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SOME NEW PROPERTIES OF SEQUENCE SPACES AND APPLICATION TO THE CONTINUED FRACTIONS

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Abstract. We give two methods of approximation of a solution of an infinite linear system. First, we will construct a sequence of finite matrices which approaches a solution, this one being defined by an infinite sequence. Then, we will apply these results to the continued fractions.

1. Introduction

Infinite linear systems have been studied by many authors, let us cite for instance Cooke [1], Defranza, Zeller [2], Maddox [6], Pòlya [1], Reade [13]. Here, we construct a natural sequence which converges to a solution of the system, by the means of a sequence of finite matrices deduced from an infinite matrix A. This principle has been developed by Pòlya, but the space used by this author contains infinitely many solutions, and the matrices are very particular. We propose another class of matrices, a space in which the system has one and only one solution, and the possibility to do a calculus of error.

This paper is organized as follows; in section 2 we recall [1], [10], [11] the Pòlya's method, which illustrate the construction of a solution of an infinite linear system. This method consists in defining a sequence of finite matrices converging to a solution of a system. We give, further, in other spaces denoted s_c or s_r, see [4], [7] another sequence of finite matrices which converges to a solution of such a space. Under other conditions, we have a second approximation of this solution, with a calculus of the error. In Section 3, we recall (see E. Hellinger and H.S. Wall [3], [14], [15]) some results concerning the continued fractions and the bounded matrices. Finally, we apply the results of Section 2 to those. In fact, to obtain the formal expansion into a continued fraction, we have to give an approximation of the coefficient a'_{11}, in the first row, and in the first column of a particular right reciprocal matrix. a'_{11} is called the leading coefficient, we have in fact $a'_{11} = a'_{11}(z)$, and it has been shown [14] that $a'_{11}(z) = \frac{1}{z} + \frac{O(1)}{z\,\mathrm{Im}(z)}$, where $O(1)$ represents a function of z which is numerically less than a constant independent of z for all z with $\mathrm{Im}(z) > 0$.

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2. Some definitions and properties of infinite linear systems

2.1. Linear infinite systems and Pòlya matrices

In this work, we shall study linear infinite systems

$$\sum_{m=1}^{+\infty} a_{nm} x_m = b_n, \qquad n = 1, 2, \dots, \tag{1}$$

where the sequences (a_{nm}) and (b_n) are given, (x_n) being the unknown sequence. This system is equivalent to the single matrix equation

$$AX = B, \tag{2}$$

where $A = (a_{nm})$, n being the index of the n-th row, m the one of the m-th column, n and m being integers greater than 1; $X = (x_n)$ and $B = (b_n)$ are one-column matrices. Define, now, for every p, by $[A]_p$ the matrix, whose elements of the p first rows, and of the p first columns are those of A. In this section the goal is to give a method permitting to calculate an approximation of a solution of an infinite linear system, by the means of a sequence of finite matrices $\{[A]_p\}$. This principle has been developed by Pòlya, see [1], but, as we shall see, the matrices used and the results obtained are totally different.

Let us recall that a Pòlya matrix A satisfies the following conditions $a_{1m} \neq 0$ for infinitely many values of m, and

$$\liminf_{m \to \infty} \sum_{k=1}^{n-1} \left| \frac{a_{km}}{a_{nm}} \right| = 0, \qquad n = 2, 3, \dots$$

Pòlya's theorem, [1], [10], is formulated as follows:

Theorem 1. *If A is a Pòlya matrix, then the equation $AX = B$, admits for any B, a solution such that the series $\sum_m a_{nm} x_m$ are absolutely convergent, for each value of n.*

Recall briefly the well-known construction of a solution of such a system, (see [1] for a detailed study). We consider, here, the particular case where the finite matrices deducted from A, are successively

$$\widetilde{A_1} = (a_{11}), \quad \widetilde{A_2} = \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}, \quad \dots,$$

$$\widetilde{A_n} = \begin{pmatrix} a_{1,\alpha_n} & \cdot & \cdot & a_{1,\alpha_n+n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,\alpha_n} & \cdot & \cdot & a_{n,\alpha_n+n-1} \end{pmatrix}, \quad \dots$$

with $\alpha_n = (n^2 - n + 2)/2$, for all $n \geq 2$, and are invertible; then a solution of (1) can be determined by the following method:

$\widetilde{A}_1 X_1 = \widetilde{B_1}$, with $X_1 = (x_1)$, $\widetilde{B_1} = (b_1)$ admits $x_1 = b_1/a_{11}$ as solution; in the same way, with x_1 defined by the preceding equation, we let $\widetilde{B_2}$, such that ${}^t\widetilde{B_2} = (0, b_2 - a_{21} x_1)$, and consider the equation:

$$\widetilde{A}_2 X_2 = \widetilde{B_2},$$

with ${}^tX_2 = (x_2, x_3)$. This one has $X_2 = (\widetilde{A_2})^{-1}\widetilde{B_2}$, as its unique solution. Step by step, setting, for $i \geq 3$

$$ {}^t\widetilde{B_i} = (0, 0, \ldots, b_i - \sum_{m=1}^{\alpha_i - 1} a_{i,m} x_m), $$

where $x_1, \ldots, x_{\alpha_i - 1}$, are determined by $X_j = (\widetilde{A_j})^{-1}\widetilde{B_j}$, $1 \leq j \leq i-1$, it has been proved that the vector Z, defined by $Z = (X_n)_{n \geq 1}$ is a solution of (1) satisfying Theorem 1.

Remark 1. In the case where the matrices $\widetilde{A_n}$ are not all invertible, it is necessary to consider the first integer p, such that $\widetilde{A_p}$ is not invertible; it has been proved by Pòlya that there exists an integer $k_1 > p$, such that the matrix

$$ \widetilde{A'_1} = \begin{pmatrix} a_{1,\alpha_p} & \cdot & \cdot & a_{1,\alpha_p + k_1 - 1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{k_1,\alpha_p} & \cdot & \cdot & a_{k_1,\alpha_p + k_1 - 1} \end{pmatrix} $$

is invertible. So, we obtain by induction a strictly increasing sequence of integers $k_1 < k_2 < \cdots < k_n < \cdots$, and a sequence of corresponding matrices. Using a reasoning analogous to the preceding one, we can construct a solution satisfying Theorem 1.

A Pólya's system admits, indeed, infinitely many solutions, see [1]. In this work we introduce spaces in which the system contains a unique solution, and we impose other hypotheses on A. Let us recall [4], [7] the spaces that we shall use.

2.2. The spaces S_c and s_c

For a sequence $c = (c_n)$, with $c_n > 0$ for every n, we define the Banach algebra S_c by

$$ S_c = \Big\{ A = (a_{nm}) \;\Big|\; \sup_n \Big(\sum_m |a_{nm}| \frac{c_m}{c_n} \Big) < \infty \Big\}, \tag{3} $$

normed by $\|A\|_{S_c} = \sup_n \Big(\sum_m |a_{nm}| \frac{c_m}{c_n} \Big)$. We also define the Banach space s_c of one-row matrices, by

$$ s_c = \Big\{ (x_n) \mid \sup_n \Big(\frac{|x_n|}{c_n} \Big) < \infty \Big\}, \tag{4} $$

normed by

$$ \|X\|_{s_c} = \sup_n \left(\frac{|x_n|}{c_n} \right). \tag{5} $$

If $c = (c_n)$, and $c' = (c'_n)$ are two sequences, such that $0 < c_n < c'_n$ $\forall n$, then:

$$ s_c \subset s_{c'}. $$

A special, very useful case is the one where $c_n = r^n$, $r > 0$. Then we denote by S_r and s_r, the spaces S_c, and s_c. When $r = 1$, we obtain the space of the bounded sequences $l^\infty = s_1$. Finally, φ is the set of all sequences that have only a finite number of nonvanishing terms.

If $\|I - A\|_{S_c} < 1$, we shall say that A satisfies the condition Γ_c. If $c = (r^n)$, Γ_c is replaced by Γ_r.

S_c being a unit algebra, we have the useful result: if A satisfies the condition Γ_c, A is invertible in the space S_c, and for every $B \in s_c$, the equation $AX = B$ admits one and only one solution in s_c, given by

$$X = \sum_{n=0}^{\infty} (I - A)^n B. \tag{6}$$

We have seen [4] that a matrix A, which verifies the condition Γ_c, for a given sequence $c = (c_n)$ is not necessarily of Pòlya type. In fact, the matrix $A = (\zeta^{|m-n|})$, with $0 < \zeta < 1/3$ satisfies the condition Γ_1; but: $\sum_{k=1}^{n-1} \frac{|a_{km}|}{|a_{nm}|} \geq \zeta > 0$, which shows that this matrix is not of Pòlya type.

