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SUFFICIENT CONDITIONS FOR ELLIPTIC PROBLEM OF OPTIMAL CONTROL IN $\mathbf{R}^n$ IN ORLICZ SOBOLEV SPACES

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Abstract. This paper is concerned with the local minimization problem for a variety of non Frechet-differentiable Gâteaux functional $J(f) \equiv \int_\Omega v(x,u,f)\,dx$ in the Orlicz-Sobolev space $(W_0^1 L_M^*(\Omega), \|.\|_M)$, where u is the solution of the Dirichlet problem for a linear uniformly elliptic operator with nonhomogenous term f and $\|.\|_M$ is the Orlicz norm associated with an N-function M.

We use a recent extension of Frechet-Differentiability (approach of Taylor mappings see [2]), and we give various assumptions on v to guarantee a critical point is a strict local minimum.

Finally, we give an example of a control problem where classical Frechet differentiability cannot be used and their approach of Taylor mappings works.

1. Preliminaries

1.1. Some definitions and well-know facts from Orlicz space. We begin by listing briefly some definitions and well-known facts from Orlicz space theory (see [1]).

Let Ω be an open subset of $\mathbf{R}^n$, with Lebesgue measure dx, and let M be an N-function (i.e. a real-valued continuous, convex, even function of $t \in \mathbf{R}$ satisfying $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$).

The Orlicz class $L_M(\Omega)$ is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))\,dx < +\infty$, and the Orlicz space $L_M^*(\Omega)$ as the linear hull of $L_M(\Omega)$.

$L_M^*(\Omega)$ is a Banach space with respect to the Luxembourg norm:

$$\|u\|_{(M)} = \inf\left\{ k > 0 : \int_\Omega M\Big(\frac{u(x)}{k}\Big)\,dx \le 1 \right\}.$$

$L_M^*(\Omega)$ is a Banach space with respect to the Orlicz norm:

$$\|u\|_M = \sup\left\{ \left|\int_\Omega u(x)v(x)\,dx\right| : \int_\Omega \overline{M}(v(x))\,dx \le 1 \right\},$$

where $\overline{M}$ is the N-function conjugate to M.

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The Orlicz norm $\|.\|_M$ is equivalent to $\|.\|_{(M)}$: $\|.\|_{(M)} \le \|.\|_M \le 2\|.\|_{(M)}$.

Let $W^m L^*_M(\Omega)$ be the Orlicz-Sobolev space of functions u such that u and its distribution derivatives up to order m lie $L^*_M(\Omega)$.

$W^m L^*_M(\Omega)$ is a Banach space with respect to the norm:

$$\|u\|_{m,M} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^2_{(M)} \right)^{\frac{1}{2}}.$$

1.2. Orlicz-Sobolev Spaces. We define a further Orlicz-Sobolev space $W^m_0 L^*_M(\Omega)$ to be the closure of $C^\infty_0(\Omega)$ in $\big(W^m L^*_M(\Omega), \sigma(\Pi L_M, \Pi E_{\bar{M}})\big)$.

1.3. Description of the optimization problem. Let A be an elliptic operator of second order:

$$Au \equiv \sum_{|l| \le 1, |s| \le 1} (-1)^l \mathcal{D}^l (a_{ls}(x) \mathcal{D}^s u),$$

where $a_{ls}(x) \in \mathcal{D}(\overline{\Omega})$. Suppose that Ω is sufficiently smooth and bounded domain in $\mathbf{R}^n$.

Let us consider the problem :

$$Au = f, \tag{1.1}$$

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

For this problem, let us state Agmon-Douglis-Niremberg's theorem:

Theorem 1.1. (Agmon-Douglis-Niremberg) *Let $1 < q < \infty$; then we have that $\forall f \in L^q(\Omega)$, there exists a unique solution $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ of problem (1.1), (1.2). Moreover, $\forall m \ge 0$ if $f \in W^{m,q}(\Omega)$, then $u \in W^{m+2,q}(\Omega)$ and $\|u\|_{W^{m+2,q}(\Omega)} \le c\|f\|_{W^{m,q}(\Omega)}$.*

Let M be an N-function such that $|t|^p \le M(t)$ for $t \ge t_0$, where $p > n$ and $t_0 > 0$. Let $f \in F \subset W^1_0 L^*_M(\Omega)$ be a control and let u the solution of problem (1.1), (1.2) in $W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$ associated to f.

Let us consider $J_k(f) = \int_\Omega v_k(x,u,f)\,dx + c_k\|f\|^2_{W^{1,2}(\Omega)}$, $(k = 0,1,2,\dots,s_1)$ and $J_k(f) = \int_\Omega v_k(x,u,f)\,dx$, $(k = s_1+1, s_1+2, \dots, s_1+s_2)$, where the sequence of functions $v_k : \Omega \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ is measurable on $\Omega \times \mathbf{R} \times \mathbf{R}$ and has second derivative with respect to (u,f) on $\mathbf{R} \times \mathbf{R}$ for almost all $x \in \Omega$.

We consider three problems of minimizing the functional $J_0(f)$:

$$i) \quad J_0(f) \to \min, \tag{1.3}$$

$$ii) \quad J_0(f) \to \min, \quad J(f) = 0, \ \text{where } J = (J_{s_1+1}, \dots, J_{s_1+s_2}), \tag{1.4}$$

$$iii) \quad J_0(f) \to \min, \quad J(f) = 0, \ \ J_k(f) \le 0, \ \ (k = 1,2,\dots,s_1). \tag{1.5}$$

We must choose a control f^0 in order that the solution u^0 of the problem (1.1), (1.2) with $f = f^0$ satisfies the inequality of the type: $J_k(f) \le 0, (1 \le k \le s_1)$ and the equality of the type: $J_k(f)=0, (s_1+1\le k\le s_1+s_2)$ and the functional $J_0(f)$ takes a minimum value. This control f^0 will be called optimal.

