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BILINEAR EXPANSIONS OF THE KERNELS OF SOME NONSELFADJOINT INTEGRAL OPERATORS

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Abstract. Let H and S be integral operators on $L^2(0,1)$ with continuous kernels. Suppose that $H>0$ and let $A=H(I+S)$. It is shown that if the (nonselfadjoint) operator S is small in a certain sense with respect to H, then the corressponding Fourier series of functions from $R(A)$ (or $R(A^*)$) converges uniformly on $[0,1]$.

1. Introduction

Let H and S be integral operators on $L^2(0,1)$ (with inner product $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$) with continuous kernels $\mathcal{H}(x,y)$ and $\mathcal{S}(x,y)$ on $[0,1]\times[0,1]$. Suppose that $H>0$ and let

$$A = H(I+S). \tag{1}$$

Classical theorems (case $S=0$, see [5]) state that the kernel $\mathcal{H}$ can be expanded into a uniformly convergent (on $[0,1]\times[0,1]$) bilinear series.

A consequence of this is that every function $f\in R(H)$ has the uniformly convergent Fourier series with respect to the system of eigenfunctions of H. ($R(H)$ denotes the image of H in $L^2(0,1)$).

Similar results hold in some cases when $S\neq 0$. Namely, if $S=S^*$, it was proved in [1] and [2] that the corresponding variant of Mercer's theorem holds. The proof was based on the spectral theorem for an operator on $L^2(0,1)$ with the definite or indefinite inner product generated by the formula

$$[f,g] = \langle (I+S)f,g\rangle.$$

In [4], a series of nice results was obtained which were related to bilinear expansions of smooth Carleman's kernels of Mercer type.

A natural question is about bilinear expansions when $S\neq S^*$. We shall show that if the operator S is small in a certain sense with respect to H, then the corresponding Fourier series of functions from $R(A)$ (or $R(A^*)$) converges uniformly

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on $[0,1]$. (Note that, because of the continuity of the kernel of A, we have $R(A) \subset C[0,1]$).

In the sequel, $\mathcal{A}(\cdot,\cdot)$ will denote the kernel of the operator A (defined by (1)).

2. The results

THEOREM. *Let, for the operators H and S from (1), there exists $\omega > 0$ such that $S^*S \leqslant \omega H^2$. If s_k are singular values of A and f_k the normalized eigenvectors of A^*A (i.e., $A^*Af_k = s_k^2 f_k$) and $g_k = (s_k)^{-1}Af_k$, then the series $\sum_{k\geqslant 1} s_k \overline{f_k(y)} g_k(x)$ is absolutely convergent on $[0,1]^2$ and uniformly convergent on $[0,1]$ with respect to arbitrary variable and its sum is equal to $\mathcal{A}(x,y)$. Also, for every $f \in R(A)$ (resp. $g \in R(A^*)$) the series $\sum_{k\geqslant 1}\langle f, g_k\rangle g_k$ (resp. $\sum_{k\geqslant 1}\langle g, f_k\rangle f_k$) converges uniformly on $[0,1]$ to f (resp. g).*

In the proof of this assertions we need the following two Lemmas.

LEMMA 1. [5] *If $T\colon L^2(0,1) \to L^2(0,1)$ is the linear operator defined by $Tf(x) = \int_0^1 M(x,y)f(y)\,dy$ and if $M \in C([0,1]^2)$ and $\langle Tf, f\rangle \geqslant 0$ for all $f \in L^2(0,1)$, then $M(x,x) \geqslant 0$ for all $x \in [0,1]$.*

LEMMA 2. *If $A = H(I+S)$, $H > 0$, $S^*S \leqslant \omega H^2$, then there exists a constant $c > 0$ such that*

$$\sqrt{A^*A} \leqslant cH, \qquad \sqrt{AA^*} \leqslant cH.$$

Proof. Since

$$A^*A = H^2 + S^*H^2 + H^2S + S^*H^2S \tag{2}$$

we have to estimate $\langle S^*H^2f, f\rangle$ and $\langle H^2Sf, f\rangle$.

