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CHARACTERIZATIONS OF CONVEXITIES OF NORMED SPACES BY MEANS OF g-ANGLES

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Abstract. The notion of g-angle has been defined by the author [8]. This notion appeared to be very useful in characterisations of inner product spaces [8]. Here we use the same notion in characterizations of strictly convex spaces, uniformly convex spaces, locally uniformly convex spaces, and normed spaces that are uniformly convex in every direction. Some corollaries of this characterizations are described as well. For example, we show that the so called quasi-inner product spaces are uniformly convex spaces, what has not been noted earlier.

0. Introduction and definitions

Let X be a real normed space, $S(X)$ the unit sphere in X and $B(X)$ the unit ball in X. The functionals

$$\tau_{\pm}(x,y) := \lim_{t\to\pm 0} t^{-1}(\|x+ty\| - \|x\|),$$

$$g(x,y) := \frac{\|x\|}{2}\left(\tau_-(x,y) + \tau_+(x,y)\right), \quad (x,y\in X)$$

always exist on X^2. The functional g is a natural generalization of the inner product and reduces to it in an inner product space (cf. [7]). In any normed space it has following properties

$$g(x,x) = \|x\|^2 \quad (x\in X), \tag{1}$$

$$g(\alpha\, x, \beta\, y) = \alpha\,\beta\, g(x,y) \quad (x,y\in X;\ \ \alpha,\beta\in \mathbf{R}) \tag{2}$$

$$g(x, x+y) = \|x\|^2 + g(x,y) \quad (x,y\in X), \tag{3}$$

$$|g(x,y)| \le \|x\|\,\|y\| \quad (x,y\in X) \qquad \text{(cf. [6])}. \tag{4}$$

By means of the functional g we defined the following notions.

Definition 1. ([8]) For $x,y\in X\setminus\{0\}$, the number

$$\measuredangle(x,y) := \arccos\frac{g(x,y)+g(y,x)}{2\,\|x\|\,\|y\|}, \tag{5}$$

is called the g-angle between x and y.

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In accordance with (4), the angle between x and y $(x \neq 0, y \neq 0)$ can be defined also by

$$\measuredangle'(x,y) := \arccos \frac{g(x,y)}{\|x\|\,\|y\|}.$$

However, the advantage of definition (5) is reflected in the fact that $\measuredangle(x,y) = \measuredangle(y,x)$.

Definition 2. ([9]) A normed space X in which the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8\,(\|x\|^2\, g(x,y) + \|y\|^2\, g(y,x)) \quad (x, y \in X)$$

holds is called a quasi-inner product space (q.i.p. space). The space of sequences l^4 is a q.i.p. space [9].

Now we quote some standard definitions.

Definition 3. ([2]) A space X is strictly convex (SC), if no open interval from $B(X)$ crosses $S(X)$.

Definition 4. ([1]) X is uniformly convex (UC) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in S(X)$ we have

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon)$$

whenever $\|x - y\| \geq \varepsilon$.

Definition 5. ([5]). X is locally uniformly convex (LUC) if for any $x_0 \in S(X)$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, x_0) > 0$ such that with $y \in S(X)$, the inequality $\|x_0 - y\| \geq \varepsilon$ implies

$$\left\| \frac{x_0+y}{2} \right\| \leq 1 - \delta(\varepsilon, x_0).$$

Definition 6. ([4]) X is uniformly convex in every direction (UCED) if for every $z \in X \backslash \{0\}$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, z)$ such that $|\lambda| < \varepsilon$ if $x, y \in S(X)$, $x - y = \lambda\, z$ and

$$\left\| \frac{x+y}{2} \right\| \geq 1 - \delta(\varepsilon, z).$$

Definition 7. The point $x \in S(X)$ is an extremal point of the ball $B(X)$ if for $x_1, x_2 \in B(X)$ the equality $x = \frac{1}{2}(x_1 + x_2)$ implies $x_1 = x_2 = x$.