2.3. Construction of a sequence converging to a solution of an infinite linear system

In the following, A is a matrix in S_c with $a_{nn} \neq 0$ for all n. For any positive integer q let A'_q denote the matrix with entries $a'_{nm} = a_{nm}$ for $1 \leq n, m \leq q$ and $a'_{nm} = 0$ otherwise. We denote by B_q the matrix deduced from B by the same way. As we have seen in 2-1 we associate to the matrix A'_q, the finite matrix $[A]_q$ of order q, whose entries in the first q rows and columns are equal to those in A. In the same way we define $[B]_q$ from B. When $[A]_q$ is invertible, we denote by $\hat{A}'_q$ the matrix

$$\begin{pmatrix} [A]_q^{-1} & \\ & O \end{pmatrix}.$$

When $A = I$, A'_q is denoted by I'_q. Then we have

$$A'_q \hat{A}'_q = \hat{A}'_q A'_q = I'_q.$$

Proposition 2. *There exists a non-invertible infinite matrix A such that all the finite matrices $[A]_q$, $q \geq 1$, are invertible.*

Proof. Consider, indeed, the matrix A defined for $\alpha > 1$ by $a_{11} = 1$, $a_{1m} = 0$ if $m \geq 2$; $a_{2m} = 1$ for all values of m; $a_{n,n-2} = -1$ if $n \geq 3$, $a_{nn} = \frac{1}{\alpha^2}$ if $n \geq 3$; the other elements being equal to 0. That is,

$$A = \begin{pmatrix} 1 & 0 & 0 & . & . & . \\ 1 & 1 & 1 & . & . & . \\ -1 & 0 & \frac{1}{\alpha^2} & 0 & O & \\ 0 & -1 & 0 & \frac{1}{\alpha^2} & . & \\ & O & . & . & . & . \end{pmatrix}. \tag{7}$$

Then A is an element of S_r, with $0 < r < 1$. The matrices $[A]_1$, $[A]_2$ and $[A]_3$ are obviously invertible; and for $q \geq 4$, the determinant of $[A]_q$ is equal to

$$\frac{1}{(\alpha^2)^{q-3}} \left(1 + \frac{1}{\alpha^2}\right),$$

and is different from 0. If A were invertible, the equation $AX = B$, where $B \in \varphi$, would have a solution. Denote then by C the matrix obtained from A, by deleting the two first rows and by D the matrix obtained from B in the same way. Let us describe the set of the solutions of $CX = D$. First we remark that C is invertible, being upper triangular. Considering the matrix obtained from C by adding of the rows $e'_1 = (1, 0, \dots)$ and $e'_2 = (0, 1, 0, \dots)$, we see that it is lower triangular with non-zero elements on the main diagonal, and then invertible. We conclude [4], [7], that the solutions of $CX = D$ form a linear space of dimension 2. The function $-1 + \frac{1}{\alpha^2}z^2$ admitting $z = \pm\alpha$ as roots, the solutions of $CX = 0$ are given by $X = (\lambda\alpha^n + \mu(-\alpha)^n)_{n\geq 1}$. Then the solutions of $CX = D$ are

$$X = (\lambda\alpha^n + \mu(-\alpha)^n)_{n\geq 1} + C^{-1}D,$$

where λ and μ are arbitrary scalars. Finally the evaluation of the sum of the series $\sum_m a_{2m}x_m$ shows then, that the product AX cannot converge, which is contradictory. ∎

We see that we must give additional conditions for A, so that the sequence defined by $\hat{A}'_q B_q$ converges to a limit, as q tends to infinity, in a given space. So, we impose the following hypotheses on A.

Definition 3. Let $c = (c_n)$ be a decreasing sequence, such that for all n: $0 < c_n \leq 1$; $A \in S_c$ (3), is called c-invertible, if the following conditions are satisfied:

1. For any q, $[A]_q$ is invertible, (we shall denote the elements of this inverse by $a'_{nm}(q)$).

2. Letting, for all $q \geq 2$:

$$k_q = \sup_{n\geq q}\Big(\sum_{m=1}^{q-1}\frac{|a_{nm}|}{c_n}\Big),$$

the series $\sum_q k_q$ is convergent.

