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SOME THEOREMS ON COSYMPLECTIC HYPERSURFACES OF SIX-DIMENSIONAL HERMITIAN SUBMANIFOLDS OF CAYLEY ALGEBRAS

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Abstract. Diverse properties of cosymplectic hypersurfaces in six-dimensional Hermitian submanifolds of Cayley algebra are considered.

Dedicated to professor Vadim F. Kirichenko
on his 55th bithday

1. Introduction

One of the most important properties of a hypersurface of an almost Hermitian manifold is the existence on a such hypersurface determined in a natural way an almost contact metric structure. This structure has been studied mainly in the case of Kählerian [9], [20] and quasi-Kählerian [15], [18] manifolds. In the case the embedding manifold is Hermitian, however, comparatively little is known about the geometry of its hypersurfaces. In the present article certain results obtained in this direction by using the Cartan structural equations of such hypersurfaces are given.

2. Preliminaries

We consider an almost Hermitian manifold, i.e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞.

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The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [1]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$$(p, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}}),$$

where $p \in M^{2n}$, ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here $a = 1, \dots, n$; $\hat{a} = a + n$.

Therefore, the matrix of the operator of the almost complex structure written in an A-frame looks as follows:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right)$$

where I_n is the identity matrix; $k, j = 1, \dots, n$.

We recall that the fundamental (or Kählerian [2]) form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in an A–frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right)$$

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it is well-known [11], two non-isomorphic 3-vector cross products are defined on it by

$$P_1(X, Y, Z) = -X(\overline{Y}Z) + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$
$$P_2(X, Y, Z) = -(X\overline{Y})Z + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{O}$ and $X \mapsto \overline{X}$ is the conjugation operator. Moreover, any other 3-vector cross product in the octave algebra is isomorphic to one of the above-mentioned.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle \cdot, \cdot \rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [11]. The submanifold $M^6 \subset \mathbf{O}$ is called Hermitian, if the almost Hermitian structure induced on it is integrable. The point $p \in M^6$ is called general [12], if

$$e_0 \notin T_p(M^6) \quad \text{and} \quad T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$ consisting only of general points is called a general-type submanifold [12]. In what follows, all submanifolds M^6 to be considered are assumed to be of general-type.

3. Cosymplectic hypersurfaces of Hermitian $M^6 \subset \mathbf{O}$

Let N be an oriented hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ and let σ be the second fundamental form of the immersion of N into M^6. As it is well-known [18], [20], the almost Hermitian structure on M^6 induces an almost contact metric structure on N. We recall [15], [18] that an almost contact metric structure on the manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of a type $(1,1)$ and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$
$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called cosymplectic [18], if

$$\nabla \eta = \nabla \Phi = 0.$$

(Here ∇ is the Levi-Civita connection of the metric g). The first group of the Cartan structural equations written in an A-frame of a hypersurface of a Hermitian manifold looks as follows [19]:

$$\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b + (\sqrt{2}B^{a3}{}_b + i\sigma^a_b)\omega^b \wedge \omega + \\
&\quad + \left(-\tfrac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^{ab}\right)\omega_b \wedge \omega, \\
d\omega_a &= -\omega^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b + (\sqrt{2}B_{a3}{}^b - i\sigma^b_a)\omega_b \wedge \omega + \\
&\quad + \left(-\tfrac{1}{\sqrt{2}}B_{ab}{}^3 - i\sigma_{ab}\right)\omega^b \wedge \omega, \\
d\omega &= (\sqrt{2}B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b)\omega^b \wedge \omega_a + (B_{3b}{}^3 + i\sigma_{3b})\omega \wedge \omega^b + \\
&\quad + (B^{3b}{}_3 - i\sigma^b_3)\omega \wedge \omega_b,
\end{aligned} \tag{1}$$

where $a, b, c = 1, 2$; $\hat{a} = a + 3$. Here B are Kirichenko structural tensor of Hermitian manifold. This tensors form a complete system of first-order differential-geometrical invariants of an arbitrary almost Hermitian structure [2], [6]. Taking into account that the first group of the Cartan structural equations of the cosymplectic structure must look as follows [10]:

$$\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b, \\
d\omega_a &= -\omega^b_a \wedge \omega_b, \\
d\omega &= 0,
\end{aligned} \tag{2}$$

we get the conditions whose simultaneous fulfilment is a criterion for the hypersurface N to be cosymplectic:

$$\begin{gathered}
1)\ B^{ab}{}_c = 0, \quad 2)\ \sqrt{2}B^{a3}{}_b + \sigma^a_b = 0, \quad 3)\ -\frac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^a_b = 0, \\
4)\ B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b = 0, \quad 5)\ B^{3b}{}_3 - i\sigma^b_3 = 0
\end{gathered} \tag{3}$$

and the formulas, obtained by comlpex conjugation (no need to write them down explicitly).

Now, we analyse the obtained conditions. From $(3)_3$ it follows that

$$\sigma^{ab} = -\frac{1}{\sqrt{2}} B^{ab}{}_3.$$

By alternating of this relation we have

$$0 = \sigma^{[ab]} = -\frac{i}{\sqrt{2}} B^{[ab]}{}_3 = -\frac{i}{2\sqrt{2}} (B^{ab}{}_3 - B^{ba}{}_3) = -\frac{i}{\sqrt{2}} B^{ab}{}_3.$$

Therefore $B^{ab}{}_3 = 0$ and consequently $\sigma^{ab} = 0$. From $(3)_2$ we get that $B^{3a}{}_b = \frac{i}{\sqrt{2}} \sigma^a_b$. We substitute this value in $(3)_4$. As a result we have

$$\sigma^a_b = i\sqrt{2} B_{3b}{}^a.$$

Now, we use the relation for the Kirichenko structure tensors of six-dimensional Hermitian submanifolds of Cayley algebra [2], [4]:

$$B^{\alpha\beta}{}_\gamma = \frac{1}{\sqrt{2}} \varepsilon^{\alpha\beta\mu} D_{\mu\gamma}, \quad B_{\alpha\beta}{}^\gamma = \frac{1}{\sqrt{2}} \varepsilon_{\alpha\beta\mu} D^{\mu\gamma},$$

where

$$D_{\mu\gamma} = \pm T^8_{\mu\gamma} + i T^7_{\mu\gamma}, \quad D^{\mu\gamma} = D_{\widehat{\mu}\widehat{\gamma}} = \pm T^8_{\widehat{\mu}\widehat{\gamma}} - i T^7_{\widehat{\mu}\widehat{\gamma}}.$$

Here T^φ_{kj} are components of the configuration tensor (in A.Gray's notation [10], or of the Euler curvature tensor [8]) of the Hermitian $M^6 \subset \mathbf{O}$; $\alpha, \beta, \gamma, \mu = 1, 2, 3$; $\widehat{\mu} = \mu + 3$; $k, j = 1, 2, 3, 4, 5, 6$; $\varphi = 7, 8$; $\varepsilon^{\alpha\beta\mu} = \varepsilon^{\alpha\beta\mu}_{1\,2\,3}$, $\varepsilon_{\alpha\beta\mu} = \varepsilon^{1\,2\,3}_{\alpha\beta\mu}$ are components of the third-order Kronecker tensor [17].

From $(3)_1$ we obtain

$$B^{ab}{}_c = 0 \;\Leftrightarrow\; \frac{1}{\sqrt{2}} \varepsilon^{ab\gamma} D_{\gamma c} = 0 \;\Leftrightarrow\; \frac{1}{\sqrt{2}} \varepsilon^{ab3} D_{3c} = 0 \;\Leftrightarrow\; D_{3c} = 0.$$

