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APPROXIMATE SOLUTION OF A BOUNDARY PROBLEM FOR A LINEAR COMPLEX DIFFERENTIAL EQUATION OF THE SECOND ORDER

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Abstract. The paper considers the problem

$$D^{(2)}w + a(z,\bar{z})Dw + b(z,\bar{z})w = f(z,\bar{z}),$$

with boundary conditions

$$c_1(z)\alpha_{g(z)}w + c_2\alpha_{g(z)}Dw = c_3(z),$$
$$d_1(z)\alpha_{h(z)}w + d_2\alpha_{h(z)}Dw = d_3(z).$$

The problem is solved approximately, by using the formulas

$$2\frac{z^2}{h^2}(w_{i+1} - 2w_i + w_{i-1}) + a_i\frac{z}{h}(w_{i+1} - w_{i-1}) + b_i w_i = f_i, \quad i = 1,\dots,n-1,$$
$$c_1(z)w_0 + c_2(z)\frac{z}{h}(-w_2 + 4w_1 - 3w_0) = c_3(z),$$
$$d_1(z)w_n + d_2(z)\frac{z}{h}(3w_n - 4w_{n-1} + w_{n-2}) = d_3(z).$$

1. Introduction

Many systems consisting of two partial equations with two unknown functions $u(x,y)$ and $v(x,y)$, which arise in various problems of physics, techniques and mechanics, can be reduced to one complex differential equation with an unknown complex function $w(z,\bar{z}) = w(x,y) = u(x,y) + iv(x,y)$. The theory of complex differential equations is currently researched with success by W. Tutschke, V. Gabrinovič, G. Manžnavidze and others.

S. Fempl and J. Kečkić ([1], [2]) studied the following linear complex differential equation of the n-th order

$$D^{(n)}w + a_1(z)D^{(n-1)}w + \dots + a_{n-1}(z)Dw + a_n(z)w = 0,$$

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where

$$Dw = (u'_x - v'_y) + i(u'_y + v'_x) = 2w'_{\bar{z}}$$

is the known Kolosov differential operator, while $a_k(z)$, $k = 1, 2, \dots, n$, are given analytic functions. If $r_k(z)$, $k = 1, 2, \dots, n$, are the solutions of the corresponding characteristic equation

$$\xi^n + a_1\xi^{n-1} + \dots + a_{n-1}\xi + a_n = 0$$

for which it is assumed that they are all mutually different, the authors proved that the general solution of the differential equation (1) has the form

$$w(z, \bar{z}) = \sum_{k=1}^{n} \varphi_k(z) \exp(\tfrac{r_k(z)}{2}\,\bar{z}), \tag{1}$$

where $\varphi_k(z)$ are arbitrary analytic functions. M. Čanak [3] obtained the same result after several years, by means of the method of complex areolar series. Later on, that result was reached by D. Dimitrovski and B. Ilijevski, using the calculus of residues and contour Laplace integral, respectively.

In this paper, we consider the following linear, complex differential equation of the second order

$$D^{(2)}w + a(z, \bar{z})Dw + b(z, \bar{z})w = f(z, \bar{z}),$$

where it is assumed that $a(z, \bar{z})$, $b(z, \bar{z})$ and $f(z, \bar{z})$ are complex continuous functions.

2. Differential equation with analytic coefficients

Let us consider the linear, complex differential equation of the second order with analytic coefficients

$$D^{(2)}w + a(z)Dw + b(z)w = f(z, \bar{z}). \tag{2}$$

The general solution of this equation (bearing in mind (1)) is

$$w(z, \bar{z}) = \varphi_1 \exp(\tfrac{r_1}{2}\bar{z}) + \varphi_2 \exp(\tfrac{r_2}{2}\bar{z}) + w_p, \tag{3}$$

where r_1 and r_2 are the roots of the characteristic equation

$$r^2 + a(z)r + b(z) = 0,$$

and $\varphi_1(z)$ and $\varphi_2(z)$ are arbitrary analytic functions, while w_p is a particular solution of equation (2). Since the general solution contains two arbitrary analytic functions, their determination requires two additional conditions. It is of great importance, both theoretically and practically, to consider the following boundary problem.

