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MEASURES OF NON-STRICT-SINGULARITY AND NON-STRICT-COSINGULARITY

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Abstract. In this paper we investigate a new measure of non-strict-singularity and a new measure of non-strict-cosingularity. Measures of non-strict-singularity and of non-strict-cosingularity have been investigated in [11], [8], [12], [7], [9], [15].

1. Introduction and preliminaries

In this paper X, Y and Z are complex Banach spaces, $B(X,Y)$ ($K(X,Y)$) the set of all bounded (compact) linear operators from X into Y. We shall write $B(X)$ ($K(X)$) instead of $B(X,X)$ ($K(X,X)$).

An operator $T \in B(X,Y)$ is in $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) if the range $R(T)$ is closed in Y and the dimension $\alpha(T)$ of the null space $N(T)$ of T is finite (the codimension $\beta(T)$ of $R(T)$ in Y is finite). Operators in $\Phi_+(X,Y) \cup \Phi_-(X,Y)$ are called semi-Fredholm operators. We set $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. The operators in $\Phi(X,Y)$ are called Fredholm operators. We shall write $\Phi_+(X)$ (resp. $\Phi_-(X)$, $\Phi(X)$) instead of $\Phi_+(X,X)$ (resp. $\Phi_-(X,X)$, $\Phi(X)$).

Let B_X denote the closed unit ball of X. Let $T \in B(X,Y)$ and

$$m(T) = \inf\{\, \|Tx\| : \|x\| = 1 \,\}$$

be the *minimum modulus* of T, and let

$$q(T) = \sup\{\, \varepsilon \geqslant 0 : \varepsilon B_Y \subset TB_X \,\}$$

be the *surjection modulus* of T.

If M is a subspace of X, then J_M will denote the embedding map of M into X, and if V is a subspace of Y, then Q_V will denote the canonical map of Y onto the quotient space Y/V.

An operator $T \in B(X,Y)$ is *strictly singular* ($T \in S(X,Y)$) if, for every infinite dimensional (closed) subspace M of X, the restriction of T to M, $T|_M$, is not a

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homeomorphism, i.e., $m(T|_M) = 0$. An operator $T \in B(X,Y)$ is *strictly cosingular* ($T \in SC(X,Y)$) if, for every infinite codimensional closed subspace V of Y the composition $Q_V T$ is not surjective. It is well known that

$$K(X,Y) \subset S(X,Y) \quad \text{and} \quad K(X,Y) \subset SC(X,Y). \tag{1.1}$$

If Ω is a non-empty bounded subset of X, then the Hausdorff measure of noncompactness of Ω is denoted by $\chi(\Omega)$, and defined as follows

$$\chi(\Omega) = \inf\{\, \varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net in } X\,\}.$$

For $A \in B(X,Y)$ the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by

$$\|A\|_\chi = \inf\{\, k \geqslant 0 : \chi_Y(A\Omega) \leqslant k_{\chi_X}(\Omega),\ \Omega \subset X \text{ is bounded}\,\}.$$

Recall that ([2])

$$\|A\|_\chi = \inf\{\, \|Q_V A\| : V \text{ is a subspace of } Y,\ \dim V < \infty\,\}.$$

For $A \in B(X,Y)$, set (see [6])

$$\|A\|_\mu = \inf\{\, \|A J_L\| : L \text{ closed subspace of } X,\ \operatorname{codim} L < \infty\,\}.$$

Recall that

$$\|A\|_\chi = 0 \iff \|A\|_\mu = 0 \iff A \in K(X,Y). \tag{1.2}$$

For $A \in B(X,Y)$, set

$$G_M(A) = \inf_{N \subset M} \|A J_N\|, \quad G(A) = G_X(A),$$
$$\Delta_M(A) = \sup_{N \subset M} G_N(A), \quad \Delta(A) = \Delta_X(A),$$

where M, N denote closed infinite dimensional subspaces of X (see [11]). Δ is a measure of non-strict-singularity of operators, i.e.,

$$\Delta(A) = 0 \iff A \in S(X,Y). \tag{1.3}$$

Weis [8] introduced for $A \in B(X,Y)$ the following functions

$$K_V(A) = \inf_{W \supset V} \|Q_W A\|, \quad K(A) = K_{\{0\}}(A),$$
$$\nabla_V(A) = \sup_{W \supset V} K_W(A), \quad \nabla(A) = \nabla_{\{0\}}(A),$$

where V, W denote closed infinite codimensional subspaces of Y.