The similar reasoning can be applied in reference to the condition

$$B^{ab}{}_3 = 0$$

obtained above:

$$B^{ab}{}_3 = 0 \;\Leftrightarrow\; \frac{1}{\sqrt{2}} \varepsilon^{ab\gamma} D_{\gamma 3} = 0 \;\Leftrightarrow\; \frac{1}{\sqrt{2}} \varepsilon^{ab3} D_{33} = 0 \;\Leftrightarrow\; D_{33} = 0.$$

So, $D_{3c} = D_{33} = 0$, that is

$$D_{3\alpha} = 0. \tag{4}$$

From $(3)_5$ we get

$$\sigma^b_3 = \sigma_{3\widehat{b}} = -iB^{3b}{}_3 = -i\frac{1}{\sqrt{2}} \varepsilon^{3b\gamma} D_{\gamma 3} = 0.$$

We have $\sigma_{ab} = \sigma_{\widehat{a}\widehat{b}} = \sigma_{3b} = \sigma_{3\widehat{b}} = 0$. We shall compute the rest of the components of the second fundamental form using $(3)_2$:

$$\sigma_{\widehat{a}b} = \sigma^a_b = i\sqrt{2} B^{a3}{}_b = i\sqrt{2} \frac{1}{\sqrt{2}} \varepsilon^{a3\gamma} D_{\gamma b} = i\varepsilon^{a3c} D_{cb}.$$

Then

$$\begin{gathered}
\sigma_{\hat{1}1} = i\varepsilon^{13c} D_{c1} = i\varepsilon^{132} D_{21} = -iD_{21};\\
\sigma_{\hat{1}2} = i\varepsilon^{13c} D_{c2} = i\varepsilon^{132} D_{22} = -iD_{22};\\
\sigma_{\hat{2}1} = i\varepsilon^{23c} D_{c1} = i\varepsilon^{231} D_{11} = iD_{11};\\
\sigma_{\hat{2}2} = i\varepsilon^{23c} D_{c2} = i\varepsilon^{231} D_{12} = iD_{12};\\
\sigma_{1\hat{1}} = \overline{\sigma_{\hat{1}1}} = iD^{12};\\
\sigma_{1\hat{2}} = \overline{\sigma_{\hat{1}2}} = iD^{22};\\
\sigma_{2\hat{1}} = \overline{\sigma_{\hat{2}1}} = -iD^{11};\\
\sigma_{2\hat{2}} = \overline{\sigma_{\hat{2}2}} = -iD^{12}.
\end{gathered}$$

We obtian that the matrix of the second fundamental form of the immersion of the cosymplectic hyperspace N into $M^6 \subset \mathbf{O}$ looks as follows:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & iD^{12} & -iD^{11} \\ 0 & 0 & 0 & iD^{22} & -iD^{12} \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ -iD_{12} & -iD_{22} & 0 & 0 & 0 \\ iD_{11} & iD_{22} & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Using the identities from [2]

$$D^{11}D^{22} = (D^{12})^2 \quad \text{and} \quad D_{11}D_{22} = (D_{12})^2,$$

we obtain that each of matrices

$$\begin{pmatrix} -iD_{12} & -iD_{22} \\ iD_{11} & iD_{12} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} iD^{12} & -iD^{11} \\ iD^{22} & -iD^{12} \end{pmatrix}$$

is degenerated. Therefore the matrix σ is also degenerated ($\operatorname{rank}\sigma \leq 3$). Hence, we obtain the following result.

THEOREM 1. *Every cosymplectic hypersurface of a six-dimensional Hermitian submanifold of Cayley algebra is a ruled manifold.*

Studying the matrix σ, we come to another result. Indeed, the criterion of the minimality of the hypersurface is the following identity [1], [2]

$$g^{kj}\sigma_{kj} = 0.$$

Knowing how the matrix of the metric tensor looks [18]

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} g^{kj}\sigma_{kj} &= g^{ab}\sigma_{ab} + g^{\widehat{a}\widehat{b}}\sigma_{\widehat{a}\widehat{b}} + g^{\widehat{a}b}\sigma_{\widehat{a}b} + g^{a\widehat{b}}\sigma_{a\widehat{b}} + g^{33}\sigma_{33} \\ &= iD_{12} - iD_{12} + iD^{12} - iD^{12} + \sigma_{33} = \sigma_{33}. \end{aligned}$$