1.4. Taylor mappings and lower semi-Taylor mappings. Let M be an N-function, $\|.\|_{W^1L^*_M(\Omega)}$ the usual norm in $W^1_0L^*_M(\Omega)$, F a subset of $W^1_0L^*_M(\Omega)$, τ a topology in F, Y a normed space, and $\|.\|_Y$ a norm in Y. According to [2], a mapping $r\colon F\longrightarrow Y$ (respectively, $r\colon F\longrightarrow \mathbf{R}$) is said to be infinitesimally $(\tau,\|.\|_{W^1L^*_M(\Omega)})$-small (respectively, infinitesimally lower $(\tau,\|.\|_{W^1L^*_M(\Omega)})$-semismall) of order p_1 at $f\in F$ if: $\forall\varepsilon>0$, $\exists\ O_f\in\tau$, $\forall h\in W^1_0L^*_M(\Omega)$ we have

$$f+h\in O_f\Rightarrow \|r(f+h)\|_Y\le\varepsilon\|h\|^{p_1}_{W^1L^*_M(\Omega)},$$

(respectively, $\forall\ \varepsilon>0$, $\exists\ O_f\in\tau$, $\forall\ h\ \in W^1_0L^*_M(\Omega)$ we have

$$f+h\in O_f\Rightarrow \|r(f+h)\|_Y\ge-\varepsilon\|h\|^{p_1}_{W^1L^*_M(\Omega)});$$

here and below, O_f is a neighborhood of f in (F,τ).

A mapping $J\colon F\to Y$ (respectively, $J\colon F\to\mathbf{R}$) is called a $(\tau,\|.\|_{W^1L^*_M(\Omega)})$-Taylor (respectively, lower $(\tau,\|.\|_{W^1L^*_M(\Omega)})$–semi-Taylor) mappings of order p_1 at $f\in F$ if there exist k linear symmetric (not necessarily continuous) mappings $J^{(k)}(f)\colon (W^1_0L^*_M(\Omega))^k\to Y$ (respectively, $J^{(k)}(f)\colon (W^1_0L^*_M(\Omega))^k\to\mathbf{R}$, $k=1,\dots,p_1$, such that

$$J(f+h)-J(f)=$$
$$=J^{(1)}(f)h+2^{-1}J^{(2)}(f)(h,h)+\cdots+(p_1)!^{-1}J^{(p_1)}(f)(h,\dots,h)+r(f+h),$$

where $r\colon F\longrightarrow Y$ (respectively, $r\colon F\longrightarrow\mathbf{R}$) is an infinitesimally $(\tau,\|.\|_{W^1L^*_M(\Omega)})$–small (respectively, infinitesimally lower $(\tau,\|.\|_{W^1L^*_M(\Omega)})$-semismall) mapping of order p_1 at $f\in F$.

We note that $J^{(1)}(f),\dots,J^{(p_1)}(f)$ are not in general single-valued. The set of tuples $(J^{(1)}(f),\dots,J^{(p_1)}(f))$ is denoted by $S_n(J,f)$.

Let us solve the problems (1.3), (1.4) and (1.5). For the problem (1.5) let us introduce the Lagrange functions:

$$\mathcal{L}(f,y^*,\lambda,\lambda_0)=\sum_{k=0}^{s_1}\lambda_kJ_k(f)+\langle y^*,J(f)\rangle, \tag{1.6}$$

$$\mathcal{L}_f(f,y^*,\lambda,\lambda_0)=\sum_{k=0}^{s_1}\lambda_kJ^{(1)}_k(f)+\langle y^*,J^{(1)}(f)\rangle, \tag{1.7}$$

$$\mathcal{L}_{ff}(f,y^*,\lambda,\lambda_0)=\sum_{k=0}^{s_1}\lambda_kJ^{(2)}_k(f)+\langle y^*,J^{(2)}(f)\rangle, \tag{1.8}$$

where $\lambda_0\in\mathbf{R}$, $y^*\in(\mathbf{R}^{s_2})^*$, $\lambda\in(\mathbf{R}^{s_1})^*$.

Also for the problem (1.4), let us introduce the Lagrange functions:

$$\mathcal{L}(f,y^*,\lambda_0)=\lambda_0J_0(f)+\langle y^*,J(f)\rangle, \tag{1.9}$$

$$\mathcal{L}_f(f,y^*,\lambda_0)=\lambda_0J^{(1)}_0(f)+\langle y^*,J^{(1)}(f)\rangle, \tag{1.10}$$

$$\mathcal{L}_{ff}(f,y^*,\lambda_0)=\lambda_0J^{(2)}_0(f)+\langle y^*,J^{(2)}(f)\rangle, \tag{1.11}$$

where $\lambda_0\in\mathbf{R}$, $y^*\in(\mathbf{R}^{s_2})^*$.

Let us give the following theorem where the proof can be traced back to [1].

THEOREM 1.2. *Let M be an N-function, Ω a bounded domain in $\mathbf{R}^n$. Suppose that $u \in L^*_M(\Omega)$ and $\|u\|_M < 1$, then $\int_\Omega M(u(x))\,dx < \infty$.*

Let us give also the following lemma where the proof can be traced back to [2].

LEMMA 1.1. *Let (Ω, Σ, μ) be a measure space with σ-finite measure, and let X be a complete linear metric space continuously imbedded in the metric space $M(\Omega)$ of equivalence classes of measurable almost everywhere finite functions $x\colon \Omega \longrightarrow \mathbf{R}$, with the metrizable topology $\tau(meas)$ of convergence in measure on each set of Σ finite measure.*

Suppose that X contains with each element $x(s)$ the function $|x(s)|$, the metric in X is translation-invariant, and $\rho(x,0) = \rho(|x|,0)$ for each $x \in X$. Then for each sequence $x_n \to 0$ in X there exist a subsequence x_{n_k} and an element $z \in X$ such that: $|x_{n_k}(s)| \leq z(s)$, k=1,2, ... in the sense of the natural order on classes of functions.

Using the same argument as in Lemma 1.1 and the result of Theorem 1.2, we obtain the following lemma.