∇ is a measure of non-strict-cosingularity, i.e.,

$$\nabla(A) = 0 \iff A \in SC(X,Y). \tag{1.4}$$

Recall that

$$\nabla(A + T) = \nabla(A) \quad \text{for all } T \in SC(X,Y), \tag{1.5}$$

and
$$K_V(A) = \inf_{W \supset V} \|Q_W A\|_\chi, \tag{1.6}$$
where V, W denote closed infinite codimensional subspaces of Y (see [12, Summary and discussion, Remark 2] or [7, Example 5.3] or [15, Lemma 2.21).

Recall that ([11], [13])
$$\begin{aligned} G(A) > 0 &\iff A \in \Phi_+(X,Y), \\ K(A) > 0 &\iff A \in \Phi_-(X,Y). \end{aligned} \tag{1.7}$$

2. Results

Schechter [11] proved the next theorem.

THEOREM 2.1. *$A \in \Phi_+(X,Y)$ if and only if for each Banach space Z there is a constant c, $0 < c < \infty$, such that*
$$\Delta(T) \leqslant c\Delta(AT), \quad T \in B(Z,X).$$

We can prove the dual theorem.

THEOREM 2.2. *$A \in \Phi_-(X,Y)$ if and only if for each Banach space Z there is a constant c, $0 < c < \infty$, such that*
$$\nabla(T) \leqslant c\nabla(TA), \quad T \in B(Y,Z). \tag{2.2.1}$$

Proof. Let $A \in \Phi_-(X,Y)$. By [6, Theorem 5.5 and Theorem 3.1] it follows that there is a constant c, $0 < c < \infty$, such that for each Banach space Z
$$\|T\|_\chi \leqslant c\|TA\|_\chi, \quad T \in B(Y,Z). \tag{2.2.2}$$
Let V be a closed subspace of Z with $\operatorname{codim} V = \infty$ and $\varepsilon > 0$. From (1.6) it follows that there is a closed subspace W of Z such that $W \supset V$, $\operatorname{codim} W = \infty$ and
$$\|Q_W TA\|_\chi < K_V(TA) + \varepsilon. \tag{2.2.3}$$
From (1.6), (2.2.2) and (2.2.3) it follows that
$$\begin{aligned} K_V(T) &\leqslant \|Q_W T\|_\chi \leqslant c\|Q_W TA\|_\chi \leqslant c(K_V(TA) + \varepsilon) \\ &\leqslant c(\nabla(TA) + \varepsilon). \end{aligned}$$
Hence $\nabla(T) \leqslant c(\nabla(TA) + \varepsilon)$.

Assume $A \notin \Phi_-(X,Y)$. By [1, Theorem 4.4.10] it follows that there is an operator $C \in K(X,Y)$ such that $\operatorname{codim} \overline{R(A-C)} = \infty$. Let $V = \overline{R(A-C)}$. Since $Q_V(A-C) = 0$, from (1.1) and (1.5) we get $\nabla(Q_V A) = \nabla(Q_V(A-C)) = 0$. Let M

and N be closed subspaces of Y/V with $\operatorname{codim} M=\infty$, $N\supset M$ and $\operatorname{codim} N=\infty$. Since $\|Q_N Q_V\|=1$ we get

$$\nabla(Q_V)=\sup_{\substack{M\subset Y/V\\ \operatorname{codim} M=\infty}}\ \inf_{\substack{N\supset M\\ \operatorname{codim} N=\infty}}\|Q_N Q_V\|=1.$$

Thus, there is no constant c, $0<c<\infty$, such that (2.2.1) holds. ∎

Let S be a subset of a Banach space A. The perturbation class associated with S is denoted by $P(S)$ and

$$P(S)=\{\, a\in A : a+s\in S \text{ for all } s\in S\,\}.$$

The perturbation class associated with $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) is denoted by $P(\Phi_+(X,Y))$ ($P(\Phi_-(X,Y))$).

For $T\in B(X,Y)$, set (see [10], [14])

$$\begin{aligned} n_{P\Phi_+}&=\|T\|_{P\Phi_+}=\inf\{\,\|T-P\| : P\in P(\Phi_+(X,Y))\,\},\\ n_{P\Phi_-}&=\|T\|_{P\Phi_-}=\inf\{\,\|T-P\| : P\in P(\Phi_-(X,Y))\,\},\end{aligned}$$

The next theorem is inspired by [3, Example 1].

