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ESTIMATION IN UNIFORM MINIFICATION PROCESSES AND THEIR TRANSFORMATIONS

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Abstract. Unknown parameters of the uniform minification processes and their transformations are estimated in this paper. Three different methods of estimating are considered. The first one is supported by the properties of the process (maximum or minimum of the quotient of successive values of the sequence), the second one is based on the probabilities $P\{X_n > X_{n-1}\}$ and $P\{X_n < X_{n-1}\}$ and the last one is the method of moments. Some numerical examples are presented.

1. Introduction

Time series with uniform marginals have been defined at the beginning of the eightieth. Chernik [2] considered the first order autoregressive time series with positive correlations with uniform marginal distribution. Chernik and Davis [3] considered autoregressive time series of the same order and the same marginal distribution, but with negative correlations. At the beginning of ninetieth, Lewis and McKenzie [5] defined the uniform minification processes and also some processes which were produced as the transforms of the minification processes. At the end of ninetieth, Ristić and Popović [6] have defined the new uniform autoregressive time series of order one, NUAR(1), which was obviously the generalization of all up to that time defined uniformly distributed autoregressions of order one.

Ristić and Popović in [7] considered the problem of estimation the parameter k of the uniform autoregressive processes of the first-order with positive and negative lag one autocorrelation functions. The same authors in [6] estimated the unknown parameters of the NUAR(1) process and suggested two estimators of (α, β). The first estimator is based on the minimum of the ratios $\{X_n/X_{n-1}\}$, while the second estimator is obtained by solving equations

$$P\{X_n > X_{n-1}\} = \frac{(1-\alpha)(1+\alpha-\beta)}{2(1-\beta)},$$
$$Corr(X_n, X_{n-1}) = \alpha^2 + (1-\alpha)\beta.$$

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The uniform minification processes have been defined [5] in the following way:

Definition 1.1. Let $K > 1$ be a constant. Let X_0 has $U(0,1)$ distribution and $\{Z_n\}$ be the innovation sequence of independent identically distributed random variables with the following density function

$$g(z;K) = \begin{cases} \frac{K-1}{(1-z)^2} & , \ z \in (0, 1/K), \\ 0 & , \ \text{otherwise}, \end{cases}$$

and let the sequences $\{X_n\}$ and $\{Z_n\}$ be semiindependent, i.e. let the random variables X_n and Z_m are independent if and only if is $n < m$. Then the time series defined as

$$X_n = K \min\{X_{n-1}, Z_n\}, \ n \geq 1 \tag{1.1}$$

is the stationary one with $U(0,1)$ marginal distribution named uniform minification process.

The autocovariance and the autocorrelation functions of such process are given by $\gamma_X(j) = 1/(12K^j)$, $j = 0, 1, \ldots$, and $\rho_X(j) = 1/K^j$, $j = 0, 1, \ldots$, respectively.

Monotone transformations of the uniform minification process are given in [5] also. By means of the monotone increasing transformation the process $\{Y_n\}$ is defined as

$$Y_n = \min\{1 - (1 - Y_{n-1})^K, 1 - (1 - W_n)^K\}, \ n \geq 1 \tag{1.2}$$

and by means of monotone decreasing transformation $\{Y_n\}$ is defined as

$$Y_n = \max\{Y_{n-1}^K, W_n^K\}, \ n \geq 1 \tag{1.3}$$

where $K > 1$, Y_0 has $U(0,1)$ distribution, $\{W_n\}$ is the sequence of independent identically distributed random variables and $\{Y_n\}$ and $\{W_n\}$ are semiindependent. The distribution of W_n depends on the transformation. So, if it is a decreasing one, respective density function of W_n is

$$h(w) = (K-1)w^{K-2}, \ w \in (0,1),$$

and if it is increasing, density function is

$$h(w) = (K-1)(1-w)^{K-2}, \ w \in (0,1).$$

But both of them have the same autocovariance and autocorrelation function:

$$\gamma_Y(j) = \frac{1}{4(2K^{|j|}+1)}, \ j \geq 0,$$

and

$$\rho_Y(j) = \frac{3}{2K^{|j|}+1}, \ j \geq 0,$$

respectively.

In the following sections the authors will consider some estimates of unknown K for all above mentioned uniformly distributed time series.