LEMMA 1.2. *Let M be an N-function, Ω a bounded domain in $\mathbf{R}^n$. Then for each sequence $u_n \to 0$ in $(L^*_M(\Omega), \|.\|_M)$ there exist a subsequence u_{n_k} and an element $z \in L^*_M(\Omega)$ such that $|u_{n_k}(s)| \leq z(s)$, $k = 1, 2, \ldots$, in the sense of the natural order on classes of functions. Moreover, $2z \in L_M(\Omega)$ i.e. $\int_\Omega M(2z(x))\,dx < +\infty$.*

Proof. Suppose that $u_n \to 0$ in $(L^*_M(\Omega), \|.\|_M)$. Then, there exists a subsequence u_{n_k} such that $\|u_{n_k}\|_M \leq \frac{1}{2^k}$. Let us put $S_n(x) = \sum_{k=0}^{n} |u_{n_k}(x)|$. We show that the sequence $S_n(x)$ is Cauchy in $(L^*_M(\Omega), \|.\|_M)$. Let $m > n$. Then

$$\|S_n(x) - S_m(x)\|_M = \left\| \sum_{k=n+1}^{m} |u_{n_k}(x)| \right\|_M \leq 2 \sum_{k=n+1}^{m} \|u_{n_k}(x)\|_M \leq 2 \sum_{k=n+1}^{m} \frac{1}{2^k}.$$

Since $(L^*_M(\Omega), \|.\|_M)$ is complete, $S_0(x) = \sum_{k=1}^{+\infty} |u_{n_k}(x)| \in L^*_M(\Omega)$. Consequently, there exists $k_0 \in \mathbf{N}$ such that $\| \sum_{k=k_0}^{+\infty} |u_{n_k}(x)|\|_M < \frac{1}{2}$.

It can be assumed that $S_n(x) \to S_0(x)$ almost everywhere in Ω. Let us put $Z(x) = \sum_{k=k_0}^{+\infty} |u_{n_k}(x)|$. Obviously, $\forall k \geq k_0$ $|u_{n_k}(x)| \leq Z(x)$ almost everywhere in Ω. Further, $\|Z(x)\|_M < \frac{1}{2}$. By theorem (1.2), it follows that $\int_\Omega M(2z(x))\,dx < +\infty$. Thus we achieve the proof. ∎

LEMMA 1.3. *Suppose that Ω is sufficiently smooth and bounded domain in $\mathbf{R}^n$. Let M be an N-function such that $|t|^p \leq M(t)$ for $t \geq t_0$, where $p > n$ and $t_0 > 0$. Then, $\exists\ c > 0$ $\forall f \in W^1_0 L^*_M(\Omega)$ $\|R(f)\|_{C(\bar{\Omega})} \leq c\|f\|_{W^1 L^*_M(\Omega)}$ and $\|f\|_{C(\bar{\Omega})} \leq c\|f\|_{W^1 L^*_M(\Omega)}$, where $R(f)$ is the solution of problem (1.1), (1.2) in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$.*

Proof. Let $f \in W_0^1 L_M^*(\Omega)$. Since $|t|^p \le M(t)$, then $f \in L^p(\Omega)$. Thus, there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of problem (1.1), (1.2). Moreover, $\|u\|_{W^{1,p}(\Omega)} \le c\|f\|_{L^p(\Omega)}$, where $c > 0$.

On the other hand, $W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$. Therefore, $u \in C(\bar{\Omega})$. Let us put $R(f) = u$. So, since $W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$, then

$$\|R(f)\|_{C(\bar{\Omega})} \le c_1\|R(f)\|_{W^{1,p}(\Omega)} \le c_2\|f\|L^p(\Omega) \le c_3\|f\|_{L_M^*(\Omega)} \le c_4\|f\|_{W^1 L_M^*(\Omega)},$$

where $c_1, c_2, c_3, c_4 > 0$.

On the other hand, we have $W_0^1 L_M^*(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$. Thus, there exists $c_5 > 0$ such that $\|f\|_{C(\bar{\Omega})} \le c_5\|f\|_{W^1 L_M^*(\Omega)}$. ■

2. Sufficient conditions of local minimum for Gâteaux functional of second order Dirichlet problem

Suppose that Ω is sufficiently smooth and bounded domain in $\mathbf{R}^n$. Let F be a subset of $W_0^1 L_M^*(\Omega)$, M an N-function such that $|t|^p \le M(t)$ for $t \ge t_0$, where $p > n$ and $t_0 > 0$. Let G be the functional defined on $W_0^1 L_M^*(\Omega)$ by: $G(f) = \int_\Omega v(x, u(x), f(x))\,dx$, where $u(x)$ is the solution of problem (1.1), (1.2) in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and the function $v\colon \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is measurable on $\Omega \times \mathbf{R} \times \mathbf{R}$ and has second derivative with respect to (u, f) on $\mathbf{R} \times \mathbf{R}$ for almost all $x \in \Omega$. Let τ_M be the topology generated by the Orlicz norm $\|.\|_M$. Henceforth in this paragraph $a = const$.

Theorem 2.1. *Suppose that the following condition is added to the conditions of paragraph (1) and (2): $v, v_{uf}^{(2)}$, $v_{fu}^{(2)}$ are continuous in $\Omega \times \mathbf{R} \times \mathbf{R}$. Let us suppose also that*

$$|v(x,u,f)| + |v_u^{(1)}(x,u,f)| + |v_f^{(1)}(x,u,f)| \le a\big(M(u) + M(f)\big) + b_5(x),$$
$$|v_{uu}^{(2)}(x,u,f)| + 2|v_{uf}^{(2)}(x,u,f)| + |v_{ff}^{(2)}(x,u,f)| \le a\big(M(u) + M(f)\big) + b_6(x),$$

where $b_5 \in L^1(\Omega)$, $b_6 \in L^1(\Omega)$. Then, G is a $(\tau_M, \|.\|_{W^1 L_M^*(\Omega)})$-Taylor mapping of first and second order at each point $f \in F$. Moreover, $G^{(2)}(f) \in \mathcal{B}((W_0^1 L_M^*(\Omega), \|.\|_{W^1 L_M^*(\Omega)}), \mathbf{R})$, $G^{(1)}(f) \in \mathcal{L}((W_0^1 L_M^*(\Omega), \|.\|_{W^1 L_M^*(\Omega)}), \mathbf{R})$.

Proof. Let us prove first that the functional G is finite. We have

$$\begin{aligned} |G(f)| &= \left|\int_\Omega v(x,u,f)\,dx\right| \le \int_\Omega |v(x,u,f)|\,dx \\ &\le a\left(\int_\Omega M(u(x))\,dx + \int_\Omega M(f(x))\,dx\right) + \int_\Omega b_5(x)\,dx < \infty. \end{aligned}$$

Indeed, we have $f \in F \subset W_0^1 L_M^*(\Omega) \hookrightarrow C(\bar{\Omega})$. Consequently, $f \in C(\bar{\Omega})$. On the other hand, $u \in W_0^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$. Therefore, $\int_\Omega M(u(x))\,dx < \infty$ and $\int_\Omega M(f(x))\,dx < \infty$. Thus, the functional G is finite.

