), & x \in [x_{2n}, x_{2n+1}] \\ L'(x) - \sum_{i=2n}^{\infty}(-1)^{i}L'(x_i), & x \in [x_{2n+1}, x_{2n+2}]. \end{cases} \tag{8}$$

From (8) it follows that F' is negative nondecreasing because it is equal to a negative constant on the intervals $[x_{2n}, x_{2n+1}]$ and differs by a constant from L' on the intervals $[x_{2n+1}, x_{2n+2}]$, and G' has a similar form, only it is constant on the intervals $[x_{2n+1}, x_{2n+2}]$. Thus the functions F and G (defined by (8) and the conditions $\lim_{x\to+\infty} F(x) = 0$ and $\lim_{x\to+\infty} G(x) = 0$) are convex decreasing and uniquely determined.

We have from (8) for $x \in [x_{2n-1}, x_{2n}]$

$$F'(x) - F'(x_{2n}) = L'(x) - L'(x_{2n}),$$

from which (by integrating from $x_{2n} - y$ to x_{2n}) it follows that

$$\mathcal{D}_{-y}F(x_{2n}) = \mathcal{D}_{-y}L(x_{2n}). \tag{9}$$

From (8) it follows that

$$\begin{aligned} F(x_{2n}) &= F(x_{2n+1}) - F'(x_{2n+1})(x_{2n+1} - x_{2n}) \\ &< L(x_{2n+1}) - L'(x_{2n+1})x_{2n+1} < \mathcal{D}_{-y}L(x_{2n}). \end{aligned} \tag{10}$$

For the last inequality we have used condition (7). We have, since $x_{2n} - y \in [x_{2n-1}, x_{2n}]$,

$$F(x_{2n} - y) - F(x_{2n}) > \mathcal{D}_{-y}F(x_{2n}) = \mathcal{D}_{-y}L(x_{2n}) \tag{11}$$

by (9). Thus we have by (10) and (11)

$$\frac{F(x_{2n} - y) - F(x_{2n})}{F(x_{2n})} > \frac{\mathcal{D}_{-y}L(x_{2n})}{\mathcal{D}_{-y}L(x_{2n})} = 1$$

which proves that F is not additively slowly varying. In the same way, by considering the points x_{2n+1}, it can be seen that G is not slowly varying either.

In the case when $\lim_{x\to\infty} L(x) = c > 0$, then at least one of the summands has a positive limit too and must be slowly varying. By applying the same construction as above to the function $L(x) - c$ we can find a decomposition of $L(x) - c = F(x) + G(x)$, such that $L(x) = F(x) + G_1(x)$, where $G_1(x) = G(x) + c$ is additively slowly varying and F is not. ∎

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