Boundary problem. Let us consider two simple, smooth closed contours L_1 and L_2 whose equations are L_1: $\bar{z} = g(z)$ and L_1: $\bar{z} = h(z)$. It is necessary to find the solution of differential equation (2) which satisfies, on the contours L_1 and L_2, the following boundary conditions

$$\begin{aligned} c_1(z)\alpha_{g(z)}w + c_2\alpha_{g(z)}Dw &= c_3(z),\\ d_1(z)\alpha_{h(z)}w + d_2\alpha_{h(z)}Dw &= d_3(z), \end{aligned} \tag{4}$$

where $c_i(z)$ and $d_i(z)$ $(i = 1, 2, 3)$ are given analytic functions. In the special case $c_2(z) = d_2(z) = 0$ the boundary problem (4) reduces to the Dirichlet-Schwarz problem, while in the case $c_1(z) = d_1(z) = 0$ it reduces to the Neumann problem.

REMARK 1. The meaning of the operator $\alpha_{g(z)}$ is the following.

Let $W = \{\, w \mid w = w(z, \bar{z})\,\}$ be the set of complex functions which are continuous in z and $\bar{z}$ in some region T, and let $\Omega = \{\, \omega \mid \omega = \omega(z)\,\}$ be the set of analytic functions in T. The function $\omega(z) = \alpha_{g(z)}w$ is obtained from the function $w(z, \bar{z})$ by replacing $\bar{z}$ with $g(z)$, while the value of z remains unchanged. The function $\omega = \alpha_{g(z)}w$ does not contain variable $\bar{z}$, so in the general case it is analytic. It can be easily proved that the operator $\alpha_{g(z)}$ maps the set W onto the set Ω, and that one and only one image ω corresponds to each original w.

In the geometrical sense, this operator means the following.

If $\bar{z} = g(z)$ is the equation of a simple, smooth closed contour, then the functions $w(z, \bar{z})$ and $\alpha_{g(z)}w$ have the same boundary value on this contour.

If the operators $\alpha_{g(z)}$ and $\alpha_{h(z)}$ are applied to the solutions of equation (2) given by (3) and to its areolar derivative

$$Dw = r_1(z)\varphi_1(z)\exp(\tfrac{r_1(z)}{2}\bar{z}) + r_2(z)\varphi_2(z)\exp(\tfrac{r_2(z)}{2}\bar{z}) + Dw_p,$$

we obtain

$$\begin{aligned} \alpha_{g(z)}w &= \varphi_1(z)\exp(\tfrac{r_1(z)}{2}g(z)) + \varphi_2(z)\exp(\tfrac{r_2(z)}{2}g(z)) + \alpha_{g(z)}w_p,\\ \alpha_{h(z)}w &= \varphi_1(z)\exp(\tfrac{r_1(z)}{2}h(z)) + \varphi_2(z)\exp(\tfrac{r_2(z)}{2}h(z)) + \alpha_{h(z)}w_p,\\ \alpha_{g(z)}Dw &= r_1(z)\varphi_1(z)\exp(\tfrac{r_1(z)}{2}g(z)) + r_2(z)\varphi_2(z)\exp(\tfrac{r_2(z)}{2}g(z)) + \alpha_{g(z)}Dw_p,\\ \alpha_{h(z)}Dw &= r_1(z)\varphi_1(z)\exp(\tfrac{r_1(z)}{2}h(z)) + r_2(z)\varphi_2(z)\exp(\tfrac{r_2(z)}{2}h(z)) + \alpha_{h(z)}Dw_p. \end{aligned} \tag{5}$$

Further, using (4) and (5), we obtain

$$\begin{aligned} &\varphi_1(z)\exp(\tfrac{r_1(z)}{2}g(z))[c_1(z) + r_1(z)c_2(z)] + \varphi_2(z)\exp(\tfrac{r_2(z)}{2}g(z))[c_1(z) + r_2(z)c_2(z)]\\ &\qquad = c_3(z) - c_1(z)\alpha_{g(z)}w_p - c_2(z)\alpha_{g(z)}Dw_p,\\ &\varphi_1(z)\exp(\tfrac{r_1(z)}{2}h(z))[d_1(z) + r_1(z)d_2(z)] + \varphi_2(z)\exp(\tfrac{r_2(z)}{2}h(z))[d_1(z) + r_2(z)d_2(z)]\\ &\qquad = d_3(z) - d_1(z)\alpha_{h(z)}w_p - d_2(z)\alpha_{h(z)}Dw_p. \end{aligned} \tag{6}$$