Let $R\colon W_0^1L_M^*(\Omega) \longrightarrow W_0^1L_M^*(\Omega)$, $h \longmapsto (R(h))(x)$, where $(R(h))(x)$ is the solution of problem

$$Au = h, \tag{2.1}$$

$$u|_{\partial\Omega} = 0. \tag{2.2}$$

Such a solution exists $\forall h \in W_0^1L_M^*(\Omega)$.

Let $G^{(1)}(f)$ be defined by:

$$\begin{aligned}
G^{(1)}(f)h &= \lim_{\lambda\to 0} \lambda^{-1}\big(G(f+\lambda h) - G(f)\big)\\
&= \lim_{\lambda\to 0} \lambda^{-1} \int_\Omega \big[v(x,u+\lambda R(h), f+\lambda h) - v(x,u,f)\big]\,dx\\
&= \lim_{\lambda\to 0} \lambda^{-1} \int_\Omega \big[v(x,u+\lambda R(h), f+\lambda h) -\\
&\qquad - v(x,u,f+\lambda h) + v(x,u,f+\lambda h) - v(x,u,f)\big]\,dx\\
&= \lim_{\lambda\to 0} \int_\Omega \bigg[\int_0^1 v_u^{(1)}(x,u+\theta\lambda R(h), f+\lambda h)R(h)\,d\theta\\
&\qquad + \int_0^1 v_f^{(1)}(x,u,f+\rho\lambda h)h\,d\rho\bigg]dx\\
&= \lim_{\lambda\to 0} \int_\Omega \bigg[\int_0^1 \big[v_u^{(1)}(x,u+\theta\lambda R(h), f+\lambda h) - v_u^{(1)}(x,u,f)\big]R(h)\,d\theta\\
&\qquad + \int_0^1 v_u^{(1)}(x,u,f)R(h)\,d\theta + \int_0^1 \big[v_f^{(1)}(x,u,f+\rho\lambda h) - v_f^{(1)}(x,u,f)\big]h\,d\rho\\
&\qquad + h\int_0^1 v_f^{(1)}(x,u,f)d\rho\bigg]\,dx\\
&= \int_\Omega v_u^{(1)}(x,u,f)R(h)\,dx + \int_\Omega h v_f^{(1)}(x,u,f)\,dx.
\end{aligned}$$

Then

$$G^{(1)}(f) = \int_\Omega v_u^{(1)}(x,u,f)R(h)\,dx + \int_\Omega v_f^{(1)}(x,u,f)h\,dx.$$

Let $G^{(2)}(f)$ be defined by:

$$\begin{aligned}
G^{(2)}(f)(h_1,h_2) &= \lim_{\lambda\to 0} \lambda^{-1}\big[G^{(1)}(f+\lambda h_2) - G^{(1)}(f)\big]h_1\\
&= \lim_{\lambda\to 0} \lambda^{-1}\bigg[\int_\Omega \Big[v_u^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_u^{(1)}(x,u,f)\Big]R(h_1)\,dx\\
&\qquad + \int_\Omega \big[v_f^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_f^{(1)}(x,u,f)\big]h_1\,dx\bigg]\\
&= \lim_{\lambda\to 0} \lambda^{-1}\bigg[\int_\Omega \big[v_u^{(1)}(x,u+\lambda R(h_2), f+\lambda h_2) - v_u^{(1)}(x,u,f+\lambda h_2)\\
&\qquad + v_u^{(1)}(x,u,f+\lambda h_2) - v_u^{(1)}(x,u,f)\big]R(h_1)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \big[v_f^{(1)}(x,u+\lambda R(h_2),f+\lambda h_2)-v_f^{(1)}(x,u,f+\lambda h_2)\\
&+v_f^{(1)}(x,u,f+\lambda h_2)-v_f^{(1)}(x,u,f)\big]h_1 dx\Big]\\
=&\lim_{\lambda\to 0}\lambda^{-1}\Big[\int_\Omega\Big[\int_0^1 v_{uu}^{(2)}(x,u+\theta\lambda R(h_2),f+\lambda h_2)\lambda R(h_2)\,d\theta\\
&+\int_0^1 v_{fu}^{(2)}(x,u,f+\rho\lambda h_2)\lambda h_2\,d\rho\Big]R(h_1)\,dx\\
&+\int_\Omega\Big[\int_0^1 v_{uf}^{(2)}(x,u+\theta\lambda R(h_2),f+\lambda h_2)\lambda R(h_2)\,d\theta\\
&+\int_0^1 v_{ff}^{(2)}(x,u,f+\rho\lambda h_2)\lambda h_2\,d\rho\Big]h_1\,dx\Big]\\
=&\int_\Omega v_{uu}^{(2)}(x,u,f)R(h_1)R(h_2)\,dx+\int_\Omega v_{uf}^{(2)}(x,u,f)R(h_1)h_2\,dx\\
&+\int_\Omega v_{fu}^{(2)}(x,u,f)h_1R(h_2)\,dx+\int_\Omega v_{ff}^{(2)}(x,u,f)h_1h_2\,dx.
\end{aligned}
$$

The linearity and bilinearity of $G^{(1)}(f)$ and $G^{(2)}(f)$ are obvious.

Let us prove now that they are bounded.

$$
\begin{aligned}
|G^{(1)}(f)h|&\le\int_\Omega |v_u^{(1)}(x,u,f)||R(h)|\,dx+\int_\Omega |v_f^{(1)}(x,u,f)||h|\,dx\\
&\le\int_\Omega\big[a\big(M(u)+M(f)\big)+|b_5(x)|\big]\big[|R(h)|+|h)|\big]\,dx\\
&\le\int_\Omega\big[a\big(M(u)+M(f)\big)+|b_5(x)|\big]\big[\max_{x\in\bar\Omega}|[R(h)](x)|+\max_{x\in\bar\Omega}|h(x)|\big]\,dx\\
&=\int_\Omega\big[a\big(M(u)+M(f)\big)+|b_5(x)|\big]\big[\|R(h)\|_{C(\bar\Omega)}+\|h\|_{C(\bar\Omega)}\big]\,dx\\
&\le c_2\|h\|_{W_1L_M^*(\Omega)},
\end{aligned}
$$

where $c_2>0$. For the last inequality, see Lemma 1.3. Consequently, $G^{(1)}(f)\in\mathcal{L}((W_0^1L_M^*(\Omega),\|.\|_{W^1L_M^*(\Omega)}),\mathbf{R})$.

