МАТЕМАТИЧКИ ВЕСНИК
54 (2002), 57–70

UDK 517.984.3
оригинални научни рад
research paper

ASCENT, DESCENT, QUASI-NILPOTENT PART AND ANALYTIC CORE OF OPERATORS

Abstract. This paper concerns a localized version of the single valued extension property of a bounded operator $T \in L(X)$, where X is a Banach space, at a point $\lambda_0 \in \mathbb{C}$. We shall relate this property to the ascent and the descent of $\lambda_0 I - T$, as well as to some spectral subspaces as the quasi-nilpotent part and the analytic core of $\lambda_0 I - T$. We shall also describe all these notions in the setting of an abstract shift condition, and in particular for weighted right shift operators on $\ell^p(\mathbb{N})$, where $1 \le p < \infty$.

1. The single-valued extension property

One of basic properties in local spectral theory is the so-called single valued extension property for bounded operators on Banach spaces. This property is enjoyed by several classes of operators as the decomposable operators, as well as other classes of operators; we refer to the excellent monograph by Laursen and Neumann [16] for a modern treatment of the theory of decomposable operators.

In this paper we shall consider the following local version of this property, introduced by Finch [13] and studied later by several authors [18, 19, 26, 1, 2, 3, 5, 6].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}_{\lambda_0}$ centered at λ_0, the only analytic function $f\colon \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \qquad (1)$$

is the function $f \equiv 0$.

AMS Subject Classification: 47 A 10, 47 A 11, 47 A 53, 47 A 55

Keywords and phrases: Single valued extension property, quasi-nilpotent part and analytic core, property (Q), weighted right shift operators.

Communicated at the 5th International Symposium on Mathematical Analysis and its Applications, Niška banja, Yugoslavia, October, 2–6, 2002.

The research was supported by the International Cooperation Project between the University of Palermo (Italy) and Conicit-Venezuela.

An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

The SVEP of $T \in L(X)$ may be also defined as follows: Let U be an open subset of $\mathbb{C}$ and let $H(U, X)$ denote the space of X-valued functions on U equipped with the topology of uniform convergence on compact subsets of U. Then $H(U, X)$ is a Fréchet space and every $T \in L(X)$ induces a continuous mapping T_U on $H(U, X)$, defined by

$$T_U(f)(\lambda) := (\lambda I - T)f(\lambda) \quad \text{for all } f \in H(U, X). \tag{2}$$

The operator $T \in L(X)$ has the SVEP precisely when T_U is injective.

The SVEP may be characterized by means of some typical tools originating from the local spectral theory. Recall that, for a bounded operator $T \in L(X)$, the *local resolvent set* of T at the point $x \in X$, is defined as the union of all open subsets U of $\mathbb{C}$ such that there exists an analytic function $f : U \to X$ which satisfies

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in U . \tag{3}$$

The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and obviously $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T.

Clearly, any analytic function which verifies (3) on this union is a local extension of the analytic function $R(\lambda, T)x := (\lambda I - T)^{-1}x$ defined on the resolvent set $\rho(T)$ of T. Generally, the analytic solutions of (3) are not uniquely determined. It is clear from the definition that, if T has the SVEP at λ_0, then the analytic solution of (3) is uniquely determined in an open disc centered at λ_0.

For every subset F of $\mathbb{C}$, let us denote by $X_T(F)$ the *analytic spectral subspace* of T associated with Ω:

$$X_T(F) := \{\, x \in X : \sigma_T(x) \subseteq F \,\}.$$

For an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C}\backslash F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash F$. Note that T has SVEP if and only if $X_T(F) = \mathcal{X}_T(F)$ for all closed sets $F \subseteq \mathbb{C}$, see Proposition 3.3.2 of [16].

