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ON B-ALGEBRAS

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Abstract. In this paper we introduce and investigate a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, a digraph on algebras defined below demonstrates a rather interesting connection between B-algebras and groups.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The present authors ([8]) introduced the notion of d-algebras, i.e., (I) $x * x = 0$; (V) $0 * x = 0$; (VI) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced a new notion, called an BH-algebra, i.e., (I), (II) $x * 0 = x$ and (VI), which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. In this paper we introduce and investigate a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, a digraph on algebras defined below demonstrates a rather interesting connection between B-algebras and groups.

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2. *B*-algebras

A *B-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = x * (z * (0 * y))$

for all x, y, z in X.

Example 2.1. Let $X := \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X : *, 0)$ is a B-algebra.

Example 2.2. Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra.

Example 2.3. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra (see [10]).

Example 2.4. Let $F\langle x, y, z\rangle$ be the free group on three elements. Define $u * v := vuv^{-2}$. Thus $u * u = e$ and $u * e = u$. Also $e * u = u^{-1}$. Now, given $a, b, c \in F\langle x, y, z\rangle$, let

$$\begin{aligned} w(a,b,c) &= ((a * b) * c)(a * (c * (e * b))^{-1} \\ &= (cbab^{-2}c^{-2})(b^{-1}cb^2a^{-1}cbcb^2)^{-1} \\ &= cbab^{-2}c^{-2}b^{-2}c^{-1}b^{-1}c^{-1}ba^{-1}b^{-2}c^{-1}b. \end{aligned}$$

Let $\mathcal{N}(*)$ be the normal subgroup of $F\langle x,y,z\rangle$ generated by the elements $w(a,b,c)$. Let $G = F\langle x,y,z\rangle/\mathcal{N}(*)$. On G define the operation "$\cdot$" as usual and define

$$(u\mathcal{N}(*)) * (v\mathcal{N}(*)) := (u * v)\mathcal{N}(*).$$

It follows that $(u\mathcal{N}(*)) * (u\mathcal{N}(*)) = e\mathcal{N}(*)$, $(u\mathcal{N}(*)) * (e\mathcal{N}(*)) = u\mathcal{N}(*)$ and

$$w(a\mathcal{N}(*), b\mathcal{N}(*), c\mathcal{N}(*)) = w(a,b,c)\mathcal{N}(*) = e\mathcal{N}(*).$$

Hence $(G; *, e\mathcal{N}(*))$ is a B-algebra.

If we let $y := x$ in (III), then we have

$$(x * x) * z = x * (z * (0 * x)). \tag{a}$$

If we let $z := x$ in (a), then we obtain also

$$0 * x = x * (x * (0 * x)). \tag{b}$$

Using (I) and (a), it follows that

$$0 = x * (0 * (0 * x)). \tag{c}$$

We observe that the three axioms (I), (II) and (III) are independent. Let $X := \{0,1,2\}$ be a set with the following left table:

$*$	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

$*$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Then the axioms (I) and (III) hold, but not (II), since $2 * 0 = 0 \neq 2$.

Similarly, the set $X := \{0,1,2\}$ with the above right table satisfies the axioms (II), (III), but not (I), since $1 * 1 = 1 \neq 0$. Let $X := \{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	1
3	3	0	0	0

Then $(X; *, 0)$ satisfies the axioms (I), (II), but not (III), since $(2*3)*0 = 1 \neq 2 = 2 * (0 * (0 * 3))$.

Lemma 2.5. *If $(X;*,0)$ is a B-algebra, then $y*z = y*(0*(0*z))$ for any $y,z \in X$.*

Proof. This follows from the axioms (II) and (III), i.e.,

$$\begin{aligned} y*z &= (y*z)*0 && \text{[by (II)]} \\ &= y*(0*(0*z)). \quad \blacksquare && \text{[by (III)]} \end{aligned}$$

Lemma 2.6. *If $(X,*,0)$ is a B-algebra then $(x*y)*(0*y) = x$ for any $x,y \in X$.*

Proof. From axiom (III) with $z = 0*y$ we find that

$$(x*y)*(0*y) = x*((0*y)*(0*y)).$$

Hence axiom (I) yields

$$(x*y)*(0*y) = x*0,$$

so that from axiom (II) it follows that $(x*y)*(0*y) = x$ as claimed. $\blacksquare$

Lemma 2.7. *If $(X,*,0)$ is a B-algebra then $x*z = y*z$ implies $x = y$ for any $x,y,z \in X$.*

Proof. If $x*z = y*z$, then $(x*z)*(0*z) = (y*z)*(0*z)$ and thus by Lemma 2.6 it follows that $x = y$. $\blacksquare$

Proposition 2.8. *If $(X;*,0)$ is a B-algebra, then*

$$x*(y*z) = (x*(0*z))*y \tag{IV}$$

for any $x,y,z \in X$.