3. The sequence of general term $\tau_q = \sup_{n,m\leq q}\{|a'_{nm}(q)|\}$ is bounded.

When $c_n = r^n$, with $r \in]0, 1]$ we shall say that A is r-invertible. Denote, now, for every q, by $X_q = (x_n(q))_{n\geq 1}$ the vector $\hat{A}'_q B_q$, we have the following result.

Theorem 4. *When A is c-invertible, for every $B \in \varphi$ the sequence (X_q) converges in s_c to a limit Z, a solution of $AX = B$.*

Proof. Let $B = (b_n)$ be an element of φ, there exists an integer N such that $b_n = 0$, for $n > N$. First let us show that (X_q) is a Cauchy sequence in s_c. For $q > N$, we have

$$X_q - X_{q-1} = \hat{A}'_q[A'_{q-1} - A'_q]X_{q-1}.$$

In fact, this last expression is equal to

$$\hat{A}'_q[I'_{q-1}B_q - A'_q\hat{A}'_{q-1}B_q],$$

where $I'_{q-1}B_q = B_q$ and $\hat{A}'_q A'_q = I'_q$. We know that for $n \geq q+1$, the terms δ_n, of $X_q - X_{q-1}$, are zero; and if $n \leq q$:

$$\delta_n = -a'_{nq}(q)\sum_{m=1}^{q-1} a_{qm}x_m(q-1).$$

Elsewhere $x_n(q-1) = \sum_{m=1}^{N} a'_{nm}(q-1)b_m$ for $n \leq q$. Then we have

$$\frac{|\delta_n|}{c_n} \leq \frac{|a'_{nq}(q)|}{c_n}\sum_{m=1}^{q-1}\sum_{k=1}^{N} |a_{qm}|\,|a'_{mk}(q-1)|\,|b_k|\,.$$

If $|\tau_q| \leq M$ for every q, then

$$\frac{|\delta_n|}{c_n} \leq \frac{M^2}{c_n}c_q\sum_{m=1}^{q-1}\frac{|a_{qm}|}{c_q}\sum_{1}^{N}|b_k|,$$

(c_n) being decreasing, there exists a constant $C > 0$, such that $\frac{|\delta_n|}{c_n} \leq CM^2k_q$, then

$$\|X_q - X_{q-1}\|_{s_c} \leq CM^2k_q.$$

We conclude easily that (X_q) is a Cauchy sequence in s_c. In fact $\forall k, l$, with $k \geq N$:

$$\|X_{k+l} - X_k\|_{s_c} \leq \sum_{q=k+1}^{k+l}\|X_q - X_{q-1}\|_{s_c} \leq CM^2\sum_{q=k+1}^{k+l}k_q.$$

Since s_c is a Banach space, the Cauchy sequence (X_q) has a limit Z in s_c. Now we have to verify that $AZ = B$. For this, let $\Delta_q = (A - A'_q)Z_q$. The coordinates of Δ_q, whose indices n are less than q, are equal to 0, and for $n \geq q+1$ they are equal to: $\sum_{m=1}^{q} a_{nm}(\sum_{l=1}^{N} a'_{ml}b_l)$. We have

$$\left\|(A - A'_q)Z_q\right\|_{s_c} \leq M\Big[\sup_{n\geq q+1}(\sum_{m=1}^{q}|a_{nm}|\frac{1}{c_n})\Big](\sum_{l=1}^{N}|b_l|).$$

Hence $\left\|(A - A'_q)Z_q\right\|_{s_c} \leq KMk_{q+1}$, with $K = \sum_{l=1}^{N}|b_l|$; then Δ_q converges to 0, in s_c. Furthermore, since the map $X \mapsto AX$ from s_c into itself is continuous, we conclude that the sequence (AZ_q) converges to AZ, as $q \to \infty$, which concludes the proof. ■