The SVEP, as well as the SVEP at a point $\lambda_0 \in \mathbb{C}$, may be characterized in a very simple way:

THEOREM 1.2. *Let $T \in L(X)$, X a Banach space. Then*

(i) T has the SVEP at λ_0 if and only if $\ker\,(\lambda_0 I - T) \cap X_T(\emptyset) = \{0\}$ *[1, Theorem 1.9];*

(ii) T has the SVEP if and only if $X_T(\emptyset) = \{0\}$, and this is the case if and only if $X_T(\emptyset)$ is closed; see [16, Proposition 1.2.16].

The basic role of SVEP arises in local spectral theory, since every decomposable operator enjoys this property. Recall that a bounded operator $T \in L(X)$ is said to have the *Bishop's property* (β) if for every open set U the operator T_U defined in (2) is injective and has closed range, while $T \in L(X)$ is said to have the *decomposition property* (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. The decomposability of $T \in L(X)$ may be defined in several ways, for instance as the union of the property (β) and the property (δ), see [16, Theorem 2.5.19]. Note that the property (β) implies that T has SVEP, while the property (δ) implies SVEP for $T^\star$, see [16, Theorem 2.5.19]. The class of decomposable operators contains, for instance, all normal operators on Hilbert spaces, all spectral operators, all operators with a non-analytic functional calculus and any operator with a totally disconnected spectrum, [16]. Examples of non-decomposable operators which have the SVEP may be found among the class of all multipliers of a commutative semi-prime Banach algebra, [16, Proposition 4.2.1]

We shall now introduce two important subspaces in local spectral theory and in Fredholm theory:

Definition 1.3. Let X be a Banach space and $T \in L(X)$. The *analytic core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and $\delta > 0$ for which:

(a) $x = u_0$, and $Tu_{n+1} = u_n$ for every $n \in \mathbb{N}$.

(b) $\|u_n\| \le \delta^n \|x\|$ for every $n \in \mathbb{N}$.

It easily follows, from the definition, that $K(T)$ is a linear subspace of X and that $T(K(T)) = K(T)$. In general, $K(T)$ is not closed and $K(T) \subseteq T^\infty(X)$, where $T^\infty(X) := \bigcap_{n=1}^\infty T^n(X)$ is the *hyperrange* of T. Furthermore, if T is quasi-nilpotent then $K(T) = \{0\}$, see [18].

Definition 1.4. Let $T \in L(X)$, X a Banach space. The *quasi-nilpotent part* of T is the set

$$H_0(T) := \{ x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0 \}.$$

Also $H_0(T)$ is a linear subspace of X, generally not closed. Furthermore, $\mathcal{N}^\infty(T) \subseteq H_0(T)$, where $\mathcal{N}^\infty(T) := \bigcap_{n=1}^\infty \ker T^n$ is the *hyperkernel* of T, and T is quasi-nilpotent if and only if $H_0(T) = X$, [27, Theorem 1.5].

The systematic investigation of the spaces $K(T)$ and $H_0(T)$ was initiated by Mbekhta [18], after an earlier work of Vrbová [27]. In particular, these authors established the following local spectral characterizations of $K(T)$ and $H_0(T)$.

Theorem 1.5. *For a bounded operator $T \in L(X)$, X a Banach space we have*

(i) $K(\lambda_0 I - T) = X_T(\mathbb{C} \setminus \{\lambda_0\})$.

(ii) $H_0(\lambda_0 I - T) = \mathcal{X}_T(\{\lambda_0\})$, so, if T has SVEP, $H_0(\lambda_0 I - T) = X_T(\{\lambda_0\})$.