It is obvious that (6) is a system of two linear equations with two unknowns $\varphi_1(z)$ and $\varphi_2(z)$. The determinant of this system is

$$\begin{aligned} D &= \begin{vmatrix} \exp(\frac{r_1 g}{2})[c_1(z) + r_1(z)c_2(z)] & \exp(\frac{r_2 g}{2})[c_1(z) + r_2(z)c_2(z)] \\ \exp(\frac{r_1 h}{2})[d_1(z) + r_1(z)d_2(z)] & \exp(\frac{r_2 h}{2})[d_1(z) + r_2(z)d_2(z)] \end{vmatrix}\\ &= \exp(\tfrac{r_1 g + r_2 h}{2})(c_1 + r_1c_2)(d_1 + r_2d_2) - \exp(\tfrac{r_1 h + r_2 g}{2})(d_1 + r_1d_2)(c_1 + r_2c_2) \neq 0, \end{aligned}$$

and therefore the system has a unique solution $\varphi_1(z)$ and $\varphi_2(z)$. Replacing their values in relation (3), we obtain the solution of boundary problem (4) for the complex differential equation (2).

3. Differential equation with non-analytic coefficients

Let us consider the complex, linear differential equation of the second order

$$D^{(2)}w + a(z,\bar{z})Dw + b(z,\bar{z})w = f(z,\bar{z}) \tag{7}$$

whose coefficients $a(z,\bar{z})$, $b(z,\bar{z})$ and $f(z,\bar{z})$ are continuous, non-analytic, complex functions (in the region G). It is necessary to find the solution of equation (7) which satisfies boundary conditions (4) on the given circles L_1: $\bar{z} = a/z$, L_2: $\bar{z} = b/z$, $a < b$, where $c_i(z)$ and $d_i(z)$ $(i = 1, 2, 3)$ are given analytic functions.

Let us take a finite interval $[a, b]$, which can be divided into n equal parts $a = x_0$, $x_1 = x_0 + h$, ... , $x_k = x_0 + kh$, ... , $x_n = x_0 + nh = b$. Further, a system of circles k_i: $\bar{z} = x_i/x$ $(i = 0, 1, \dots, n)$ can be constructed. The non-analytic complex function, which is the solution of boundary problem (7)–(4), can be expressed in an explicit form. However, when this is not possible, it can be represented in a tabular form, if its boundary values on the given circles $w(z,\bar{z})|_{k_i}$ are known. In some cases, by using operator α, we can compute a series of analytic functions $w_0(z)$, $w_1(z)$, ... , $w_n(z)$ whose boundary values on circles k_i are identical with the boundary values of non-analytic function $w(z,\bar{z})$ on the same circles. The representation of non-analytic function $w(z,\bar{z})$ by the following tabular form

k_i	k_0: $\bar{z} = \dfrac{x_0}{z}$	k_1: $\bar{z} = \dfrac{x_1}{z}$	$\cdots$	k_n: $\bar{z} = \dfrac{x_n}{z}$
$w_i(z)$	$w_0(z)$	$w_1(z)$	$\cdots$	$w_n(z)$

(8)

is called α-representation of the function $w(z,\bar{z})$ on the system of circles k_i. If $\alpha_{x_i/z}w(z,\bar{z}) = w_i(z)$ $(i = 0, 1, \dots, n)$, then representation (8) is unique, i.e., one and only one analytic function $w_i(z)$ corresponds to each value (each circle) $\bar{z} = x_i/z$. This α-representation, which is very important in the theory of complex interpolation, appeared for the first time in paper [4]. In that paper the author constructed the so-called α-interpolation, complex polynomial which corresponds to table (8), and which approximates non-analytic function $w(z,\bar{z})$.

By using a similar idea, we can search for an approximate solution of problem (7)–(4) in the form (8), that is, in the form of a series of analytic functions $w_i(z)$ $(i = 0, 1, \dots, n)$ whose boundary values on the circles $\bar{z} = x_i/z$ are approximately equal to the boundary values of the exact solution of the boundary problem on these circles. By using the notation

$$\begin{aligned} a_i(z) &= \alpha_{x_i/z}a(z,\bar{z}), & b_i(z) &= \alpha_{x_i/z}b(z,\bar{z}),\\ Dw_i &= \alpha_{x_i/z}Dw, & D^{(2)}w_i &= \alpha_{x_i/z}D^{(2)}w,\\ w_i(z) &= \alpha_{x_i/z}w(z,\bar{z}), & f_i(z) &= \alpha_{x_i/x}f(z,\bar{z}), \end{aligned}$$

and substituting the values of areolar complex derivatives by approximate ψ differences, defined in the following way