In what follows we shall write $\cos(x,y)$ instead of $\cos \measuredangle(x,y)$. Some additional properties of functional g are quoted below.

Lemma 1. *For $x, y \in X$ we have*

$$-\|x\|\,\|y\| \leq \|x\|\,(\|x\| - \|x-y\|) \leq g(x,y) \leq \|x\|\,(\|x+y\| - \|x\|) \leq \|x\|\,\|y\|. \quad (6)$$

Proof. Using (2), (3) and (4) we obtain

$$g(x, x \pm y) = \|x\|^2 \pm g(x,y) \leq \|x\|\,\|x \pm y\|.$$

Hence (6) is true. ∎

For $x, y \in S(X)$ the definition (5) and inequalities (6) imply

$$1 - \|x - y\| \le \cos(x, y) \le \|x + y\| - 1. \tag{7}$$

LEMMA 2. *For $x, y \in X \setminus \{0\}$, the implication*

$$\cos(x, y) = 1 \implies g(x, y) = g(y, x) = \|x\| \, \|y\| \tag{8}$$

is true.

Proof. $\dfrac{g(x,y) + g(y,x)}{2\,\|x\|\,\|y\|} = 1 \implies g(x,y) + g(y,x) = 2\,\|x\|\,\|y\|$. To complete the proof it is sufficient to use (4). ∎

LEMMA 3. *We have*

$$\begin{aligned} &(\forall x \in S(X))(\forall y \in X)\, \|x \pm y\| \le 1 \implies \\ &\qquad \implies (g(x,y) = 0 \wedge (\forall t \in [-1,1])\, \|x + ty\| = 1), \end{aligned} \tag{9}$$

$$\begin{aligned} &(\forall x, y \in S(X))\|x \pm y\| \le 1 \implies \\ &\qquad \implies (g(x,y) = g(y,x) = 0 \wedge (\forall t \in [-1,1])\, \|x + ty\| = \|y + tx\| = 1). \end{aligned} \tag{10}$$

Proof. Let $x \in S(X)$ and $\|x \pm y\| \le 1$. Then using (3) and (4) we get $g(x, x \pm y) = 1 \pm g(x,y) \le 1$. From here $g(x,y) = 0$ and $|g(x, x+ty)| = 1 \le \|x+ty\|$ for any $t \in [-1,1]$. Accordingly $\|x \pm y\| = 1$. In addition for $t \in [-1,1]$ we have

$$\begin{aligned} g(x, x \pm ty) = 1 \le \|x \pm ty\| &= \|x - tx + tx \pm ty\| \\ &= \|(1-t)x + t(x \pm y)\| \ (\text{resp. } \|(1+t)x + (-t)(x \mp y)\|) \\ &\le (1-t) + t = 1 \ \text{ if } \ 0 \le t \le 1 \\ &\qquad (\text{resp. } (1-t) + (-t) = 1 \ \text{ if } \ -1 \le t \le 0). \end{aligned}$$

Hence, $\|x \pm ty\| = 1$ for $t \in [-1,1]$.

If in the above considerations we assume that $y \in S(X)$, then for x and y the same arguments can be applied, what means that (10) holds. ∎

1. Characterization of the property (SC) by g-angle

It is known that strict convexity can be characterized by extremal points. Namely, if every point $x \in S(X)$ is an extremal point of the ball $B(X)$, then X is strictly convex. The converse of this statement also holds.

