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JOVAN KARAMATA
(1902–1967)

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The great mathematician, one of the internationally best known Serbian scientists, Jovan Karamata was born into a rich merchants family. Living in the city of Zemun on the borders of the Austrian and Turkish empires and Serbia, several generations of the family developed a flourishing business through almost three centuries. Even in 1772 they acquired an 18 century baroque mansion in which eight generations of the family have been living. They belonged to the upper class so that even the Austrian emperor Joseph the 2nd when visiting Zemun in 1778—the time of the siege of Belgrade in the Austrian-Turkish war—stayed in their home. This has been witnessed by a huge Austrian coat of arms painted across the ceiling of his room by the well known artist Dimitrije Bačević, [6]. Here the story about many known, even famous European families repeats: the ancestors were soldiers, merchants, landowners etc., usually scientists, authors, artists, etc. J. Karamata was born in Zagreb on February 1, 1902, but spent the major part of his life (until 1951) in Zemun. There he completed the first three classes of high school, and then was send by his father, because of the First World War to Lausanne. Upon graduating in 1920 he returned to Belgrade, enrolled the Faculty of Engineering of Belgrade University but in 1922 following his interest and talent, he moved to the Faculty of Philosophy—Department of Mathematics where he graduated in 1925. At the beginning he was influenced by his professor Mihajlo Petrović, the famous Serbian mathematician, who himself was a student of celebrated French mathematicians Ch. Hermite, E. Picard, H. Poincaré and transferred from Paris to his students in Belgrade important new mathematical ideas and theories.

Moreover, with his colleagues M. Milanković and B. Gavrilović he created an inspiring atmosphere, fruitful for creative work. But, Karamata considered himself as a self-taught man, acquiring his education by pursuing some books and research papers by masters, already at his early student days. An outstanding position in this respect occupies the famous book by G. Pólya and G. Szegö, "Aufgaben und

Lehrsätze aus der Analysis". As a result, three months after the graduation he submitted his doctoral thesis "O jednoj vrsti granica sličnih odredjenim integralima" and obtained his PhD. degree already by March 26, 1926, [5].

Then he had an usual but fast academic career, disturbed only by the war.

In 1927/28 a Rockefeller Foundation fellow in Paris, in 1928 teaching assistant at the Faculty of Philosophy, University of Belgrade, in 1930 assistant professor, in 1937 associate one and in 1950 the full professor. In 1951 he became a full professor at the University of Geneva.

Karamata, however, being by family position deeply rooted to his country, left it reluctantly; financial motives if any, were at the tail of his decision. By the communist regime he was considered as a "reactionary bourgeois capitalist", not friendly to the socialism, although pretending otherwise [5]. That minimized his professional influence at the Department and made his life frustrating so he decided to leave. But he never broke ties with his country. He frequently visited the Academy, lectured at the Mathematical Institute, the University of Novi Sad, and the Faculty of Economics in Belgrade, and cooperated with his former students.

In addition, claim to his early international reputation, he was elected in 1933 Corresponding Member of the Yugoslav Academy of Sciences and Arts (Zagreb), and in 1936 a Corresponding Member of Regia Societas Scientiarum Bohemica (Prague). In 1939 he became a Corresponding Member, and in 1948 a Full Member of the Serbian Academy of Sciences and Arts.

He was one of the founders of the Mathematical Institute of Serbian Academy of Sciences and Arts in 1946, and the first Editor-in-Chief of both of the journals Matematički Vesnik and Publications de l'Institut Mathématique.

In 1954 he became a member of the Editorial Board and Director of the journal L'Enseignement Mathématique from Geneva. His influence to the form and content of this journal was significant.

In 1931 he married Emilija Nikolajević, who gave birth to their two sons and a daughter. His wife died in 1959.

After a long illness, Jovan Karamata died on August 14, 1967 in Geneva. His ashes rest in Zemun.

The work of Jovan Karamata belongs to the classical analysis which he knew to perfection in the time of its prime. He had the characteristics and the power of the great analysts of that epoch. By the words of M. Tomić, he not only shared the opinion of G.H. Hardy that the mathematics was young people's game, but demonstrated it. Indeed, the three most famous of Karamata's papers belong to the time of his youth, and attract more attention today. They appeared in 1930–31, [7].