Let us prove now that $G^{(2)}(f)$ is also bounded. We have

$$
\begin{aligned}
|G^{(2)}(f)(h_1,h_2)|\le&\int_\Omega\big[a\big(M(u)+M(f)\big)+|b_6(x)|\big]\times\\
&\times\big[|R(h_1)||R(h_2)|+|R(h_1)||h_2|+|h_1||R(h_2)|+|h_1||h_2|\big]\,dx\\
\le&\big[a\big(M(u)+M(f)\big)+\|b_6(x)\|_{L_1(\Omega)}\big]\times\\
&\times\big[\max_{x\in\bar\Omega}|[R(h_1)](x)|\max_{x\in\bar\Omega}|[R(h_2)](x)|+\max_{x\in\bar\Omega}|[R(h_1)](x)|\max_{x\in\bar\Omega}|h_2(x)|\\
&+\max_{x\in\bar\Omega}|h_1(x)|\max_{x\in\bar\Omega}|[R(h_2)](x)|+\max_{x\in\bar\Omega}|h_1(x)|\max_{x\in\bar\Omega}|h_2(x)|\big]
\end{aligned}
$$

$$\leq c_3\big[\|R(h_1)\|_{C(\bar\Omega)}\|R(h_2)\|_{C(\bar\Omega)}+\|R(h_1)\|_{C(\bar\Omega)}\|h_2\|_{C(\bar\Omega)}$$
$$+\|h_1\|_{C(\bar\Omega)}\|R(h_2)\|_{C(\bar\Omega)}+\|h_1\|_{C(\bar\Omega)}\|h_2\|_{C(\bar\Omega)}\big]$$
$$\leq c_4\|h_1\|_{W^1L^*_M(\Omega)}\|h_2\|_{W^1L^*_M(\Omega)}.$$

For the last inequality see Lemma 1.3. Thus,

$$G^{(2)}(f)\in\mathcal{B}((W^1_0L^*_M(\Omega),\|.\|_{W^1L^*_M(\Omega)}),\mathbf{R}).$$

Let us prove now that G is a $(\tau_M,\|.\|_{W^1L^*_M(\Omega)})$-mapping. Let $f\in F$. We show that the mapping

$$r(h)\equiv G(f+h)-G(f)-G^{(1)}(f)h-2^{-1}G^{(2)}(f)(h,h)$$

is $(\tau_M,\|.\|_{W^1L^*_M(\Omega)})$-of second order at zero.

Assume that this is not so. Then there exist a sequence $\tilde h_m\in F$ and a number $\varepsilon>0$ such that $\tilde h_m\to 0$ in $L^*_M(\Omega)$, but

$$|r(\tilde h_m)|\geq\varepsilon\|\tilde h_m\|^2_{W^1L^*_M(\Omega)}.$$

On the other hand, using the fact that $W^{1,p}_0(\Omega)\hookrightarrow C(\bar\Omega)$ and the regularity of solution of the problem (1.1), (1.2), we obtain $R(\tilde h_m)\to 0$ in $(L^*_M(\Omega),\|.\|_M)$. Using Lemma (1.2), we deduce that $\exists\tilde z\in L^*_M(\Omega)\ \forall m\ |(R(\tilde h_m))(x)|\leq\tilde z(x)$, where $2\tilde z\in L_M(\Omega)$.

Analogously, for $\tilde h_m\in W^1_0L^*_M(\Omega)$, we obtain $|\tilde h_m(x)|\leq\tilde{z_1}(x)$, where $2\tilde{z_1}\in L_M(\Omega)$. We have

$$\begin{aligned}
r(h)=\int_\Omega\big[&v(x,u+R(h),f+h)-v(x,u,f)-v^{(1)}_u(x,u,f)R(h)-v^{(1)}_f(x,u,f)h\\
&-2^{-1}\big[v^{(2)}_{uu}(x,u,f)R^2(h)+2v^{(2)}_{uf}(x,u,f)R(h)h+v^{(2)}_{ff}(x,u,f)h^2\big]\big]\,dx\\
=\int_\Omega\int_0^1&\big[v^{(1)}_u(x,u+\theta R(h),f+h)R(h)-v^{(1)}_u(x,u,f)R(h)\\
&-2^{-1}v^{(2)}_{uu}(x,u,f)R^2(h)\big]\,d\theta\,dx+\int_\Omega\int_0^1\big[v^{(1)}_f(x,u,f+\lambda h)h-v^{(1)}_f(x,u,f)h\\
&-2^{-1}v^{(2)}_{ff}(x,u,f)h^2\big]\,d\lambda\,dx-\int_\Omega\int_0^1 v^{(2)}_{uf}(x,u,f)R(h)h\,d\lambda\,dx\\
&+\int_\Omega\int_0^1\int_0^1 v^{(2)}_{uf}(x,u+\theta R(h),f+\lambda h)hR(h)\,d\lambda\,d\theta dx\\
=\int_\Omega\int_0^1&\big[v^{(1)}_u(x,u+\theta R(h),f)-v^{(1)}_u(x,u,f)-\theta v^{(2)}_{uu}(x,u,f)R(h)\big]R(h)\,d\theta dx\\
&+\int_\Omega\int_0^1\big[v^{(1)}_f(x,u,f+\lambda h)-v^{(1)}_f(x,u,f)-\lambda v^{(2)}_{ff}(x,u,f)h\big]h\,d\lambda\,dx\\
&-\int_\Omega\int_0^1 v^{(2)}_{uf}(x,u,f)R(h)h\,d\lambda\,dx
\end{aligned}$$

$$+\int_\Omega\int_0^1\int_0^1 v_{uf}^{(2)}(x,u+\theta R(h),f+\lambda h)hR(h)\,d\lambda\,d\theta\,dx.$$