That is why $g^{kj}\sigma_{kj} = 0 \ \Leftrightarrow\ \sigma_{33} = 0$. The equality $\sigma_{33} = 0$ means that $\sigma(\xi, \xi) = 0$. We have proved

THEOREM 2. *The cosymplectic hypersurface of a six-dimensional Hermitian submanifold of Cayley algebra is minimal if and only if its second fundamental form satisfies the condition $\sigma(\xi, \xi) = 0$.*

As the hypersurface N is a totally geodesic submanifold of a Hermitian $M^6 \subset$ **O** precisely when the matrix σ vanishes, we can conclude that the conditions

$$D_{11} = D_{12} = D_{22} = D^{11} = D^{12} = D^{22} = \sigma_{33} = 0 \tag{6}$$

are a criterion for N to be a totally geodesic submanifold of M^6.

We recall [5] that the almost Hermitian manifold satisfies the g-cosym- plectic hypersurfaces axiom if through every point of this manifold passes a totally geodesic cosymplectic hypersurface. That is why for the Hermitian $M^6 \subset$ **O** satisfying the g-cosymplectic hypersurfaces axiom the equalities (5) are correct for every point of M^6. We have proved previously [2], [4] that the matrix D of a six-demensional Hermitian sumbanifold of the octave algebra looks as follows:

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D^{11} & D^{12} & D^{13} \\ 0 & 0 & 0 & D^{21} & D^{22} & D^{23} \\ 0 & 0 & 0 & D^{31} & D^{32} & D^{33} \end{pmatrix}.$$

If M^6 satisfies the g-cosymplectic hypersurfaces axiom, then taking into account (4) and (6) we get that this matrix vanishes. But the matrix D vanishes at every point of a six-dimensional almost Hermitian submanifold of Cayley algebra precisely when the given submanifold is Kählerian [2], [3], [4], [13]. That is why we have

THEOREM 3. *Every six-dimensional Hermitian submanifold of Cayley algebra satisfying the g-cosymplectic hypersurfaces axiom is a Kählerian manifold.*

4. The type number of cosymplectic hypersurfaces of Hermitian $M^6 \subset$ O

When we give a Riemannian manifold and its submanifold, the rank of determined second fundamental form is called the type number.

THEOREM 4. *The type number of a cosymplectic hypersurface N of a Hermitian submanifold $M^6 \subset$ **O** is at most three.*

Proof. As each of matrices

$$\begin{pmatrix} -iD_{12} & -iD_{22} \\ iD_{11} & iD_{12} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} iD^{12} & -iD^{11} \\ iD^{22} & -iD^{12} \end{pmatrix}$$

is degenerated, the sum of their ranks is at most two. Taking into account that form (5) we have

$$\operatorname{rank}\sigma \leq \operatorname{rank}\begin{pmatrix} -iD_{12} & -iD_{22} \\ iD_{11} & iD_{12} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} iD^{12} & -iD^{11} \\ iD^{22} & -iD^{12} \end{pmatrix} + 1,$$

we conclude $\operatorname{rank}\sigma \leq 3$. ■

By force of Theorem 2 and Theorem 4, we get:

COROLLARY 1. *The type number of a minimal cosymplectic hypersurface of a Hermitian submanifold $M^6 \subset \mathbf{O}$ is at most two.*

COROLLARY 2. *If $M^6 \subset \mathbf{O}$ is a Kählerian submanifold, then the type number of its cosymplectic hypersurface is at most one.*

COROLLARY 3. *If $M^6 \subset \mathbf{O}$ is a Kählerian submanifold, and N is its cosymplectic hypersurface, then the following statements are equivalent:*

1) N is a minimal hypersurface of M^6;

2) N is a totally geodesic hypersurface of M^6;

3) N is a totally umbilical hypersurface of M^6;

4) the type number of N vanishes.

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