These three results entered several standard monographs on the subject— even some textbooks, and we chose to present these briefly, among many others appearing in his 131 scientific papers, (see, Bibliography in [7]).

1. New method in Tauberian theory

We recall classical N. Abel's (direct) theorem:

Put $s_n = \sum\limits_0^{\infty} s_\nu$; *then* $s_n \to s$, $n \to \infty$, *implies* $\sum\limits_0^{\infty} a_\nu x^\nu \to s$, $x \to 1-0$, *the latter series being convergent for* $|x| < 1$.

I.e., the convergence of the series implies Abel summability. The inverse statement is in general false. For example, $\sum\limits_0^{\infty}(-1)^\nu x^\nu = \frac{1}{1+x} \to \frac{1}{2}$ as $x \to 1-0$, i.e., the series $\sum(-1)^\nu$ is Abel summable but it obviously diverges. But the inverse statement holds with the additional condition (1) $na_n \to 0$, $n \to \infty$, as proved by A. Tauber in 1897. Feeling something important Hardy and Littlewood named such inverse results: Tauberian theorems. Tauber's proof was quite elementary, but in 1910 J.E. Littlewood proved the following theorem which is now classical:

Let the series $\sum a_\nu$ *be Abel summable to a sum* s *and (2)* $|na_n| \le M$, *then it converges to* s.

The generalization of convergence condition (1) to (2) to a non-specialists does not seem to be significant. But the importance of Tauberian theorems emanated from this theorem of Littlewood which became the starting point of a new branch of mathematical analysis—the Tauberian theory. His proof, however, was extremely involved and long. In spite of the attempts of a number of reputed analysts no significant simplifications were made. Then Karamata published his spectacular proof in two pages: ("Über die Hardy-Littlewoodschen Umkehrung des Abelschen Stetigkeitssatzes", Math. Zeit. **32** (1930), 319–320).

It runs as follows.

Karamata first noticed that Abel summability of the series $\sum a_\nu$ to the sum s implies that for any power $g(x^\nu) = x^{\alpha\nu}$, $\alpha \ge 1$,

$$(1-x)\sum_0^{\infty} s_\nu g(x^\nu)x^\nu \to s\int_0^1 g(t)\,dt, \quad x \to 1-0. \tag{3}$$

This is then also true for every polynomial $P(t)$. Then, by applying Weierstrass approximation theorem, Karamata concludes that (3) holds for any bounded R-integrable function $g(x)$. By choosing $x = e^{-1/n}$, $g(x) = 1/x$ for $1/e \le x \le 1$ and zero otherwise, the C-summability (i.e. that $\sum\limits_0^{n} s_\nu/(n+1) \to s$, $n \to \infty$) of the series follows. Hence by a simple Hardy's lemma, its convergence, due to (2). Much later, in 1952 N. Wielandt shared that the C-summability step, that is the use of Hardy's lemma can be avoided. On the other hand, in 1984 V. Marić and M. Tomić proved that lemma in a couple of lines.

The proof was immediately internationally recognized. R. Schmidt call it "an important discovery", K. Knopp said that it was "a surprising proof", N. Wiener named it “elegant”, E.C. Titchmarsh “extremely elegant” etc. In 1978, on the occasion of the 60th anniversary of Mathematische Zeitschrift Karamata’s paper was chosen among the first 50 most important ones selected among all of these published in this journal in that period, [7].

Nowadays the proof can be found in the well-known treatises of Knopp, G. Doetsch, Titchmarsh, Hardy, D.V. Widder and J. Favard, [5].

The following true story is nice. When visiting the St. Andrews university in Scotland I was introduced to E.T. Copson (the author of widely used textbook on the theory of complex functions). He told me: “I have known only one Yugoslavian mathematician—Karamata. When cooperating with Hardy at Oxford I found him one day nervously pacing up and down his office. Not responding to greetings he abruptly said: ‘I just received a letter from a young man in Belgrade who claims he can prove Hardy-Littlewood theorem in just two pages. This is simply impossible!’ ”

To understand properly why such an excitement about a proof one has to have in mind that at that time Tauberian theory was one of the highlights of the analysis. Many famous analysts worked on that subject. Among others, Hardy, Littlewood in England, E. Landau, F. Hausdorff, Knopp in Germany, M. Riesz, L. Féjer in Hungary etc. The significance and the depth of the theory is best seen from the fact that S. Ikehara’s Tauberian theorem (1931) contains the celebrated prime numbers one: $\pi(x) \sim x/\ln x$, as $x \to \infty$, where $\pi(x)$ is the number of primes not exceeding x.