Proof. Using Lemma 2.5 and (II) we obtain:

$$\begin{aligned} (x*(0*z))*y &= x*(y*(0*(0*z))) && \text{[by (II)]} \\ &= x*(y*z). \quad \blacksquare && \text{[by Lemma 2.5]} \end{aligned}$$

Lemma 2.9 *Let $(X;*,0)$ be a B-algebra. Then for any $x,y \in X$,*

(i) *$x*y = 0$ implies $x = y$,*

(ii) *$0*x = 0*y$ implies $x = y$,*

(iii) *$0*(0*x) = x$.*

Proof. (i) Since $x*y = 0$ implies $x*y = y*y$, by Lemma 2.7, it follows that $x = y$.

(ii) If $0*x = 0*y$, then $0 = x*x = (x*x)*0 = x*(0*(0*x)) = x*(0*(0*y)) = (x*y)*0 = x*y$, and thus by (i), $x = y$.

(iii) For any $x \in X$, we obtain $0*x = (0*x)*0 = 0*(0*(0*x))$ by axioms (II) and (III). By (ii) it follows that $x = 0*(0*x)$ as claimed. $\blacksquare$

Note that Lemma 2.9 is proven in [1] based on Lemmas 2.5, 2.6, 2.7 and Proposition 2.8 above.

Let $(X;*,0)$ be a B-algebra and let $g \in X$. Define $g^n := g^{n-1} * (0 * g)$ $(n \geq 1)$ and $g^0 := 0$. Note that $g^1 = g^0 * (0 * g) = 0 * (0 * g) = g$ by Lemma 2.9.

LEMMA 2.10. *Let $(X;*,0)$ be a B-algebra and let $g \in X$. Then $g^n * g^m = g^{n-m}$ where $n \geq m$.*

Proof. If X is a B-algebra then note that by Lemma 2.9 it follows that $g^2 * g = (g^1 * (0 * g)) * g = (g * (0 * g)) * g = g * (g * (0 * (0 * g))) = g * (g * g) = g * 0 = g$. Assume that $g^{n+1} * g = g^n$ $(n \geq 1)$. Then

$$\begin{aligned} g^{n+2} * g &= (g^{n+1} * (0 * g)) * g \\ &= g^{n+1} * (g * (0 * (0 * g))) && \text{[by (III)]} \\ &= g^{n+1} * 0. && \text{[by (c)]} \\ &= g^{n+1}. && \text{[by (II)]} \end{aligned}$$

Assume $g^n * g^m = g^{n-m}$ where $n - m \geq 1$. Then

$$\begin{aligned} g^n * g^{m+1} &= (g^n * (g^m * (0 * g)) \\ &= (g^n * g) * g^m && \text{[by (III)]} \\ &= g^{n-1} * g^m \\ &= g^{n-(m+1)}, && \text{[since } n - m - 1 \geq 0] \end{aligned}$$

proving the lemma. ∎

LEMMA 2.11. *Let $(X;*,0)$ be a B-algebra and let $g \in X$. Then $g^m * g^n = 0 * g^{n-m}$ where $n > m$.*

Proof. If X is a B-algebra then, by applying (III), (I) and Lemma 2.9, we have $g * g^2 = g * (g^1 * (0 * g)) = (g * g) * g^1 = 0 * g$. Assume that $g * g^n = g^{n-1}$ where $(n \geq 1)$. Then

$$\begin{aligned} g * g^{n+1} &= g * (g^n * (0 * g)) \\ &= (g * g) * g^n && \text{[by (III)]} \\ &= 0 * g^n. && \text{[by (I)]} \end{aligned}$$

Assume that $g^m * g^n = g^{n-m}$ where $n - m \geq 1$. Then

$$\begin{aligned} g^{m+1} * g^n &= (g^m * (0 * g)) * g^n \\ &= g^m * (g^n * g) && \text{[by (IV)]} \\ &= g^m * g^{n-1} \\ &= 0 * g^{n-m-1}, \end{aligned}$$

proving the lemma. ∎

We summarize the above Lemmas:

THEOREM 2.12. *Let $(X;*,0)$ be a B-algebra and let $g \in X$. Then*

$$g^m * g^n = \begin{cases} g^{m-n} & \textit{if } m \geq n, \\ 0 * g^{n-m} & \textit{otherwise.} \end{cases}$$

Proposition 2.13. *If* $(X;*,0)$ *is a* B*-algebra, then* $(a*b)*b = a*b^2$ *for any* $a,b \in X$.