LEMMA 4. *$x \in S(X)$ is an extremal point of the ball $B(X)$ if and only if*

$$(\forall y \in X)\ \|x \pm y\| \le 1 \implies y = 0. \tag{11}$$

Proof. Let $x \in S(X)$ be an extremal point of the ball $B(X)$ and let $\|x \pm y\| \le 1$. Let $u = x + y$, $v = x - y$. Then $u, v \in B(X)$ and $x = (u+v)/2$, thus $u = v = x$. There from we have $y = 0$. Let now $x \in S(X)$, $x = (x_1 + x_2)/2$, $x_1, x_2 \in B(X)$. Let us put $y = (x_1 - x_2)/2$. Then $x + y = x_1$, $x - y = x_2$ and $\|x \pm y\| \le 1$. Therefore by (11) we have $y = 0$ what means that $x_1 = x_2 = x$. ∎

LEMMA 5. *The following statements are equivalent*

$$(\forall x, y \in S(X))\; x \ne y \implies g(x,y) < 1, \tag{12}$$

$$(\forall x, y \in S(X))\; x \ne y \implies \cos(x,y) < 1. \tag{13}$$

Proof. (12) ⇒ (13). For $x, y \in S(X)$ and $x \ne y$, from $g(x,y) < 1$ and $g(y,x) < 1$ we get $\cos(x,y) < 1$ by (5), hence (13) holds.

(13) ⇒ (12). Suppose that (12) does not hold. This means that there exist $x, y \in S(X)$ with $x \ne y$ such that $g(x,y) = 1$. For these x and y we have $\|x+y\| \le 2$ and $g(x, x+y) = 1+1 \le \|x+y\|$ wherefrom $\|x+y\| = 2$. Setting $x+y = u$, $x-y = v$ we get $x = (u+v)/2$, $y = (u-v)/2$, $\|u/2 \pm v/2\| = 1$ and $u/2 \in S(X)$. Therefore by (3) we have $g(x+y, x-y) = g(x+y, x+y-2y) = \|x+y\|^2 - 2g(x+y, y) = 0$ and $g(x+y, y) = 2$. Hence $g(x+y, x) = g(x+y, x+y-y) = \|x+y\|^2 - g(x+y, y) = 2$. So we have that $\cos(x, (x+y)/2) = 1$. In view of $x \ne (x+y)/2$, this means that (13) fails. ∎

THEOREM 1. *The following statements are equivalent:*

(a) *X is strictly convex,*

(b) *The implication (12) holds,*

(c) *The implication (13) holds.*

Proof. In view of Lemma 5, it is sufficient to show that X is strictly convex if and only if the implication (12) holds. Suppose that X is SC and (12) does not hold. This means that there exist $x, y \in S(X)$ such that $x \ne y$ and $g(x,y) = 1$. Then by (7) we have $g(x,y) \le \|x+y\| - 1$ wherefrom we get $\|x+y\| = 2$. By letting $x + y = u$ and $x - y = v$ we have $x = (u+v)/2$, $y = (u-v)/2$, $u/2 \in S(X)$ and $\|u/2 \pm v/2\| = 1$. From Lemma 4 we conclude $v = 0$, i.e. $x = y$, contrary to the hypothesis.

Suppose now that (12) holds and X is not SC. Then, by Lemma 4, there exists $x \in S(X)$ and $y \ne 0$ such that $\|x \pm y\| \le 1$. Applying Lemma 3 we conclude that $g(x,y) = 0$ and $\|x - y\| = 1$. Therefore by (12) we have $g(x, x-y) < 1$. On the other hand we have $g(x, x-y) = g(x,x) - g(x,y) = 1$, which yields $1 < 1$ what is impossible. Hence, (12) implies strict convexity of the space X. ∎

COROLLARY 1. *X is strictly convex if and only if one of the following two implications holds:*

$$(\forall x, y \in S(X))\; \cos(x,y) = 1 \implies x = y,$$

$$(\forall x, y \in S(X))\; g(x,y) = 1 \implies x = y.$$

2. Characterization of the property (UC) and the property (LUC) by g-angles

LEMMA 6. *The following statements hold.*

(a) *Let* (x_n) *and* (y_n) *be sequences from* $S(X)$ *such that* $\cos(x_n, y_n) \to 1$ $(n \to \infty)$. *Then* $\|x_n + y_n\| \to 2$ $(n \to \infty)$.