This theorem shows not only the existence of a solution for the system, but mainly gives a natural method to approximate it. Here, we see, once more, that a Pòlya matrix does not necessarily satisfy the preceding hypotheses; consider the lower triangular matrix A, defined by $a_{nm} = 0$ if $m \neq n$, $n-1$, $a_{nn} = 1$ and $a_{nn-1} = \zeta^{n-1}$, where $0 < \zeta < 1$. We have $k_q = \zeta^{q-1}$, then the series $\sum_q k_q$ is convergent; A verifies the condition Γ_1, which proves that the sequence (τ_q) is convergent, and we see that all the matrices $[A]_q$ are invertible. Hence A is 1-invertible, but is not of Pòlya type, since it is lower triangular. When we used the condition Γ_r, (see 2-2), we also had the existence (and uniqueness) of the solution, but this one was written in a more complicated form of a sum of a series of infinite matrices (6). When A is r-invertible, and satisfies the condition Γ_r, which is possible, as we shall see, the unique solution in s_r can be written as

$$Z = \lim_{q\to\infty}\widehat{A}'_q B_q = \sum_{n=0}^{\infty}(I - A)^n B.$$

Denote by E_r the set of r-invertible matrices, and by K_u, $u>0$, the set of the matrices, whose entries on the main diagonal are equal to 1, (case which can be referred considering the product DA, where $D=(\delta_{nm}/a_{nm}))$, and which satisfy the condition Γ_u. For every pair $(r,u)\in]0,1]\times R^{+*}$, the set $E_r\cap K_u$ is not empty, and contains the unit matrix I. If T denotes the set of all upper triangular infinite matrices, whose diagonal elements are all equal to 1, then we have the following result

PROPOSITION 5. *For every pair* (r,u) *of reals, with* $0<r\leq 1$ *and* $u\geq 1$, *one has* $K_u\cap T\subset E_r$.

Proof. Let A be a matrix of $K_u\cap T$, we have here

$$\sup_n\Big(\sum_{m=n+1}^{\infty}|a_{nm}|\,u^{m-n}\Big)<1.$$

Since $r\leq u$, A belongs to S_r. The matrix A'_q being upper triangular with non zero entries on the main diagonal, is invertible for all values of q. Furthermore, $k_q=0$ for all q; we have now to verify that (τ_q) is bounded. For this, let us consider the matrix A^*_q, whose entries are those of A'_q, except those of the main diagonal, whose indices are larger than q which are equal to 1, that is

$$A^*_q=\begin{pmatrix}[A]_q & & & \\ & 1 & & O\\ O & & 1 & \\ & & & .\end{pmatrix}. \tag{8}$$

Then A^*_q is invertible, and $(A^*_q)^{-1}$ can be written under the form

$$\begin{pmatrix}[A]_q^{-1} & & & \\ & 1 & & O\\ O & & 1 & \\ & & & .\end{pmatrix}. \tag{9}$$

The sequence of general term $(A^*_q)^{-1}$ is then bounded in S_1. Indeed, for n less than $q-1$

$$\sum_{m=n+1}^{q}|a_{nm}|\leq\|I-A\|_{S_u},$$

since $u\geq 1$. Hence $\|I-A^*_q\|_{S_1}\leq\|I-A\|_{S_u}<1$ and

$$\|(A^*_q)^{-1}\|_{S_1}\leq\frac{1}{1-\|I-A\|_{S_u}}.$$

Finally, $\|\hat{A}'_q\|_{S_1}$ being less than $\|(A^*_q)^{-1}\|_{S_1}$, we conclude that for every n, m, q,

$$|a'_{nm}(q)|\leq\sup_i\Big(\sum_{j=1}^{q}|a'_{ij}(q)|\Big)\leq\frac{1}{1-\|I-A\|_{S_u}}.$$

Then the sequence (τ_q) is bounded; this completes the proof. ■

In the same way, considering the case where $c_n=1$, for every n, k_q is, then defined by $\sup_{n\geq q}\Big(\sum_{m=1}^{q-1}|a_{nm}|\Big)$, we have the following result.

Corollary 6. *If A is a matrix belonging to K_1, such that the series $\sum_q k_q$ is convergent, then A belongs to E_1.*

Proof. In fact $[A]_q$ is invertible for every q, and $|\tau_q| \leq \frac{1}{1-\|I-A\|_{S_u}}$, for any u strictly larger than 1. ∎

Example 7. If $A = (\sigma^{|m-n|m})$, with $\sigma \in]0,1[$ and $\sum_{m\geq 1, m\neq n} \sigma^{|m-n|m} < 1$, then we see that A satisfies the condition Γ_1 and

$$k_q \leq \sigma^{q-1} + \sigma^{2(q-2)} + \cdots + \sigma^{q-1} \leq (q-1)\sigma^{q-1}.$$

The second condition of Definition 3 is then satisfied, and A is 1-invertible. We see that this matrix is of Pòlya, which proves that this system has infinitely many solutions; and here we have determined a space in which we have one and only one solution approximated by the sequence X_q.