The operator H^2 is positive and thus, by the Cauchy inequality, we have

$$\begin{aligned}|\langle S^*H^2f, f\rangle|^2 &= |\langle H^2f, Sf\rangle|^2 \leqslant \langle H^2f, f\rangle\langle H^2Sf, Sf\rangle\\ &= \|Hf\|^2\|HSf\|^2 \leqslant \|Hf\|^2\|H\|^2\|Sf\|^2\\ &= \|Hf\|^2\|H\|^2\langle S^*Sf, f\rangle \leqslant \|Hf\|^2\|H\|^2\omega\langle H^2f, f\rangle\\ &= \omega\|H\|^2\|Hf\|^4.\end{aligned}$$

Therefore

$$|\langle S^*H^2f, f\rangle| \leqslant \sqrt{\omega}\,\|H\|\langle H^2f, f\rangle \tag{3}$$

and hence we get

$$|\langle H^2Sf, f\rangle| \leqslant \sqrt{\omega}\,\|H\|\langle H^2f, f\rangle. \tag{4}$$

Since

$$\langle S^*H^2Sf, f\rangle = \|HSf\|^2 \leqslant \|H\|^2\|Sf\|^2 = \|H\|^2\langle S^*Sf, f\rangle \leqslant \omega\|H\|^2\langle H^2f, f\rangle,$$

from (2), (3), (4) it follows that $\langle A^*Af, f\rangle \leqslant (1+\sqrt{\omega}\,\|H\|)^2\langle H^2f, f\rangle$, i.e. $A^*A \leqslant (1+\sqrt{\omega}\,\|H\|)^2H^2$.

Having in mind that the function $\lambda \mapsto \sqrt{\lambda}$ is operator monotone, we get

$$\sqrt{A^*A} \leqslant (1 + \sqrt{\omega}\,\|H\|)H. \tag{5}$$

From the equality $A^* = (I + S^*)H$ we get $\|A^*f\| \leqslant \|I + S^*\| \cdot \|Hf\|$ $(f \in L^2(0,1))$, i.e.

$$\sqrt{AA^*} \leqslant \|I + S^*\| \cdot H. \tag{6}$$

From (5) and (6) we obtain the assertion of the Lemma, with

$$c = \max\{1 + \sqrt{\omega}\,\|H\|, \|I + S^*\|\}. \quad \blacksquare$$

Proof of the Theorem. From $A^*Af_k = s_k^2 f_k$ and $Af_k = s_k g_k$ it follows that f_k, $g_k \in C[0,1]$ (because A has the continuous kernel) and A has the following singular (see [3]) expansion

$$A = \sum_{k\geqslant 1} s_k \langle \cdot, f_k \rangle g_k.$$

Also, there holds

$$\left.\begin{aligned} \sqrt{A^*A}\, f &= \sum_{k\geqslant 1} s_k \langle f, f_k \rangle g_k \\ \sqrt{AA^*}\, f &= \sum_{k\geqslant 1} s_k \langle f, g_k \rangle g_k, \end{aligned}\right. \qquad f \in L^2(0,1). \tag{7}$$

The series on the right-hand side of the previous equalities converge in the norm of $L^2(0,1)$.

Consider the operators (on $L^2(0,1)$) S_n', S_n'' defined in the following way: $S_n' = cH - \sum_{k=1}^n \langle \cdot, f_k \rangle f_k$, $S_n'' = cH - \sum_{k=1}^n \langle \cdot, g_k \rangle g_k$. Since, by (7) and Lemma 2, $\langle S_n' f, f \rangle \geqslant 0$, $\langle S_n'' f, f \rangle \geqslant 0$, $f \in L^2(0,1)$ and since the operators S_n', S_n'' have continuous kernels, we get from Lemma 1

$$c\mathcal{H}(x,x) \geqslant \sum_{k=1}^n s_k |f_k(x)|^2, \quad c\mathcal{H}(x,x) \geqslant \sum_{k=1}^n s_k |g_k(x)|^2, \qquad n \in \mathbf{N}.$$

Since $\mathcal{H} \in C([0,1]^2)$, there exists $M_0 < +\infty$ such that

$$\sum_{k=1}^n s_k |f_k(x)|^2 \leqslant M_0, \qquad \sum_{k=1}^n s_k |g_k(x)|^2 \leqslant M_0. \tag{8}$$

From (8) it follows that the series

$$\sum_{k\geqslant 1} s_k \overline{f_k(y)} g_k(x)$$

is absolutely convergent for all $x, y \in [0,1]$. Let $S(x,y)$ denote its sum. Observe that form (8) it follows that the partial sums of the previous series are bounded by M_0.

Fix $x \in [0,1]$. Then we have

$$\begin{aligned} \Big| \sum_{k=p}^q s_k \overline{f_k(y)} g_k(x) \Big|^2 &\leqslant \sum_{k=p}^q s_k |f_k(y)|^2 \sum_{k=p}^q s_k |g_k(x)|^2 \\ &\leqslant M_0 \sum_{k=p}^q s_k |g_k(x)|^2 \to 0 \quad (p, q \to \infty) \end{aligned}$$

and the series $\sum_{k\geqslant 1} s_k\overline{f_k(y)}g_k(x)$ converges uniformly with respect to y on $[0,1]$, for every fixed x and hence its sum is a continuous function with respect to y.