(b) *Let* (x_n) *and* (y_n) *be sequences from* $B(X)$ *such that* $\|x_n + y_n\| \to 2$ $(n \to \infty)$. *Then* $g((x_n + y_n)/\|x_n + y_n\|, x_n) \to 1$ $(n \to \infty)$.

(c) *Let* (x_n) *be a sequence from* $S(X)$ *and* (y_n) *a sequence from* $B(X)$ *such that* $g(x_n, y_n) \to 1$ $(n \to \infty)$. *Then* $\|x_n + y_n\| \to 2$ $(n \to \infty)$ *and* $\cos((x_n + y_n)/\|x_n + y_n\|, x_n) \to 1$ $(n \to \infty)$.

(d) *Let* (x_n) *and* (y_n) *be sequences from* $B(X)$ *such that* $\|x_n + y_n\| \to 2$ $(n \to \infty)$. *If* $\|(x_n + y_n)/\|x_n + y_n\| - x_n\| \to 0$ $(n \to \infty)$, *then* $\|x_n - y_n\| \to 0$ $(n \to \infty)$.

Proof. The statement a) follows immediately from (7).

(b) By virtue of (6) we have

$$\|x_n + y_n\| \left(\|x_n + y_n\| - \|y_n\|\right) \leq g(x_n + y_n, x_n) \leq \|x_n + y_n\|$$

$$\Rightarrow \|x_n + y_n\| - 1 \leq \|x_n + y_n\| - \|y_n\| \leq g\left(\frac{x_n + y_n}{\|x_n + y_n\|}, x_n\right) \leq 1$$

$$\Rightarrow g\left(\frac{x_n + y_n}{\|x_n + y_n\|}, x_n\right) \to 1 \quad (n \to \infty).$$

(c) By virtue of (6) we have $\|x_n + y_n\| \to 2$ $(n \to \infty)$. From (b) $g\left(\frac{x_n+y_n}{\|x_n+y_n\|}, x_n\right) \to 1$ $(n \to \infty)$. By $g\left(\frac{x_n+y_n}{\|x_n+y_n\|}, x_n\right) = \frac{1+g(x_n,y_n)}{\|x_n+y_n\|}$, and $g(x_n, y_n) \to 1$ $(n \to \infty)$ we have $g\left(x_n, \frac{x_n+y_n}{\|x_n+y_n\|}\right) \to 1$ $(n \to \infty)$. So, $\cos\left(\frac{x_n+y_n}{\|x_n+y_n\|}, x_n\right) \to 1$ $(n \to \infty)$.

(d) Since $\frac{\|x_n-y_n\|-(2-\|x_n+y_n\|)\|x_n\|}{\|x_n+y_n\|} \leq \frac{\|x_n-y_n-(2-\|x_n+y_n\|)x_n\|}{\|x_n+y_n\|} = \left\|\frac{x_n+y_n}{\|x_n+y_n\|} - x_n\right\|$ from $\|x_n+y_n\| \to 2$ $(n \to \infty)$ and $\|(x_n+y_n)/\|x_n+y_n\|-x_n\| \to 0$ $(n \to \infty)$ we get $\|x_n - y_n\| \to 0$. ■

LEMMA 7. *The following implications are equivalent:*

(a) $(\forall x_n, y_n \in S(X))\ \|x_n + y_n\|/2 \to 1\ (n \to \infty) \implies \|x_n - y_n\| \to 0\ (n \to \infty)$.

(b) $(\forall x_n, y_n \in S(X))\ g(x_n, y_n) \to 1\ (n \to \infty) \implies \|x_n - y_n\| \to 0\ (n \to \infty)$.

(c) $(\forall x_n, y_n \in S(X))\ \cos(x_n, y_n) \to 1\ (n \to \infty) \implies \|x_n - y_n\| \to 0\ (n \to \infty)$.