Let us come back to the matrix A, defined by (7). Consider a real $0 < \theta < 1$ and the matrix $M_\theta = AP'_\theta$, with $P'_\theta = (\theta^{n-1}\delta_{nm})$. Denote, as in Definition 3, $[M_\theta]_q^{-1} = (\alpha_{nm}(q))_{n,m\leq q}$, we have

Corollary 8. *$\forall r \leq 1$ $M_\theta \notin E_r$, and the sequence defined by:*

$$\tau'_q = \sup_{n,m\leq q} \{|\alpha_{nm}(q)|\}$$

is not bounded.

Proof. First we show that there exists no real $r \leq 1$, such that M_θ is r-invertible. Let B be any matrix of φ; if Y were an element of s_r, such that $M_\theta Y = B$, we would conclude that $B = A(P'_\theta Y)$, where $P'_\theta Y \in s_r$ (since $\theta \leq 1$). A would be surjective from s_r into φ, which is contradictory, as we have seen in Proposition 2. By application of Theorem 4, we see that, at least one condition of Definition 3 is not verified. It is easy to prove that it is the third, where τ_q is replaced by τ'_q (notice that this property does not depend on r). In fact, the first hypothesis of Definition 3 is obviously satisfied, and for the second hypothesis, we have $\forall q \geq 4$

$$k_q = \left(\frac{\theta}{r}\right)^q \frac{1}{r^3};$$

and if $r \in]\theta, 1[$, the series $\sum_q k_q$ is convergent. ∎

2.4. A second method of approximation of a solution of an infinite system

We suppose that all the diagonal elements are equal to 1, we can give an analogous result, whose the advantage is to approximate the solution $Z = \sum_{n=0}^{\infty}(I-A)^n B$, of the equation $AX = B$, by the means of the sequence

$$Y_q = (A_q^*)^{-1}B = \sum_{n=0}^{\infty}(I - A_q^*)^n B,$$

A_q^* being defined in (8). When $B \in \varphi$, Y_q is equal to the sequence $X_q = (x_n(q))_n$, given in Definition 3. Given a real $r > 0$, let us define from the infinite matrix A

two sequences:

$$\gamma_q = \sup_{n\leq q}\Big(\sum_{m=q+1}^{\infty} |a_{nm}|\, r^{m-n}\Big), \quad \gamma'_q = \sup_{n\geq q+1}\Big(\sum_{m=1,m\neq n}^{\infty} |a_{nm}|\, r^{m-n}\Big),$$

so that to assert the following result

PROPOSITION 9. *Assume that:*

i) The sequences (γ_q) *and* (γ'_q) *converge to* 0*, as* q *tends to infinity;*

ii) A *satisfies the condition* Γ_r*.*

Then for any $B \in s_r$*,* (Y_q) *converges to* Z *in* s_r*, and*

$$\|Y_q - Z\|_{s_r} \leq \sup(\gamma_q, \gamma'_q)\frac{\|B\|_{s_r}}{(1-\rho)^2},$$

where $\rho = \|I - A\|_{S_r}$*.*

Proof. If we put $\rho_q = \|A - A^*_q\|_{S_r}$, (then $\rho_q \leq \rho$, for every q), it can be easily proved that

$$\|Y_q - Z\|_{s_r} \leq \left\|A - A^*_q\right\|_{S_r} \sum_{n=1}^{\infty}\sum_{i=1}^{n} \rho_q^{n-i}\rho^{i-1}\, \|B\|_{s_r}.$$

We can evaluate the double series in the second member of the inequality, since

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \rho_q^{n-i}\rho^{i-1} = \sum_{n=1}^{\infty} \frac{\rho_q^n}{\rho_q - \rho} - \sum_{n=1}^{\infty} \frac{\rho^n}{\rho_q - \rho} = \frac{1}{(1-\rho_q)(1-\rho)},$$

this last term being less than $1/(1-\rho)^2$. Finally ρ_q being equal to $\sup(\gamma_q, \gamma'_q)$, tends to 0 as $q \to \infty$, this complete the proof. ∎