$$\frac{\alpha_{x_1/z}w-\alpha_{x_0/z}w}{h/z}=\psi\left(\frac{x_0}{z},\frac{x_1}{z}\right),$$
$$\dots\dots\dots$$
$$\frac{\alpha_{x_n/z}w-\alpha_{x_{n-1}/z}w}{h/z}=\psi\left(\frac{x_{n-1}}{z},\frac{x_n}{z}\right),$$
$$\dots\dots\dots$$
$$\frac{\psi\left(\frac{x_{n-1}}{z},\frac{x_n}{z}\right)-\psi\left(\frac{x_{n-2}}{z},\frac{x_{n-1}}{z}\right)}{2\frac{h}{z}}=\psi\left(\frac{x_{n-2}}{z},\frac{x_{n-1}}{z},\frac{x_n}{z}\right),$$
$$\dots\dots\dots$$
$$\frac{\psi\left(\frac{x_1}{z},\frac{x_2}{z},\dots\frac{x_n}{z}\right)-\psi\left(\frac{x_0}{z},\frac{x_1}{z},\dots,\frac{x_{n-1}}{z}\right)}{n\frac{h}{z}}=\psi\left(\frac{x_0}{z},\frac{x_1}{z},\dots,\frac{x_{n-1}}{z},\frac{x_n}{z}\right),$$

we obtain

$$\begin{gathered}\psi\left(\frac{x_0}{z},\frac{z_1}{z}\right)=\frac{z}{h}(w_1-w_0),\quad\dots,\quad\psi\left(\frac{x_{n-1}}{z},\frac{x_n}{z}\right)=\frac{z}{h}(w_n-w_{n-1}),\\ \psi\left(\frac{x_0}{z},\frac{x_1}{z},\frac{x_2}{z}\right)=\frac{z^2}{2h^2}(w_2-2w_1+w_0),\quad\dots,\\ \psi\left(\frac{x_{n-2}}{z},\frac{x_{n-1}}{z},\frac{x_n}{z}\right)=\frac{z^2}{2h^2}(w_n-2w_{n-1}+w_{n-2}).\end{gathered}\tag{9}$$

It is well-known that for the areolar derivative in the sense of Vekua, we have

$$DW|_{\bar z=g_0z}=2\lim_{g_1\to g_0}\frac{\alpha_{g_1}W-\alpha_{g_0}W}{g_1-g_0}=\alpha_{g_0}DW,\tag{10}$$

where L_0: $\bar z=g_0(z)$ and L_1: $\bar z=g_1(z)$ are two simple, smooth, closed contours such that contour L_1 encircles contour L_0 and which, using some continuous transformation, can be shrinked to it. It follows form (9) and (10), where $g_i(z)=x_i/z$, that we can take the approximate values

$$\begin{aligned}Dw_i&=\frac{z}{h}(w_{i+1}-w_{i-1}),\\ D^2w_i&=\frac{2z^2}{h^2}(w_{i+1}-2w_i+w_{i-1}),\quad(i=1,2,\dots,n-1).\end{aligned}\tag{11}$$

Taking into account equation (7) and boundary conditions (4) on the circles $\bar z=x_i/z$ we obtain

$$2\frac{z^2}{h^2}(w_{i+1}-2w_i+w_{i-1})+a_i\frac{z}{h}(w_{i+1}-w_{i-1})+b_iw_i=f_i,\quad(i=1,\dots,n-1),\tag{12}$$

$$c_1(z)w_0+c_2(z)\frac{z}{h}(-w_2+4w_1-3w_0)=c_3(z),\tag{13}$$

$$d_1(z)w_n+d_2(z)\frac{z}{h}(3w_n-4w_{n-1}+w_{n-2})=d_3(z).\tag{14}$$

Relations (12), (13) and (14) form a linear system of $n+1$ equations with $n+1$ unknowns. By solving this system, we find the required analytic functions $w_0(z)$, $w_1(z), \ldots, w_n(z)$, and therefore we get the required α-representation of the approximate solution with the given boundary problem.

REMARK 2. Using the approximations for the derivatives given by (11), as well as for the derivatives in w_0 and w_n given by

$$Dw_0 = \frac{z}{h}(-w_2 + 4w_1 - 3w_0), \qquad Dw_n = \frac{z}{h}(3w_n - 4w_{n-1} + w_{n-2}),$$

it can be proved that on the circles $\bar{z} = x_i/z$ the error of the proposed method is given by

$$R = L(w) - L_h(w_n) = o(h^2).$$

We refer to [5] and [6] for more details about the proof.

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