Proof. (a) $\Rightarrow$ (c). Let (x_n) and (y_n) be sequences from $S(X)$ such that $\cos(x_n, y_n) \to 1$. Then, by virtue of (a) in Lemma 6, $\|x_n + y_n\|/2 \to 1$ $(n \to \infty)$. So, by the implication (a), $\|x_n - y_n\| \to 0$ $(n \to \infty)$.

(c) $\Rightarrow$ (b). Let (x_n) and (y_n) be sequences from $S(X)$ such that $g(x_n, y_n) \to 1$ $(n \to \infty)$. Then, by virtue of (c) in Lemma 6, $\|x_n + y_n\| \to 2$ $(n \to \infty)$ and $\cos((x_n+y_n)/\|x_n+y_n\|, x_n) \to 1$ $(n \to \infty)$, whence $\|(x_n+y_n)/\|x_n+y_n\|-x_n\| \to 0$

$(n\to\infty)$ by the implication (c). Then, by virtue of (d) in Lemma 6, $\|x_n-y_n\|\to 0$ $(n\to\infty)$.

(b) $\Rightarrow$ (a). Let (x_n) and (y_n) be sequences from $S(X)$ such that $\|x_n+y_n\|/2\to 1$ $(n\to\infty)$. Then, by virtue of (b) in Lemma 6, $g((x_n+y_n)/\|x_n+y_n\|,x_n)\to 1$ $(n\to\infty)$, whence $\|(x_n+y_n)/\|x_n+y_n\|-x_n\|\to 0$ $(n\to\infty)$ by the implication (b). So, by virtue of (d) in Lemma 6, $\|x_n-y_n\|\to 0$ $(n\to\infty)$. ∎

Theorem 2. *The following statements about X are equivalent.*

(a) *The space X is UC.*

(b) $(\forall x_n,y_n\in S(X))\ g(x_n,y_n)\to 1\ (n\to\infty) \implies \|x_n-y_n\|\to 0\ (n\to\infty)$.

(c) $(\forall x_n,y_n\in S(X))\ \cos(x_n,y_n)\to 1\ (n\to\infty) \implies \|x_n-y_n\|\to 0\ (n\to\infty)$.

(d) $(\forall\varepsilon>0)(\exists\delta>0)(\forall x,y\in S(X))\ \|x-y\|\ge\varepsilon \implies \cos(x,y)\le 1-\delta$.

Proof. By statement (4), p.189 of [2] X is UC if and only if X has the property (a) in Lemma 7. Hence, by Lemma 7 we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(d) $\Rightarrow$ (c). Suppose that (d) does hold but (c) does not hold. This means that there exist a sequence (x_n) and a sequence (y_n) from $S(X)$ such that $\cos(x_n,y_n)\to 1$ $(n\to\infty)$ and $\|x_n-y_n\|\not\to 0$ $(n\to\infty)$. This means that the following statement holds. For (x_n) and (y_n) selected as above we have

$$(\exists\varepsilon>0)(\forall n_0\in N)\ n\ge n_0 \implies \|x_n-y_n\|\ge\varepsilon.$$

By (d) there exists $\delta>0$ such that $\cos(x_n,y_n)\le 1-\delta$ what is contrary to $\cos(x_n,y_n)\to 1$ $(n\to\infty)$. So, the statement (c) holds.

(c) $\Rightarrow$ (d). If (d) does not hold, then

$$(\exists\varepsilon>0)(\forall\delta>0)(\exists x,y\in S(X))\ \|x-y\|\ge\varepsilon\wedge\cos(x,y)>1-\delta.$$

Let $\delta=\frac{1}{n}$, $n\in\mathbf{N}$. Then there exist sequences (x_n) and (y_n) from $S(X)$ such that, for $n\in\mathbf{N}$, $\|x_n-y_n\|\ge\varepsilon$ and $1-\frac{1}{n}<\cos(x_n,y_n)$. So, $\cos(x_n,y_n)\to 1$ $(n\to\infty)$ and $\|x_n-y_n\|\not\to 0$ $(n\to\infty)$. Hence, the statement (c) does not hold. ∎

Theorem 3. *The following statements about X are equivalent.*

(a) *The space X is LUC.*

(b) $(\forall x_0\in S(X))(\forall y_n\in S(X))\ \cos(x_0,y_n)\to 1\ (n\to\infty) \implies \|x_0-y_n\|\to 0$ $(n\to\infty)$.