2. Estimation of parameter K in uniform minification processes

Let us suppose that the realization $(X_0, X_1, \cdots, X_N)$ is known. The unknown parameter K of the uniformly distributed minification process will be estimated by means of:

1. maximum of a quotient X_n/X_{n-1},
2. probability $v_n \equiv P\{X_n > X_{n-1}\}$ and
3. method of moments.

The first method is the consequence of the definition of uniform minification process. It implies that $X_n \le KX_{n-1}$ and $K \ge \frac{X_n}{X_{n-1}}$. That is why we can use

$$\hat{K} = \max_{1\le n\le N} \left\{ \frac{X_n}{X_{n-1}} \right\}$$

as the estimate of K. We present some simulation results of this method in table 1. As it can be seen, the method gives very good results.

N	$K = 1.46$	$K = 8.31$	$K = 100$
500	1.46	8.31	100
1000	1.46	8.31	100
5000	1.46	8.31	100
10000	1.46	8.31	100
15000	1.46	8.31	100

Table 1. Some estimation results of parameter K

The second method for estimating K is founded on the probability

$$\begin{aligned} v_n &= P\{KX_{n-1} > X_{n-1}, KZ_n > X_{n-1}\} = P\{KZ_n > X_{n-1}\} \\ &= K + K(K-1)\ln\frac{K-1}{K}. \end{aligned}$$

As the estimate of the probability v_n we use the statistics

$$\frac{1}{N}\sum_{n=1}^{N} I(X_n > X_{n-1}).$$

Then the estimate for K can be calculated as the solution of the equation

$$K + K(K-1)\ln\frac{K-1}{K} = \frac{1}{N}\sum_{n=1}^{N} I(X_n > X_{n-1}). \qquad (2.1)$$

As the equation (2.1) can be solved only by some numerical method, we used Newton method as far the function on K was differentiable. Initial value for K is $K_0 = 1.01$.

We present in table 2 some simulation results on applying this method. As it can be seen the method is worse than the first one. It is nonstable for great values of K. Even sometimes it gives negative values for K which are not available.

N	$K = 1.46$	$K = 8.31$	$K = 100$
500	1.47084	28.2796	$---$
1000	1.44789	19.0212	$---$
5000	1.45966	11.3271	64.6033
10000	1.44523	9.41827	98.5397
15000	1.45966	8.72976	71.9293

Table 2. Some estimation results of parameter K

The third method we used for estimating K is the method of moments. The estimate of K is statistic

$$\hat{K} = \frac{1}{\hat{\rho}_X(1)},$$

where $\hat{\rho}_X(1)$ is the estimate of lag-one autocorrelation.

N	$K = 1.46$	$K = 8.31$	$K = 100$
500	1.4739482	6.1965789	$---$
1000	1.4603409	5.6665281	$---$
5000	1.4933005	7.9122146	$---$
10000	1.4718437	8.5941056	$---$
15000	1.4632418	8.0751143	121.0526505

Table 3. Some estimation results of parameter K

Simulation results for estimate of K by means of method of moments are given in table 3. It can be seen that in some cases the method of moments is not valid. The reason is that the random variable X_n is obtained by the random variables Z_n which is defined on interval $(0, 1/K)$ and this interval is small for large K. This remark is valid for all the estimation methods using in this paper (in case of large values K).

3. Estimation of parameter K in monotone transformations of uniform minification processes

Let $(Y_0, Y_1, \ldots, Y_N)$ be the sample from the only one realization of the process (1.2) or (1.3). Then we can estimate K as follows:

1. using the maximum of the quotient $\frac{\ln(1-Y_n)}{\ln(1-Y_{n-1})}$ for the process (1.2),
2. using the maximum of the quotient $\frac{\ln Y_n}{\ln Y_{n-1}}$ for the process (1.3),
3. using the probabilities $v_{1n} \equiv P\{Y_n > Y_{n-1}\}$ for the process (1.2),

4. using the probabilities $v_{2n} \equiv P\{Y_n < Y_{n-1}\}$ for the process (1.3),
5. using the method of moments.