The theory culminated in 1932 by Wiener’s general Tauberian theorem, one of the keystones of Mathematical Analysis (to be quoted below). It contains Littlewood’s theorem as a special case. Yet, Karamata’s proof has kept its importance for at least two reasons.

Firstly, to derive Littlewood’s theorem from the Wiener’s one is by no means easy. Direct proof is much more convenient. Secondly, and more important, the proof offers a new method which is then used, e.g., for estimating the remainder $R_n = s_n - s$, as it was done by several authors (e.g., A.G. Postnikov, G. Frend, J. Korevaar).

2. The theory of slowly varying functions

This appears to be the most important achievement of Karamata. The idea originated in many attempts of relaxing the convergence conditions such as (1) and (2). The most important of which, in Wiener’s opinion, were the ones by R. Schmidt. He was the first to consider a group of terms of a sequence, not only a single one as in (1) and (2). Thus he introduced slowly oscillating sequences s_n (under the name “sehr langsam oscillierende Folgen”), by requiring $s_0 - s_q \to 0$, with $p \to \infty$, $q \geq \infty$ and $q/p \to 1$.

Analogously, the functions satisfying $f(y) - f(x) \to 0$, when $y > x$, $x \to \infty$ and $y/x \to 1$ (or $y - x \to 0$), are called slowly oscillating. The importance of this class one can recognize from the H.R. Pitt’s form of the mentioned Wiener’s general Tauberian theorem which reads as follows, [1]:

Let $g \in L^1(-\infty,\infty)$ and its Fourier transform $\hat{g}(u)$ be different from zero for every u. If $\int g(x-y)a(y)\,dy \to l \int g(y)\,dy$ as $x \to \infty$ then $a(y) \to l$, as $y \to \infty$, provided that $a(y)$ is slowly oscillating.

Nobody before Karamata studied these functions as a distinctive class, leaving aside convergence condition. This led him to the proper and very simple definition:

A function $L(x)$, defined for $x \geq 0$ is called slowly varying if it is positive, continuous and such that

$$L(tx)/L(x) \to 1 \text{ as } x \to \infty, \text{ for every fixed } t > 0. \tag{4}$$

Also, the function r of the form $r(x) = x^\alpha L(x)$ is called regularly varying; α is the index of regular variation.

It is Karamata's remarkable achievement to derive from this definition a whole theory covering a number of useful properties of L-functions. We mention two most important of these. The first one gives the canonical representation

$$L(x) = c(x) \exp \int_a^x (\varepsilon(u)/u)\alpha_u$$

where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

The second one states that the limit (4) holds uniformly in $t \in [a,b]$, $0 < a < b < \infty$. There are not many theories today which in spite of being of a general reach, stem from such a simple definition as (4). It is interesting that his seminal paper appeared in an rather unknown journal at that time. ("Sur une mode de croissance régulier des fonctions", Mathematica, (Cluj), **4** (1930), 38–53).

After Karamata made the first striking use of slowly varying functions to Tauberian theorems, as it will be shown below, it was soon realized by a number of authors that these have various applications in several branches of analysis such as Abelian theorems (i.e., asymptotic behaviour of integrals, Mercerian theorems, complex functions (in particular, entire ones), number theory, etc). But a real surprise came by the famous W. Feller's book on probability theory in 1966 where the great potential of slowly varying functions in that field became apparent. That was further emphasized by L. de Haan's work in 1970. A curious fact is that Karamata himself was not knowledgeable in probability theory. More recently, the applications are made to the generalized functions and to differential equations, both linear and non-linear. There are three monographs dealing with the theory and applications of slowly varying functions: by E. Seneta in 1976, [2], by J. Geluk and de Haan in 1987, [3], and the most comprehensive one, by N.H. Bingham, C.M. Goldie and J.L. Teugels in 1987, [4].