Proof. It follows from (III) that $(a*b)*b = a*(b*(0*b)) = a*b^2$. ■

Proposition 2.14. *If* $(X;*,0)$ *is a* B*-algebra, then* $(0*b)*(a*b) = 0*a$ *for any* $a,b \in X$.

Proof. It follows from (IV) and (I) that $(0*b)*(a*b) = ((0*b)*(0*b))*a = 0*a$. ■

3. Commutativity

A B-algebra $(X;*,0)$ is said to be *commutative* if $a*(0*b) = b*(0*a)$ for any $a,b \in X$. The B-algebra in Example 2.1 is commutative, while the B-algebra in Example 2.3 is not commutative, since $3*(0*4) = 2 \neq 1 = 4*(0*3)$.

Proposition 3.1. *If* $(X;*,0)$ *is a commutative* B*-algebra, then* $(0*x)*(0*y) = y*x$ *for any* $x,y \in X$.

Proof. Since X is commutative, by applying Lemma 2.5 we obtain:

$$\begin{aligned}(0*x)*(0*y) &= y*(0*(0*x)) \\ &= y*x. \quad ■\end{aligned}$$

Theorem 3.2. *If* $(X;*,0)$ *is a commutative* B*-algebra, then* $a*(a*b) = b$ *for any* $a,b \in X$.

Proof. If X is commutative, then by (IV) we obtain $a*(a*b) = (a*(0*b))*a$ $= (b*(0*a))*a = b*(a*a) = b$. ■

Corollary 3.3. *If* $(X;*,0)$ *is a commutative* B*-algebra, then the left cancellation law holds, i.e.,* $a*b = a*b'$ *implies* $b = b'$.

Proof. It follows from Theorem 3.2 that $b = a*(a*b) = a*(a*b') = b'$. ■

Proposition 3.4. *If* $(X;*,0)$ *is a commutative* B*-algebra, then* $(0*a)*(a*b) = b*a^2$ *for any* $a,b \in X$.

Proof. If X is a commutative B-algebra, then

$$\begin{aligned}(0*a)*(a*b) &= ((0*a)*(0*b))*a && \text{[by (IV)]} \\ &= (b*a)*a && \text{[by Proposition 3.1]} \\ &= b*a^2. \quad ■ && \text{[by Proposition 2.13]}\end{aligned}$$

4. Derived algebras and B-algebras

Given algebras (i.e., groupoids, binary systems) $(X;*)$ and $(X;\circ)$, it is often argued that they are "essentially equivalent" when they are not, and even if it is perfectly clear how we may proceed from one to the other and back again, it is also not clear that knowledge of one "implies" knowledge of the other in a complete enough sense as to have the statement that they are "essentially equivalent" survive closer inspection.

We proceed with an example. Usually, given the integers Z, we consider the system $(Z;+,0)$ as an abelian group with identity 0. If we consider the system $(Z;-,0)$, then we can reproduce $(Z;+,0)$ by "defining" $x+y:=x-(0-y)$, and observing that in the first case "0 is the unique element such that $x-0=x$ for all x, while in the second case "0 is the unique element such that $x+0=x$ for all x".

However, that is by no means all we might have said to identify 0 nor is it necessary what we need to say to identify 0 in this setting.

Let $(X;*,0)$ and $(X;\circ,0)$ be algebras. We denote $(X;*,0)\to(X;\circ,0)$ if $x\circ y=x*(0*y)$, for all $x,y\in X$. The algebra $(X;\circ,0)$ is said to be *derived* from the algebra $(X;*,0)$. Let V be the set of all algebras defined on X and let $\Gamma_d(V)$ be the digraph whose vertices are V and whose arrows are those described above.

A d-algebra $(X;*,0)$ is called a *$d-BH$-algebra* ([9]) if it satisfies (II).

EXAMPLE 4.1. ([9]) If we define $x*y:=\max\{0,\frac{x(x-y)}{x+y}\}$ on X, then $(X;*,0)$ is a $d-BH$-algebra.