THEOREM 2.3. *Let $T\in B(X,Y)$. Then*

$$m(T)\leqslant\|T\|_{P\Phi_+}\leqslant\|T\|, \tag{2.3.1}$$

$$q(T)\leqslant\|T\|_{P\Phi_-}\leqslant\|T\|. \tag{2.3.2}$$

Proof. (2.3.1) Assume $P\in P(\Phi_+(X,Y))$. It implies $P\notin\Phi_+(X,Y)$. By [1, Theorem 4.4.7] it follows that there is $K\in K(X,Y)$ such that $\dim N(P-K)=\infty$. Set $M=N(P-K)$ and $\varepsilon>0$. By (1.2) we get $\|PJ_M\|_\mu=\|KJ_M\|_\mu=0$. Hence there is a closed subspace $V\subset M$ such that $\dim M/V<\infty$ and $\|PJ_V\|<\varepsilon$. For $x\in V$ we have

$$\|Tx-Px\|\geqslant\|Tx\|-\|Px\|\geqslant m(T)\|x\|-\varepsilon\|x\|.$$

It implies $\|T-P\|\geqslant\|(T-P)J_V\|\geqslant m(T)-\varepsilon$. Hence $\|T-P\|\geqslant m(T)$. Thus $\|T\|_{P\Phi_+}\geqslant m(T)$.

(2.3.2) Let $P\in P(\Phi_-(X,Y))$. Then $P\notin\Phi_-(X,Y)$. From [1, Theorem 4.4.10] it follows that there is $K\in K(X,Y)$ such that $\operatorname{codim}\overline{R(P-K)}=\infty$. Set $U=\overline{R(P-K)}$. From $Q_U(P-K)=0$ and (1.2) it follows $\|Q_UP\|_\chi=\|Q_UK\|_\chi=0$. Hence for $\varepsilon>0$ there is a finite dimensional subspace $W\subset Y/U$ such that $\|Q_WQ_UP\|<\varepsilon$. There is a closed subspace $V\subset Y$ such that $V\supset U$ and $W=V/U$. It is not difficult to see that the operator $A\colon (Y/U)/(V/U)\to Y/V$ defined by

$$A((y+U)+V/U)=y+V,\quad y\in Y,$$

is an isometric isomorphism and $AQ_{V/U}Q_U=Q_V$. Hence $\|Q_VP\|=\|Q_{V/U}Q_UP\|$. It follows that $\|Q_VP\|<\varepsilon$. Hence

$$\|T-P\|\geqslant\|Q_V(T-P)\|\geqslant\|Q_VT\|-\|Q_VP\|\geqslant q(Q_VT)-\varepsilon\geqslant q(T)-\varepsilon.$$

Thus $\|T-P\|\geqslant q(T)$, and $\|T\|_{P\Phi_-}\geqslant q(T)$. ∎

Now we use the notation of [7]: let, for $T \in B(X,Y)$,

$$\begin{aligned}\mathrm{sn}_{P\Phi_+}(T) &= \sup_M n_{P\Phi_+}(TJ_M),\\ \mathrm{isn}_{P\Phi_+}(T) &= \inf_M \mathrm{sn}_{P\Phi_+}(TJ_M),\end{aligned}$$

where M denotes a closed infinite dimensional subspace of X and

$$\begin{aligned}\mathrm{sn}'_{P\Phi_-}(T) &= \sup_U n_{P\Phi_-}(Q_U T),\\ \mathrm{isn}'_{P\Phi_-}(T) &= \inf_U \mathrm{sn}'_{P\Phi_-}(Q_U T),\end{aligned}$$

where U denotes a closed infinite codimensional subspace of Y.

Zemánek [13] considered the following functions

$$\begin{aligned}u(A) &= \sup\{\, m(AJ_W) : W \text{ is a closed subspace of } X \text{ with } \dim W = \infty \,\},\\ v(A) &= \sup\{\, q(Q_V A) : V \text{ is a closed subspace of } Y \text{ with } \operatorname{codim} V = \infty \,\}.\end{aligned}$$

From the definition of the strictly singular and strictly cosingular operators it is obvious that

$$\begin{aligned}u(A) = 0 &\iff A \in S(X,Y),\\ v(A) = 0 &\iff A \in SC(X,Y).\end{aligned} \tag{2.4}$$

For $A \in B(X,Y)$ set (see [4], [5])

$$\begin{aligned}G_u(A) &= \inf\{\, u(AJ_M) : M \text{ is a closed subspace of } X,\ \dim M = \infty \,\},\\ K_v(A) &= \inf\{\, v(Q_U A) : U \text{ is a closed subspace of } Y,\ \operatorname{codim} U = \infty \,\}.\end{aligned}$$

Recall that

$$\begin{aligned}G_u(A) > 0 &\iff A \in \Phi_+(X,Y),\\ K_v(A) > 0 &\iff A \in \Phi_-(X,Y).\end{aligned} \tag{2.5}$$