Let us estimate K for the process (1.2) in the first place. From the definition itself it follows that $Y_n \le 1-(1-Y_{n-1})^K$, i.e. $K \ge \frac{\ln(1-Y_n)}{\ln(1-Y_{n-1})}$. So, the estimate of K can be defined as

$$\hat{K} = \max_{1\le n\le N}\left\{\frac{\ln(1-Y_n)}{\ln(1-Y_{n-1})}\right\}.$$

Some simulating results based on this statistics are given in table 4.

N	$K=1.46$	$K=8.31$	$K=100$
500	1.46	8.31	100
1000	1.46	8.31	100
5000	1.46	8.31	100
10000	1.46	8.31	100
15000	1.46	8.31	100

Table 4. Some estimation results of parameter K

Let us now use the probability v_{1n}. From the equation (1.2) immediately follows that

$$\begin{aligned} v_{1n} &= P\{1-(1-Y_{n-1})^K > Y_{n-1}, 1-(1-W_n)^K > Y_{n-1}\} \\ &= P\{Y_{n-1} < 1-(1-W_n)^K\} = \frac{K}{2K-1}. \end{aligned}$$

That implies that we have to solve the equation $\hat{v} = \dfrac{K}{2K-1}$, on K where

$$\hat{v} = \frac{1}{N}\sum_{n=1}^{N} I(Y_n > Y_{n-1})$$

is the estimate of v_{1n}. So, the estimate of K is $\hat{K} = \dfrac{\hat{v}}{2\hat{v}-1}$.

In table 5 we present some results of the application of the last statistics. All remarks that are valid for the corresponding method of estimating K of uniform minification process are valid here also.

N	$K=1.46$	$K=8.31$	$K=100$
500	1.3928571	14.3888889	$---$
1000	1.4727626	14.3888889	$---$
5000	1.4780908	8.3616352	250.4999999
10000	1.4659969	8.0075075	227.7727272
15000	1.4630200	7.9553684	234.8750000

Table 5. Some estimation results of parameter K

Let us consider now the process (1.3). From the definition it follows that $Y_n \geq Y_{n-1}^K$, i.e. $K \geq \frac{\ln Y_n}{\ln Y_{n-1}}$. That implies that K can be estimated by the statistics

$$\hat{K} = \max_{1 \leq n \leq N} \left\{ \frac{\ln Y_n}{\ln Y_{n-1}} \right\}.$$

This statistics is good estimate of K and some simulating results are presented in table 6.

N	$K = 1.46$	$K = 8.31$	$K = 100$
500	1.46	8.31	100
1000	1.46	8.31	100
5000	1.46	8.31	100
10000	1.46	8.31	100
15000	1.46	8.31	100

Table 6. Some estimation results of parameter K

Let us consider now the method of probabilities v_{2n}. From the definition (1.3) follows that

$$v_{2n} = P\{Y_{n-1}^K < Y_{n-1}, W_n^K < Y_{n-1}\} = P\{W_n^K < Y_{n-1}\} = \frac{K}{2K-1}.$$

So, the equation $\hat{v}^* = \dfrac{K}{2K-1}$, is to be solved on K, where

$$\hat{v}^* = \frac{1}{N} \sum_{n=1}^{N} I(Y_n < Y_{n-1})$$

is the estimate of v_{2n}. The solution mentioned above implies the statistics $\hat{K} = \dfrac{\hat{v}^*}{2\hat{v}^* - 1}$.

Some simulating results concerning this statistics are presented in table 7.

N	$K = 1.46$	$K = 8.31$	$K = 100$
500	1.4920635	8.8333333	31.7500000
1000	1.4615385	10.9266667	$---$
5000	1.4585890	8.2639752	$---$
10000	1.4652510	8.8892617	192.8076923
15000	1.4580991	7.7236842	80.2872340

Table 7. Some estimation results of parameter K

The method of moments, or preciously the method using the sampling coefficient of correlation for two successive elements of the series is the general one for both series (1.2) and (1.3). Both of them have the same correlation function

$$\rho_Y(1) = \frac{3}{2K+1}, \ j \geq 0.$$

So, we can estimate K using statistics

$$\hat{K} = \frac{1}{2}\left(\frac{3}{\hat{\rho}_Y(1)} - 1\right),$$

in both cases, where $\hat{\rho}_Y(1)$ is the estimate of corresponding coefficient of correlation.

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