Let $f \in C[0,1]$ be a fixed function. Then (because of the uniform convergence with respect to y)

$$\int_0^1 S(x,y)f(y)\,dy = \sum_{k\geqslant 1} s_k g_k(x)\int_0^1 f(y)\overline{f_k(y)}\,dy = \sum_{k\geqslant 1} s_k g_k(x)\langle f, f_k\rangle. \tag{9}$$

(The series on the right-hand side of (9) converges not only for every x but also uniformly with respect to x because

$$\begin{aligned}\Big|\sum_{k=p}^{q} s_k g_k(x)\langle f, f_k\rangle\Big|^2 &\leqslant \sum_{k=p}^{q} s_k|g_k(x)|^2 \sum_{k=p}^{q} s_k|\langle f, f_k\rangle|^2 \\ &\leqslant M_0 s_p\|f\|^2 \to 0 \quad (p,q\to\infty)\,)\end{aligned}$$

On the other hand, from the singular expansion of A we get

$$\int_0^1 \mathcal{A}(x,y)f(y)\,dy = \sum_{k\geqslant 1} s_k g_k(x)\langle f, f_k\rangle \tag{10}$$

(the series converges in the norm of $L^2(0,1)$).

Thus, from (9), (10) it follows that for every $f \in C[0,1]$ we have

$$\int_0^1 (\mathcal{A}(x,y) - \mathcal{S}(x,y))f(y)\,dy = 0.$$

Putting $f(y) = \overline{\mathcal{S}(x,y)} - \overline{\mathcal{A}(x,y)}$ $(\in C[0,1])$ we get

$$\int_0^1 |\mathcal{A}(x,y) - \mathcal{S}(x,y)|^2\,dy = 0$$

and hence $\mathcal{A}(x,y) = \mathcal{S}(x,y)$ for every $y \in [0,1]$. Since $x \in [0,1]$ was arbitrary, we have $\mathcal{A}(x,y) = \mathcal{S}(x,y)$, $x,y \in [0,1]$. So

$$\mathcal{A}(x,y) = \sum_{k\geqslant 1} s_k\overline{f_k(y)}g_k(x)$$

for every $x,y \in [0,1]$.

Let now $f \in R(A)$. Then

$$f(x) = \int_0^1 \mathcal{A}(x,y)\varphi(y)\,dy, \qquad \varphi \in L^2(0,1)$$

and thus, by the Lebesgue dominated convergence theorem, we have ($\mathcal{A}_n(x,y) = \sum_{k=1}^{n} s_k\overline{f_k(y)}g_k(x)$)

$$\begin{aligned}f(x) &= \int_0^1 \lim_{n\to\infty}\mathcal{A}_n(x,y)\varphi(y)\,dy = \lim_{n\to\infty}\int_0^1 \mathcal{A}_n(x,y)\varphi(y)\,dy \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} s_k g_k(x)\langle\varphi, f_k\rangle = \sum_{k\geqslant 1} s_k g_k(x)\langle\varphi, f_k\rangle.\end{aligned}$$

Since the preceeding series converges uniformly with the respect to $x \in [0,1]$ and since $\{g_k\}$ is an othonormal system in $L^2(0,1)$ (see [3]), we have $s_k\langle\varphi, f_k\rangle = \langle f, g_k\rangle$ and, finally, we get

$$f(x) = \sum_{k\geqslant 1} \langle f, g_k\rangle g_k(x)$$

(the series converges uniformly on $[0,1]$).

The asssertion for the function $g \in R(A^*)$ can be proved in a similar way. ■

REMARK. The second part of the Theorem was proved in [5] in a different way. The proof presented here is a consequence of the previously established bilinear expansion of the function $\mathcal{A}(x,y)$.

REFERENCES

[**1**] M. R. Dostanić, *Generalizations of Mercer's theorem for a class of nonselfadjoint operators*, Mat. Vesnik **41** (1989), 22–29.

[**2**] M. R. Dostanić, *Generalizations of the Mercer theorem*, Publ. de l'Inst. Math., Nouv. serie, **54(68)** (1993), 63–70.

[**3**] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translation of the Math. Monographs, **18**, Amer. Math. Soc., Providence, R.I., 1969.

[**4**] I. M. Novitskii, *Integral representations of linear operators by smooth Carleman kernels of Mercer type*, Proc. London Math. Soc. (3) **68** (1994), 161–177.

[**5**] F. Riesz, B. Sz.-Nagy, *Leçons d'analyse fonctionelle*, Akademiai Kiado, Budapest, 1972.

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