From (2.3.1) and (2.3.2) it follows

$$\begin{aligned}G_u(T) &\leqslant \mathrm{isn}_{P\Phi_+}(T) \leqslant G(T),\\ K_v(T) &\leqslant \mathrm{isn}'_{P\Phi_-}(T) \leqslant K(T).\end{aligned} \tag{2.6}$$

By (2.6), (1.7) and (2.5) we get

$$\begin{aligned}\mathrm{isn}_{P\Phi_+}(T) > 0 &\iff T \in \Phi_+(X,Y),\\ \mathrm{isn}'_{P\Phi_-}(T) > 0 &\iff T \in \Phi_-(X,Y).\end{aligned} \tag{2.7}$$

(2.7) follows also from [7, Theorem 2.3(2) and Theorem 3.3(2)].

For $T \in B(X,Y)$, set

$$\begin{aligned}\Delta_{P\Phi_+}(T) &= \sup_M \inf_{N\subset M} \|TJ_N\|_{P\Phi_+},\\ \nabla_{P\Phi_-}(T) &= \sup_V \inf_{W\supset V} \|Q_W T\|_{P\Phi_-},\end{aligned}$$

where M, N denote closed infinite dimensional subspaces of X and V, W denote closed infinite codimensional subspaces of Y.

Analogously as in [11] it can be proved that $\Delta_{P\Phi_+}$ ($\nabla_{P\Phi_-}$) is a seminorm.

From (2.3.1) and (2.3.2) it follows

$$\begin{aligned} u &\leqslant \Delta_{P\Phi_+} \leqslant \Delta, \\ v &\leqslant \nabla_{P\Phi_-} \leqslant \nabla. \end{aligned} \tag{2.8}$$

By (2.8), (1.3), (1.4) and (2.4) we get that $\Delta_{P\Phi_+}$ is a measure of non-strict-singularity and $\nabla_{P\Phi_-}$ is a measure of non-strict-cosingularity, i.e.,

$$\Delta_{P\Phi_+}(T) = 0 \iff T \in S(X,Y), \tag{2.9}$$

$$\nabla_{P\Phi_-}(T) = 0 \iff T \in SC(X,Y). \tag{2.10}$$

(2.9) and (2.10) follow also from [7, Theorem 2.4(2) and Theorem 3.3(2)].

It is well known that

$$S(X,Y) \subset P(\Phi_+(X,Y)) \quad \text{and} \quad SC(X,Y) \subset P(\Phi_-(X,Y)).$$

THEOREM 2.4. *Let X and Y be Banach spaces. Then:*

(2.11.1) $S(X,Y) = P(\Phi_+(X,Y))$ *if and only if from* $P \in P(\Phi_+(X,Y))$ *it follows* $PJ_M \in P(\Phi_+(M,Y))$ *for each closed infinite dimensional subspace* M *of* X*;*

(2.11.2) $SC(X,Y) = P(\Phi_-(X,Y))$ *if and only if from* $P \in P(\Phi_-(X,Y))$ *it follows* $Q_V P \in P(\Phi_-(X,Y/V))$ *for each closed infinite codimensional subspace* V *of* Y.

Proof. (2.11.1). Let $S(X,Y) = P(\Phi_+(X,Y))$. Suppose M is a closed infinite dimensional subspace of X and $P \in P(\Phi_+(X,Y))$. Then $P \in S(X,Y)$. It implies $PJ_M \in S(M,Y) \subset P(\Phi_+(M,Y))$.

Assume that for each closed infinite dimensional subspace M of X from $P \in P(\Phi_+(X,Y))$ it follows $PJ_M \in P(\Phi_+(M,Y))$. Hence for $T \in B(X,Y)$ we get

$$\begin{aligned} \|TJ_M\|_{P\Phi_+} &= \inf\{\, \|TJ_M - P_1\| : P_1 \in P(\Phi_+(M,Y))\,\} \\ &\leqslant \inf\{\, \|TJ_M - PJ_M\| : P \in P(\Phi_+(X,Y))\,\} \\ &\leqslant \inf\{\, \|T - P\| : P \in P(\Phi_+(X,Y))\,\} \leqslant \|T\|_{P\Phi_+}. \end{aligned}$$

Therefore

$$\Delta_{P\Phi_+}(T) \leqslant \|T\|_{P\Phi_+}. \tag{2.11.3}$$

If $T \in P(\Phi_+(X,Y))$, then $\|T\|_{P\Phi_+} = 0$. By (2.11.3) it follows that $\Delta_{P\Phi_+}(T) = 0$. From (2.9) we get $T \in S(X,Y)$. Thus $S(X,Y) = P(\Phi_+(X,Y))$.

(2.11.2). Analogously to (2.11.1). ■

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