EXAMPLE 10. This result can be applied to the matrix $A = (\sigma^{|m-n|m})$, with $0 < \sigma < 1/3$. So A satisfies the condition Γ_1. Considering the sequence defined by $\chi_q = \frac{\sigma^q}{1-\sigma^q}$, we have:

$$\gamma_q \leq \sup_{n\leq q}\Big\{\sum_{k=1}^{\infty} \sigma^{(q+1)(q+k-n)}\Big\} \leq \chi_{q+1},$$

$$\gamma'_q \leq \sup_{n\geq q+1}\Big\{(n-1)\sigma^{n-1} + \tfrac{\sigma^{n+1}}{1-\sigma^{n+1}}\Big\} \leq q\sigma^q + \chi_{q+2}.$$

Since the sequences (χ_q) and $(q\sigma^q)$ converge to 0, as q tends to infinity, it is the same for $\|A - A^*_q\|_{S_1}$. Furthermore there exists an integer N, such that

$$q\sigma^q + \chi_{q+2} - \chi_{q+1} = q\sigma^q\left[1 + \frac{\sigma^2}{q\,(1-\sigma^{q+2})} - \frac{\sigma}{q\,(1-\sigma^{q+1})}\right] > 0$$

for all $q \geq N$. Hence applying the proposition we get

$$\|Y_q - Z\|_{s_1} \leq (q\sigma^q + \chi_{q+2})\,\frac{\|B\|_{s_1}}{(1-\rho)^2} \quad \text{for all } q \geq N.$$

Remark 2. We note that a matrix r-invertible does not necessarily satisfy the previous proposition. Indeed, take $A \in K_1 \cap T$, defined for a given real ρ, $0 < \rho < 1$, by

$$A = \begin{pmatrix} 1 & \rho & & \\ & 1 & \rho & O \\ O & & \cdot & \cdot \\ & & & \cdot \end{pmatrix};$$

we deduct from the Proposition 5, that it belongs to E_1; but $\gamma_q = \rho$ does not converge to 0, as $q \to \infty$. This shows that the first condition of the preceding proposition cannot be satisfied.

3. An application to the continued fractions

Let us consider the system of linear equations

$$\begin{cases} (\beta_1 + z)x_1 - a_1 x_2 = b_1, \\ \quad - a_1 x_1 + (\beta_2 + z)x_2 - a_2 x_3 = b_2, \\ \quad\quad - a_2 x_2 + (\beta_3 + z)x_3 - a_3 x_4 = b_3, \\ \quad\quad \ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{10}$$

If ${}^tB = e_1' = (1, 0, 0, \dots)$, it is well known that we may write the linear equations in the form

$$x_1 = \frac{1}{\beta_1 + z - \frac{a_1 x_2}{x_1}}, \quad \frac{a_1 x_2}{x_1} = \frac{a_1^2}{\beta_2 + z - \frac{a_2 x_3}{x_2}}, \quad \frac{a_2 x_3}{x_2} = \frac{a_2^2}{\beta_3 + z - \frac{a_3 x_4}{x_3}}, \quad \dots$$

If we substitute in succession from each into the preceding, we obtain the formal expansion of x_1 into a continued fraction, also called the A-fraction, that is

$$x_1 = \frac{1}{\beta_1 + z - \frac{a_1^2}{\beta_2 + z - \frac{a_2^2}{\beta_3 + z - \ddots}}}.$$

The system defined by (10), is equivalent to the matrix equation $AX = B$. The infinite tridiagonal matrix A admitting infinitely many right inverses it is necessary to recall some results on continued fractions and bounded matrices. l^2 is the normed space of the sequences $X = (x_n)$ such that $\sum_n |x_n|^2 < \infty$, with $\|X\|_{l^2} = \left(\sum_n |x_n|^2\right)^{1/2}$. A is called bounded, if there exists a constant $M > 0$, such that for all $X, Y \in l^2$, $X = (x_n)$, $Y = (y_n)$

$$\left|{}^tXAY\right| = \left|\sum_{n,m} a_{nm} x_n y_m\right| \le M \|X\|_{l^2} \|Y\|_{l^2}.$$

We define by $\|A\|_2$ the smallest number $M > 0$, such that this inequality is verified; it is easy to see that $\|\cdot\|_2$is a norm defined in the space of the bounded matrices. It is known that there exists one bounded inverse, such that x_1 can be written, as above, in a continued fraction. If we denote by a'_{nm} the elements of this bounded inverse, B being equal to ${}^te_1'$, we have $x_1 = a'_{11}$. This number is called the leading coefficient, see [14], it is said too that "a'_{11} is formally equal to the A-fraction". We shall write more precisely $a'_{11} = a'_{11}(z)$.