Let A_m, B_m be two functions defined by:

$$A_m(x,\theta)=\begin{cases}\frac{v_u^{(1)}(x,u+\theta R(\tilde h_m),f)-v_u^{(1)}(x,u,f)}{R(\tilde h_m)}-\theta v_{uu}^{(2)}(x,u,f), & \text{if } R(\tilde h_m)\neq 0,\\ 0, & \text{if } R(\tilde h_m)=0,\end{cases}$$

$$B_m(x,\lambda)=\begin{cases}\frac{v_f^{(1)}(x,u,f+\lambda\tilde h_m)-v_f^{(1)}(x,u,f)}{\tilde h_m}-\lambda v_{ff}^{(2)}(x,u,f), & \text{if } \tilde h_m\neq 0,\\ 0, & \text{if } \tilde h_m=0.\end{cases}$$

Let F_m be defined by: $F_m(x,\theta,\lambda)=v_{uf}^{(2)}(x,u(x)+\theta R(\tilde h_m),f+\lambda\tilde h_m)-v_{uf}^{(2)}(x,u(x),f)$. Then

$$|r(\tilde h_m)|=\Big|\int_\Omega\int_0^1 A_m(x,\theta)R^2(\tilde h_m)\,d\theta\,dx+\int_\Omega\int_0^1 B_m(x,\lambda)\tilde h_m^2\,d\lambda\,dx + \int_\Omega\int_0^1\int_0^1 F_m(x,\theta,\lambda)R(\tilde h_m)\tilde h_m\,d\lambda\,d\theta\,dx\Big|.$$

Thus

$$\begin{aligned}|r(\tilde{h_m})|&\le\int_0^1\int_\Omega|A_m(x,\theta)|\,dx\,d\theta\max_{x\in\bar\Omega}|[R(\tilde h_m)](x)|^2\\&\quad+\int_0^1\int_\Omega|B_m(x,\lambda)|\,dx\,d\lambda\max_{x\in\bar\Omega}|\tilde h_m|^2\\&\quad+\int_0^1\int_0^1\int_\Omega|F_m(x,\theta,\lambda)|\,dx\,d\lambda\,d\theta\max_{x\in\bar\Omega}|[R(\tilde h_m)](x)|\max_{x\in\bar\Omega}|\tilde h_m|\\&\le c_5\Big[\int_0^1\int_\Omega|A_m(x,\theta)|\,dx\,d\theta+\int_0^1\int_\Omega|B_m(x,\lambda)|\,dx\,d\lambda\\&\quad+\int_0^1\int_0^1\int_\Omega|F_m(x,\theta,\lambda)|\,dx\,d\lambda\,d\theta\Big]\|\tilde h_m)\|^2_{W^1L^*_M(\Omega)}.\end{aligned}\tag{2.3}$$

On the other hand, $\exists k_m(x)$ such that $0\le k_m(x)\le 1$ and

$$|A_m(x,\theta)|\le|v_{uu}^{(2)}(x,u(x)+k_m(x)\theta[R(\tilde{h_m})](x),f)|+|v_{uu}^{(2)}(x,u(x),f)|.$$

So, we obtain

$$\begin{aligned}|A_m(x,\theta)|&\le a\big[M(u(x)+k_m(x)\theta[R(\tilde{h_m})](x))+M(f)\big]+|b_6(x)|\\&\quad+a\big[M(u)+M(f)\big]+|b_6(x)|\\&\le a\Big[\frac12 M(2\tilde z(x)))|+2M(f)+M(u(x))+\frac12 M(2u(x)))\Big]\\&\quad+2|b_6(x)|\in L^1(\Omega).\end{aligned}$$

Analogously, $\exists S_m(x)$ such that $0\le S_m(x)\le 1$ and

$$|B_m(x,\lambda)|\le|v_{ff}^{(2)}(x,u(x),\lambda S_m(x)\tilde h_m(x)+f)|+|v_{ff}^{(2)}(x,u(x),f)|.$$

So, we obtain

$$\begin{aligned}|B_m(x,\lambda)| &\le a\big[M(u(x)) + M(\lambda S_m(x)\tilde{h}_m(x) + f)\big] + |b_6(x)| \\ &\quad + a\big[M(u(x)) + M(f(x))\big] + |b_6(x)| \\ &\le a\left[2M(u) + \frac{1}{2}M(2\tilde{z}_1(x)) + \frac{1}{2}M(2f(x)) + M(f)\right] + 2|b_6(x)| \in L^1(\Omega).\end{aligned}$$

Analogously for F_m, we obtain

$$\begin{aligned}|F_m(x,\theta,\lambda)| &\le |v^{(2)}_{uf}(x,u(x),f)| + |v^{(2)}_{uf}(x,u(x)+\theta R(\tilde{h}_m), f+\lambda\tilde{h}_m)| \\ &\le a\left[\frac{1}{2}M(2\tilde{z}(x)) + \frac{1}{2}M(2\tilde{z}_1(x))\frac{1}{2}M(2u) + \frac{1}{2}M(2f)\right] + 2|b_6(x)| \\ &\quad + a\big[M(u) + M(f)\big] \quad \in L^1(\Omega).\end{aligned}$$

Let us remark that $A_m(x,\theta) \to 0$, $B_m(x,\lambda) \to 0$, $F_m(x,\theta,\lambda) \to 0$ almost everywhere. Thus

$$\begin{aligned}\int_0^1 \int_\Omega |A_m(x,\theta)|\, dx\, d\theta \to 0 \quad \text{as} \quad m \to +\infty, \\ \int_0^1 \int_\Omega |B_m(x,\lambda)|\, dx\, d\lambda \to 0 \quad \text{as} \quad m \to +\infty, \\ \int_0^1 \int_0^1 \int_\Omega |F_m(x,\theta,\lambda)|\, dx\, d\lambda\, d\theta \to 0 \quad \text{as} \quad m \to +\infty;\end{aligned}$$

but this contradicts (2.3).