PROPOSITION 4.2. *The derived algebra $(X;*,0)$ from a $d-BH$-algebra $(X;*,0)$ is a left zero semigroup.*

Proof. Let $(X;*,0)$ be a $d-BH$-algebra and let $(X;*,0)\to(X;\circ,0)$. Then $x\circ y=x*(0*y)$, for any $x,y\in X$. Since $(X;*,0)$ is a $d-BH$-algebra, $x*(0*y)=x*0=x$, i.e., $x\circ y=x$, proving that $(X;\circ,0)$ is a left zero semigroup. ■

Notice that such an arrow in $\Gamma_d(V)$ can always be constructed, but it is not true that a backward arrow always exists. For example, since every BCK-algebra $(X;*,0)$ is a $d-BH$-algebra, we have $(X;*,0)\to(X;\circ,0)$ where $(X;\circ,0)$ is a left zero semigroup by Proposition 4.2. Assume that $(X;\circ,0)\to(X;*,0)$, where $(X;*,0)$ is a non-trivial BCK-algebra. Then $x*y=x\circ(0\circ y)$, for any $x,y\in X$. Since $(X;\circ)$ is a left zero semigroup, we have $x*y=x$ for any $x,y\in X$, contradicting that $(X;*,0)$ is a BCK-algebra.

The most interesting result in this context may be:

THEOREM 4.3. *Let $(X;*,0)$ be a B-algebra. If $(X;*,0)\to(X;\circ,0)$, i.e., if $x\circ y=x*(0*y)$, then $(X;\circ,0)$ is a group.*

Proof. If $(X;*,0)\to(X;\circ,0)$, then $x\circ y=x*(0*y)$, for any $x,y\in X$. By Lemma 2.9 $0*(0*x)=x$ for any $x\in X$, i.e., $x=0\circ x$. Since $x\circ 0=x*(0*0)=x*0=x$, 0 acts like an identity element of X. Routine calculations show that $(X;\circ,0)$ is a group. ■

PROPOSITION 4.4. *The derived algebra from a group is that group itself.*

Proof. Let $(X;*,0)$ be a group with identity 0. If $(X;*,0) \to (X;\circ,0)$, then $x \circ y = x * (0 * y) = x * y$, since 0 is the identity, for any $x, y \in X$. This proves the proposition. ■

Thus, we can use the $\to$ mechanism to proceed from the B-algebras to the groups, but since groups happen to be sinks in this graph, we cannot use the $\to$ mechanism to return from groups to B-algebras. This does not mean that there are no other ways to do so, but it does argue for the observation that B-algebras are not only "different", but in a deep sense "non-equivalent', and from the point of view of the digraph $\Gamma_d(V)$ the B-algebra is seen to be a predecessor of the group.

Given a group $(X;\cdot,e)$, if we define $x * y := x \cdot y^{-1}$, then $(X;*,0=e)$ is seen to be a B-algebra, and furthermore, it also follows that $(X;*,0=e) \to (X;\cdot,e)$, since $x * (e * y) = x \cdot (e \cdot y^{-1})^{-1} = x \cdot (y^{-1})^{-1} = x \cdot y$.

The problem here is that there is *not a formula involving only* $(X;\cdot,e)$ which produces $x * y$, i.e., we have to introduce $(X;\cdot,\_^{-1},e)$ as the type to describe a group to permit us to perform this task. In fact, we may use this observation as another piece of evidence that B-algebras $(X;*,0)$ are not "equivalent" to groups $(X;*,0)$. If we introduce the mapping $x \to 0 * x$ as the 'inverse'. i.e., if we write $x^{-1} = 0*x$, then $(X;*,\_^{-1},0)$ becomes a species like $(X;\cdot,\_^{-1},e)$, but in the case of the B-algebra the mapping $x \to 0*x$ is *not a new item which needs to be introduced, while in the case of groups it is.*

The difficulty we are observing in the situation above is also visible in the case of "*the subgroup test*". If $(X;\cdot,e)$ is an infinite group, and if $\emptyset \neq S \subseteq X$, then if S is closed under multiplication it is not the case that S need a subgroup. Indeed, the rule is that if $x, y \in S$, then also $x \cdot y^{-1} \in S$. From what we have already seen, $x \cdot y^{-1}$ is precisely the element $x * y$ if $(X;*,e) \to (X;\cdot,e)$ in $\Gamma_d(V)$. Thus we have the *the following "subgroup test" for B-algebras*: $\emptyset \neq S \subseteq X$ is a subalgebra of the B-algebra $(X;*,0)$, precisely when $x, y \in S$ implies $x * y \in S$.

Also, suppose $(X;*,0) \to (X;\cdot,e=0)$ in $\Gamma_d(V)$ where it is given that $(X;\cdot,e)$ is a group. Then it is not immediately clear that $(X;*,0)$ must be a uniquely defined B-algebra, even if we know that there is at least one B-algebra with this property.

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