In order to give explicitly this bounded inverse we give the following result [14].

PROPOSITION 11. *If we have $\|I - A\|_2 < 1$, A admits a bounded reciprocal, and for every $B \in l^2$ the equation $AX = B$ admits only one solution in l^2.*

Writing

$$A = zI + J_0 = z\left(I + \frac{1}{z}J_0\right),$$

we deduce that for $|z| > \|J_0\|_2$, A admits the bounded inverse we have been searching for. We can, now, give an application of Proposition 9 where the sequence $Y_q = X_q = \hat{A}'^{\,t}_q e'_1$, given in 2-3 is used. Let us recall that $x_1(q)$ is the first component of this vector.

PROPOSITION 12. *Assume that*

i) $\beta_n = O(1)$, as $n \to \infty$;

ii) (a_n) is decreasing and converges to 0 as n tends to infinity. Putting $K = \sup_n(a_n, |\beta_n|)$, we deduce that if $|z| > 3K$, then $x_1(q) \to a'_{11}(z)$, as $q \to \infty$, and

$$|x_1(q) - a'_{11}(z)| \leq 2a_q \frac{1}{(|z| - 3K)^2}.$$

Proof. In order to reduce to the case where all the elements of the main diagonal are equal to 1, we consider the product DA, where $D = (\delta_{nm}/\beta_n + z)$. We have $\|I - DA\|_{S_1} = \sup(\tau_1, \tau_2)$, with

$$\tau_1 = \frac{a_1}{|\beta_1 + z|}, \quad \tau_2 = \sup_n \left(\frac{a_n + a_{n+1}}{|\beta_n + z|}\right).$$

Putting $\rho' = \|I - DA\|_{S_1}$, we have for $|z| > 3K$

$$\rho' \leq \sup_n \left(\frac{2a_n}{|z| - K}\right) \leq \frac{2K}{|z| - K} < 1.$$

Since $D \in S_1$, and $A^{-1} = (DA)^{-1} D$, we deduce that A is invertible in S_1. We have

$$\gamma_q = \frac{a_q}{|\beta_q + z|}, \quad \gamma'_q = \sup_{n \geq q} \left(\frac{a_n + a_{n+1}}{|\beta_n + z|}\right),$$

both less than $\frac{2a_q}{|z|-K}$, as $|z| > 3K$. Let us show, now, that $\|J_0\|_2 \leq 3K$. We have

$${}^tXJ_0Y = -\sum_{n=2}^{\infty} a_{n-1}x_n y_{n-1} + \sum_{n=1}^{\infty} \beta_n x_n y_n - \sum_{n=1}^{\infty} a_n x_n y_{n+1}.$$

Hence $|{}^tXJ_0Y| \leq 3K \|X\|_{l^2} \|Y\|_{l^2}$, and $\|J_0\|_2 \leq 3K$. If we take, $|z| > 3K$, then A admits a bounded reciprocal, as above, which is equal to the preceding inverse

A^{-1}. Denote now, by $x_1'(q)$ the first component of $\hat{A}_q' D^{-1}\, {}^t e_1$, and by α_{nm}' the elements of the matrix $(DA)^{-1}$, then we have

$$x_1'(q) = (\beta_1 + z) x_1(q), \ \alpha_{11}' = (\beta_1 + z) a_{11}'.$$

Hence, applying Proposition 9, we obtain

$$|x_1(q) - a_{11}'(z)| = \left| \frac{x_1'(q) - \alpha_{11}'}{\beta_1 + z} \right| \le \frac{1}{|\beta_1 + z|} \frac{2a_q}{|z| - K} \frac{1}{(1 - \rho')^2}$$

and since $\frac{1}{1-\rho'} \le \frac{|z|-K}{|z|-3K}$, we deduce the result. ■

We remark that the map $A\colon X \to AX$ defined in the previous proposition is a bijection from s_1 into itself, and its restriction to l^2 is, also unrecognized, bijective from this space into l^2. A^{-1} is a bounded inverse, and belongs to S_1.

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