Now let us prove that the functional G is a $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-Taylor mapping of first order at each point $f \in F$. We must estimate

$$r(h) \equiv G(f+h) - G(f) - G^{(1)}(f)h.$$

Assume that $\tilde{h}_m \to 0$ in $L^*_M(\Omega)$, and $|r(\tilde{h}_m)| \ge \varepsilon\|\tilde{h}_m\|_{W^1L^*_M(\Omega)}$. Thus $R(\tilde{h}_m) \to 0$ in $(L^*_M(\Omega), \|.\|_M)$ as $m \to +\infty$. Using Lemma (1.2), we deduce that there exists $\tilde{z} \in L^*_M(\Omega)$ and there exists $\tilde{z_1} \in L^*_M(\Omega)$ such that $\forall m \in \mathbf{N}$ $|[R(\tilde{h}_m)](x)| \le \tilde{z}(x)$ and $|\tilde{h}_m(x)| \le \tilde{z_1}(x)$ almost everywhere in Ω. Moreover, $2\tilde{z} \in L_M(\Omega)$ and $2\tilde{z_1} \in L_M(\Omega)$.

On the other hand, we have

$$\begin{aligned}|r(\tilde{h}_m)| &= \bigg|\int_\Omega \big[v(x,u(x)+[R(\tilde{h}_m)](x), f(x)+\tilde{h}_m(x)) - v(x,u(x),f(x)) \\ &\qquad - v^{(1)}_u(x,u(x),f(x))R(\tilde{h}_m) - v^{(1)}_f(x,u(x),f(x))\tilde{h}_m\big]\, dx\bigg| \\ &\le \bigg|\int_\Omega \big[v(x,u(x)+R(\tilde{h}_m), f(x)+\tilde{h}_m) - v(x,u(x),f(x)+\tilde{h}_m) \\ &\qquad - v^{(1)}_u(x,u(x),f(x))R(\tilde{h}_m) + v(x,u(x),f(x)+\tilde{h}_m)\end{aligned}$$

$$- v(x,u(x),f(x)) - v_f^{(1)}(x,u(x),f(x))\tilde{h}_m\big]\,dx\bigg|$$
$$= \left|\int_\Omega A_m(x)R(\tilde{h}_m)\,dx\right| + \left|\int_\Omega B_m(x)\tilde{h}_m dx\right|,$$

where

$$A_m(x) = \begin{cases} \frac{v(x,u+R(\tilde{h}_m),f+\tilde{h}_m)-v(x,u,f+\tilde{h}_m)}{R(\tilde{h}_m)} - v_u^{(1)}(x,u,f), & \text{if } R(\tilde{h}_m)\neq 0,\\ 0, & \text{if } R(\tilde{h}_m)=0,\end{cases}$$
$$B_m(x) = \begin{cases} \frac{v(x,u,f+\tilde{h}_m)-v(x,u,f)}{\tilde{h}_m} - v_f^{(1)}(x,u,f), & \text{if } \tilde{h}_m\neq 0,\\ 0, & \text{if } \tilde{h}_m=0.\end{cases}$$

Consequently,

$$\begin{aligned} |r(\tilde{h}_m)| &\leq \int_\Omega |A_m(x)|\,dx\ \max_{x\in\bar{\Omega}}\big|[R(\tilde{h}_m)](x)\big| + \int_\Omega |B_m(x)|dx\ \max_{x\in\bar{\Omega}}\big|\tilde{h}_m(x)\big|\\ &\leq c_6\left[\int_\Omega |A_m(x)|\,dx + \int_\Omega |B_m(x)|\,dx\right]\|\tilde{h}_m\|_{W^1L_M(\Omega)}. \end{aligned} \tag{2.4}$$

Let us remark that $A_m(x)\to 0$, $B_m(x)\to 0$ almost everywhere in Ω.

Using the mean value theorem, we obtain

$$\begin{aligned} |A_m(x)| &\leq a\big[M(u+R(\tilde{h}_m)) + M(f(x)+\tilde{h}_m(x))\big] + |\widehat{b}_3(x)|\\ &\quad + a\big[M(u(x)) + M(f(x))\big] + |\widehat{b}_3(x)|\\ &\leq a\left[\frac{1}{2}M(2\tilde{Z}(x)) + \frac{1}{2}M(2\tilde{Z}_1(x))\right] + 2|\widehat{b}_3(x)|\\ &\quad + a\left[\frac{1}{2}M(2u(x)) + \frac{1}{2}M(2f(x))\right] + a\big[M(u(x)) + M(f(x))\big] \in L^1(\Omega). \end{aligned}$$

Using the same reasons, we obtain

$$\begin{aligned} |B_m(x)| &\leq a\big[M(u(x)) + M(f(x)+\tilde{h}_m(x))\big] + |\widehat{b}_3(x)|\\ &\quad + a\big[M(u(x)) + M(f(x))\big] + |\widehat{b}_3(x)|\\ &\leq a\left[2M(u(x)) + \frac{1}{2}M(2\tilde{z}_1(x)) + \frac{1}{2}M(2f(x))\right]\\ &\quad + aM(f(x)) + 2|\widehat{b}_3(x)| \in L^1(\Omega). \end{aligned}$$

By the Lebesgue dominated convergence theorem, we conclude that

$$\int_\Omega |A_m(x)|\,dx \to 0 \quad \text{as} \quad m\to+\infty$$

and

$$\int_\Omega |B_m(x)|\,dx \to 0 \quad \text{as} \quad m\to+\infty;$$

but this contradicts (2.4). ■

Now let us give the sufficient conditions of optimality for the problem (1.3), (1.4) and (1.5).

THEOREM 2.2 *Suppose that, in problem (1.4), v_k satisfies the conditions of Theorem 2.1, then the functionals*

$$J_k(f) \equiv \int_\Omega v_k(x,u,f)\,dx, \quad (k = s_1+1,\dots,s_1+s_2)$$

are $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_\Omega v_k(x,u,f)\,dx + c_k\|f\|^2_{W^{1,2}(\Omega)}$, $(k = 0,\dots,s_1)$ are lower $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0,\dots,s_1+s_2)$.