(c) $(\forall x_0\in S(X))(\forall\varepsilon>0)(\exists\delta>0)(\forall y\in S(X))\ \|x_0-y\|\ge\varepsilon \implies \cos(x_0,y)\le 1-\delta$.

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). In the proof of corresponding parts in Theorem 2, we may replace x_n by x_0 for every n.

(c) $\Rightarrow$ (a). Let $x_0\in S(X)$ and let (y_n) be a sequence from $S(X)$ such that $\|x_0+y_n\|\to 2$ $(n\to\infty)$. To show (a), we have only to prove that $\|x_0-y_n\|\to 0$ $(n\to\infty)$. By virtue of (b) in Lemma 6, $g(z_n,x_0)\to 1$ $(n\to\infty)$, where $z_n=(x_0+y_n)/\|x_0+y_n\|$ for every n. Then, by virtue of (c) in Lemma 6, $\|x_0+z_n\|\to 2$

$(n \to \infty)$ and $\cos((x_0 + z_n)/\|x_0 + z_n\|, x_0) \to 1$ $(n \to \infty)$, whence $\|x_0 - (x_0 + z_n)/\|x_0 + z_n\|\,\| \to 0$ $(n \to \infty)$ by the implication (c). Hence, we get by (d) in Lemma 6 that $\|x_0 - z_n\| \to 0$ $(n \to \infty)$, and so $\|x_0 - y_n\| \to 0$ $(n \to \infty)$. Thus (a) holds. ∎

It was proved earlier (cf. [9]) that a quasi-inner product space is smooth, uniformly smooth and very smooth. Now we get the following result.

COROLLARY 1. *A q.i.p. space is UC.*

Proof. Using Definition 2 we easily show that for a q.i.p. space the following implication holds

$$(\forall x, y \in S(X))\ \|x - y\| \geq \varepsilon \implies \cos(x, y) \leq 1 - \frac{\varepsilon^4}{16}.$$

3. Characterization of the property (UCED) by g-angle

LEMMA 8. *The following implication is true*

$$(\forall x_n, y_n \in B(X))\ (\|x_n + y_n\| \to 2 \wedge x_n - y_n \to z)\ (n \to \infty) \implies$$
$$\implies x_n - \frac{x_n + y_n}{\|x_n + y_n\|} \to \frac{z}{2}\ \ (n \to \infty) \tag{14}$$

Proof. $\left\| x_n - \dfrac{x_n + y_n}{\|x_n + y_n\|} - \dfrac{z}{2} \right\| = \left\| \dfrac{(\|x_n + y_n\| - 2)x_n}{\|x_n + y_n\|} + \dfrac{x_n - y_n}{\|x_n + y_n\|} - \dfrac{z}{2} \right\| \leq$ $\dfrac{2 - \|x_n + y_n\|}{\|x_n + y_n\|} + \left\| \dfrac{x_n - y_n}{\|x_n + y_n\|} - \dfrac{z}{2} \right\| \to 2\ \ (n \to \infty).$

LEMMA 9. *If*

$$(\forall x_n \in S(X))(\forall y_n \in B(X))\ (\cos(x_n, y_n) \to 1 \wedge x_n - y_n \to z)\ \ (n \to \infty) \implies z = 0. \tag{15}$$

then

$$(\forall x_n \in S(X))(\forall y_n \in B(X))\ (g(x_n, y_n) \to 1 \wedge x_n - y_n \to z)\ \ (n \to \infty) \implies z = 0. \tag{16}$$