3. Tauberian theorem

In 1931 ("Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stietjessche Transformation betreffen", Journal für die reine und angewandte Mathematik **164** (1931), 27–39) Karamata proved the following result:

Let $\mathcal{U}(x)$ be a nondecreasing function in $[0,\infty]$ such that

$$w(x)=\int_0^\infty e^{-xs}\,d\{U(s)\}$$

is bounded for every $x>0$ and if $\rho\geq 0$, $L(x)$ is slowly varying and $w(x)=x^{-\rho}L(1/x)$ as $x\to 0+$, then $\mathcal{U}(x)\sim x^\rho L(x)/\Gamma(\rho+1)$, $x\to\infty$. Analogously, $L(1/x)$ can be replaced by $L(x)$ and $x\to 0$ by $x\to\infty$.

It is the third important discovery of Karamata that we mention, which is today known as Karamata Tauberian theorem, or sometimes as Hardy-Littlewood-Karamata one.

In the original statement only power x^ρ (or $x^{-\rho}$) appeared i.e. $L\equiv 1$. Hardy and Littlewood realized very early that a factor of slower increase could be introduced; so they put $\prod_1^n(\log_k x)t_k$, where $\log_k x$ stands for the k-th iteration of the logarithm and t_k is a real number. Then, the natural question arises, what kind of factors one could use, and also should all of these require a new proof? Then Karamata used L as a factor, obtaining thus the most general form of the theorem and with an elegant proof. Seneta considers this to be "the most famous and very widely useful theorem in probability theory" and Bingham even as "one of the major analytic results of this century", [6].

Another major Karamata's contribution to mathematics was in educating young mathematicians. He was a devoted teacher and published several textbooks some of which being quite original. More important, he had a rare ability to discover and promote talented students to start research at an early stage and then lead them to the scientific maturity. To appraise his merits properly, one has to know how difficult was the situation in our country immediately after the Second World War in many respect. It was undergoing the so called "revolutionary changes" which included political pressure even persecution of a number of students and professors. On the other hand, the well equipped library of the Department of Mathematics—the only one of the kind was destroyed by the war bombing. In spite of all that, Karamata besides of teaching, worked with his students, in the Academy of Science, in Hotel Majestic, at his home. He landed them books from his own library. For those who were considered as politically unreliable, even enemies, he was their only hope. Later in Geneva, he helped a number of them to get foreign scholarship for further studies, and positions at foreign universities. He continued to cooperate and to write joint papers with some of them to the end of his life. Thus, he created what was later named by Seneta and Bingham as the "Yugoslav school of analysis", but that was in fact Karamata's school.

We conclude with the following additional remarks. Karamata wrote his papers with an utmost care although he was able to write several pages of flawless calculations in no time and in a very legible and harmonious handwriting at that.

On the other hand, once being ill he outlined an entire proof without writing a single character, [5] ("Ein Konvergentzsatz für trigonometrische Integrale", Journal für die reine und angewandte Mathematik, **178** (1937), 29–33). His papers are characterized by clarity and elegance.

Yet, in everyday life he showed no interest in any kind of art nor was knowledgeable in the literature.

On the other hand, he exhibited a somewhat surprising interest in antroposophical doctrine of Rudolf Steiner—a mixture of philosophy and mysticism. He enjoyed talking about the subject and it was very interesting how he purified some rather nebulous statements of the doctrine through his powerful logic, to make these almost plausible.

REFERENCES

[**1**] G.H. Hardy, *Divergent Series*, Oxford 1948.

[**2**] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, 580, Springer, Berlin, Heidelberg, New-York, 1976.

[**3**] J.L. Geluk, L. de Haan, *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract, 40 Amsterdam, 1987.

[**4**] N.N. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation, Encyclopaedia of Mathematics and its Applications*, Vol. 27, Cambridge University Press, Cambridge, 1987.

[**5**] M. Tomić, S. Aljančić, *Remembering Jovan Karamata*, Publ. Inst. Math. **48** (62), 1990, 1–6.

[**6**] A. Nikolić, *Jovan Karamata – život kroz matematiku*, Zadužbina Andrejević, Beograd 1999.

[**7**] M. Tomić, *Jovan Karamata 1902-1967*, Bulletin T. CXXII de l'Académie Serbe des Sciences et des Arts, Classe Sci. Math. Not., Sciences Math. 26 (2001), 1–30.