Let us suppose also that, $J(\widehat{f}) = 0$, $J^{(1)}(\widehat{f})$ is an open mapping of

$$(W_0^1 L^*_M(\Omega), \|.\|_{W^1L^*_M(\Omega)})$$

onto $\mathbf{R}^{s_2}$, $\exists\ \widehat{y}^* \in (\mathbf{R}^{s_2})^*$, $\exists\alpha > 0$: $\mathcal{L}_f(\widehat{f},\widehat{y}^*,1) = 0$, and $\forall h \in ker J^{(1)}(\widehat{f})$ $\mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,1)(h,h) \geq 2\alpha\|h\|^2_{W^1L^*_M(\Omega)}$, where $\mathcal{L}_f(\widehat{f},\widehat{y}^*,1)$ and $\mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,1)$ are given by formulas (1.10), (1.11). Then $\widehat{f}$ is a strict τ_M-local minimum point.

Proof. All conditions of Theorem 1.5 in [2] are satisfied. Thus $\widehat{f}$ is a strict τ_M-minimum point. ■

THEOREM 2.3. *Suppose that, in problem (1.5), v_k satisfies the conditions of Theorem 2.1, then the functionals*

$$J_k(f) \equiv \int_\Omega v_k(x,u,f)\,dx, \quad (k = s_1+1,\dots,s_1+s_2)$$

are $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_\Omega v_k(x,u,f)\,dx + c_k\|f\|^2_{W^{1,2}(\Omega)}$, $(k = 0,\dots,s_1)$ are lower $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0,\dots,s_1+s_2)$.

Let us suppose also that, $\widehat{f} \in F$, $J(\widehat{f}) = 0$, $J_k(\widehat{f}) = 0$, $(k = 0,\dots,s_1)$. Let us put $L = \{h \in W_0^1L^*_M(\Omega) / J_k^{(1)}(\widehat{f})h = 0,\ k = 1,\dots,s_1,\ J^{(1)}(\widehat{f})h = 0\}$. Suppose that $J^{(1)}(\widehat{f})$ is an open map from $(W_0^1L^*_M(\Omega), \|.\|_{W^1L^*_M(\Omega)})$ onto $\mathbf{R}^{s_2}$, $\exists\widehat{\lambda} \in (\mathbf{R}^{s_1})^*$, $\exists\widehat{y}^* \in (\mathbf{R}^{s_2})^*$, $\exists\gamma \geq 0$, $\exists\widehat{\lambda}_k > 0$, $(k = 1,\dots,s_1)$: $\mathcal{L}_f(\widehat{f},\widehat{y}^*,\widehat{\lambda},1) = 0$ and $\forall h \in L$ $\mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)(h,h) \geq 2\gamma\|h\|^2_{W^1L^*_M(\Omega)}$, where $\mathcal{L}_f(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)$ and $\mathcal{L}_{ff}(\widehat{f},\widehat{y}^*,\widehat{\lambda},1)$ are defined by formulas (1.7) and (1.8). Then $\widehat{f}$ is a strict τ_M-local minimum point.

Proof. All conditions of Theorem 1.6 in [2] are satisfied. Thus $\widehat{f}$ is a strict τ_M-local minimum point. ■

THEOREM 2.4. *Suppose that, in problem (1.3), v_k satisfies the conditions of Theorem 2.1, then the functionals*

$$J_k(f) \equiv \int_\Omega v_k(x,u,f)\,dx, \quad (k = s_1+1,\dots,s_1+s_2)$$

are $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-Taylor mappings of first and second order at each point $f \in F$ and $J_k(f) = \int_\Omega v_k(x,u,f)\,dx + c_k\|f\|^2_{W^{1,2}(\Omega)}$, $(k = 0,\dots,s_1)$ are lower $(\tau_M, \|.\|_{W^1L^*_M(\Omega)})$-semi-Taylor mappings of first and second order at each point $f \in F$. Consequently, $\exists J_k^{(1)}(f)$ and $\exists J_k^{(2)}(f)$, $(k = 0,\dots,s_1+s_2)$.

Let us suppose also that, $J_0^{(1)}(\widehat{f}) = 0$ and $\exists\alpha > 0\ \forall h \in W_0^1L^*_M(\Omega)$: $J_0^{(2)}(\widehat{f})(h,h) \geq 2\alpha\|h\|^2_{W^1L^*_M(\Omega)}$. Then $\widehat{f}$ is a strict τ_M-local minimum point.

Proof. All conditions of Theorem 1.4 in [2] are satisfied. Thus $\widehat{f}$ is a strict τ_M-local minimum point. ■

REMARK. Let us remark that, in Theorem 2.1, the increasing conditions satisfied by v are not sufficient to certify the Frechet-differentiability of functional $G : (W_0^1L^*_M(\Omega), \|.\|_M) \to \mathbf{R}$.

Indeed, let us define $v\colon \Omega\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ by: $v(x,u,f) = \big[\mathrm{ch}|u| - |u| - 1\big] + \big[\mathrm{ch}|f| - |f| - 1\big]$, and put $b_0(x) \equiv 0$. Let us suppose that $a = 1$. Let us put $M(t) = e^{|t|} - |t| - 1$, and let $d_m \to +\infty$. Let $\tilde{f} \in W_0^1L^*_M(\Omega)$. By the countable additivity of Lebesgue measure,

$$\exists c > 0\ \exists\Omega' \subset \Omega,\ \mu(\Omega') > 0,\ \rho(\Omega', \partial\Omega) > 0,$$

and $\forall x \in \Omega'$ $|\tilde{f}(x)| \leq c$. Let $\Omega_m \subset \Omega'$ such that $\mu(\Omega_m) = (e^{d_m} - d_m - 1)^{-1}$. Put $\mathcal{D} \equiv \max\{|v(x,u,f)| : |u| \leq c,\ |f| \leq c,\ x \in \overline{\Omega}\} < \infty$.

Let $\tilde{h}_m$ be defined by:

$$\tilde{h}_m(x) = \begin{cases} (d_m)^{\frac{1}{2}} - \tilde{f}(x), & \text{if } x \in \Omega_m, \\ 0, & \text{if } x \in \Omega\setminus\Omega_m. \end{cases}$$

Then $\|\tilde{h}_m(x)\|_{(M)} \to 0$, but $|G(\tilde{f}(x) + \tilde{h}_m(x)) - G(\tilde{f}(x))| \geq \big[\mathrm{ch}|d_m| - |d_m| - 1\big]\mu(\Omega_m) - \mathcal{D}\mu(\Omega_m) \to \frac{1}{2}$. Thus G is not Frechet-differentiable in $(W_0^1L^*_M(\Omega), \|.\|_M)$.

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