Proof. Let the sequence (x_n) be from $S(X)$ and let the sequence (y_n) be from $B(X)$. Suppose that $g(x_n, y_n) \to 1$ $(n \to \infty)$ and $x_n - y_n \to z$ $(n \to \infty)$. By (6) we have $g(x_n, y_n) \leq \|x_n + y_n\| - 1$ and $g(x_n, y_n) \to 1$ $(n \to \infty)$ implies that $\|x_n + y_n\| \to 2$ $(n \to \infty)$. Therefore $g\left(\dfrac{x_n + y_n}{\|x_n + y_n\|}, x_n\right) \to 1$ $(n \to \infty)$ (see b) in Lemma 6).

In addition, $g(x_n, y_n) \to 1$ $(n \to \infty)$ and $\|x_n + y_n\| \to 2$ $(n \to \infty)$ imply that $g\left(x_n, \dfrac{x_n + y_n}{\|x_n + y_n\|}\right) \to 1$ $(n \to \infty)$. Since $x_n, \dfrac{x_n + y_n}{\|x_n + y_n\|} \in S(X)$, we get $\cos\left(x_n, \dfrac{x_n + y_n}{\|x_n + y_n\|}\right) \to 1$ $(n \to \infty)$. Since by Lemma 8 we have $x_n - \dfrac{x_n + y_n}{\|x_n + y_n\|} \to \dfrac{z}{2}$ $(n \to \infty)$ we obtain by (15) that $z = 0$, hence (16) holds. ∎

THEOREM 4. *X is UCED if an only if (16) holds.*

Proof. By Theorem 1 from [3] X is UCED if and only if the following statement holds

$$(\forall x_n, y_n \in B(X))\, (\|x_n + y_n\| \to 2 \,\wedge\, x_n - y_n \to z) \;\; (n \to \infty) \implies z = 0. \tag{17}$$

We shall prove that statements (16) and (17) are equivalent.

(17) $\Rightarrow$ (16). Let (x_n) be a sequence from $S(X)$ and (y_n) a sequence from $B(X)$ such that $g(x_n, y_n) \to 1$ $(n \to \infty)$ and $x_n - y_n \to z$ $(n \to \infty)$. We have already seen that $g(x_n, y_n) \to 1$ $(n \to \infty)$ implies $\|x_n + y_n\| \to 2$ $(n \to \infty)$. Hence, for given sequences (x_n) and (y_n) we have $x_n, y_n \in B(X)$, $\|x_n + y_n\| \to 2$ $(n \to \infty)$ and $x_n - y_n \to z$ $(n \to \infty)$. Therefore, according to (17), we get $z = 0$. This means that (17) $\Rightarrow$ (16).

(16) $\Rightarrow$ (17). Let sequences (x_n) and (y_n) from $B(X)$ fulfill conditions $\|x_n + y_n\| \to 2$ $(n \to \infty)$ and $x_n - y_n \to z$ $(n \to \infty)$. Since $\|y_n\| \le 1$, from the inequality

$$2 \ge g(x_n + y_n, x_n) \ge \|x_n + y_n\|(\|x_n + y_n\| - \|y_n\|) \ge \|x_n + y_n\|(\|x_n + y_n\| - 1)$$

(cf. (6)) we get $g\left(\dfrac{x_n + y_n}{\|x_n + y_n\|}, x_n\right) \to 1$ $(n \to \infty)$.

Besides $x_n, \dfrac{x_n + y_n}{\|x_n + y_n\|} \in B(X)$ and by Lemma 8 we have

$$x_n - \frac{x_n + y_n}{\|x_n + y_n\|} \to \frac{z}{2} \quad (n \to \infty)$$

Therefore by (16) we conclude that $z = 0$, i.e. the condition (17) is fulfilled. ∎

COROLLARY 2. *If the statement (15) holds, then X is UCED.*

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