Note that, for every $\lambda_0 \in \mathbb{C}$, the following inclusions hold:

$$X_T(\emptyset) \subseteq X_T(\mathbb{C} \setminus \{\lambda_0\}) = K(\lambda_0 I - T) \subseteq (\lambda_0 I - T)^\infty(X) \tag{4}$$

and

$$\ker\,(\lambda_0 I - T) \subseteq \mathcal{N}^\infty(\lambda_0 I - T) \subseteq H_0(\lambda_0 I - T) \subseteq X_T(\{\lambda_0\}). \tag{5}$$

Two important notions in Fredholm theory are those of the ascent and the descent of an operator. The *ascent* of an operator T is the smallest non-negative integer $p := p(T)$ such that ker $T^p =$ ker T^{p+1}. If such integer does not exist we put $p(T) := \infty$. The *descent* of an operator T is the smallest non-negative integer $q := q(T)$ such that $T^q(X) = T^{q+1}$, and if such integer does not exist we put $q(T) := \infty$. It is well-known that, if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$, [14, Proposition 38.3]. Furthermore, if λ_0 belongs to the spectrum $\sigma(T)$, then $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$ if and only if λ_0 is a pole of the resolvent $R(\lambda, T) := (\lambda I - T)^{-1}$, [14, Proposition 50.23]. Obviously, in this case λ_0 is an isolated point of $\sigma(T)$.

Recall that $T \in L(X)$, X is said to be *semi-Fredholm* if $T(X)$ is closed and at least one of the two defects $\alpha(T) :=$ dim ker T or $\beta(T) :=$ codim $T(X)$ is finite.

DEFINITION 1.6. An operator $T \in L(X)$, X a Banach space, is said to be *semi-regular* if $T(X)$ is closed and ker $T \subseteq T^\infty(X)$.

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T\,|M$ is semi-regular and $T\,|N$ is quasi-nilpotent.

An important case is obtained if we assume in the definition above that $T\,|N$ is nilpotent. In this case T is said to be of *Kato type*, [17]. If N is finite-dimensional then T is said to be *essentially semi-regular*, see Rakočević [24] or Müller [22]. Obviously, any semi-regular operator is of Kato type. Note that if T is of Kato type then $T^\infty(X) = K(T)$ and $K(T)$ is closed, see [2, Theorem 2.3 and Theorem 2.4]. An important class of operators of Kato type is given by the class of all semi-Fredholm operators, see West [29]. Furthermore, taking $M = \{0\}$ and $N = X$, we see that every quasi-nilpotent operator is of Kato type.

In the sequel by $M^\perp$ we shall denote the *annihilator* of the subset $M \subseteq X$; and by ${}^\perp N$ the *pre-annihilator* of the subset $N \subseteq X^\star$.

THEOREM 1.7. *For a bounded operator $T \in L(X)$, where X is a Banach space, the following implications hold:*

(i) $H_0(\lambda_0 I - T)$ closed $\Rightarrow$ $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T)$ closed $\Rightarrow$ $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$ $\Rightarrow$ T has SVEP at λ_0.

(ii) $X = H(\lambda_0 I - T) + K(\lambda_0 I - T)$ $\Rightarrow$ $T^\star$ has SVEP at λ_0.

Moreover, if $\lambda_0 I - T$ is of Kato type, then all these implications are equivalences.

Proof. Without loss of generality, we may consider $\lambda_0 = 0$.

(i) Assume that $H_0(T)$ is closed and let $\tilde{T}$ denote the restriction of T to the Banach space $H_0(T)$. Obviously, $H_0(T) = H_0(\tilde{T})$, so that $\tilde{T}$ is quasi-nilpotent and hence $K(\tilde{T}) = \{0\}$. It is easy to see that $H_0(T) \cap K(T) = K(\tilde{T})$. This shows the first implication. The second implication of (i) is an immediate consequence of Theorem 1.5. Indeed, we have

$$\ker\,(\lambda_0 I - T) \cap X_T(\emptyset) \subseteq H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T),$$

so, if the last intersection is $\{0\}$, then T has the SVEP at λ_0, by Theorem 1.2.

(ii) From [17, Proposition 1.8] we know that $H_0(T) \subseteq^{\perp} K(T^\star)$ and hence $K(T^\star) \subseteq H_0(T)^\perp$. We also have $H_0(T^\star) \subseteq K(T)^\perp$. Indeed, let $\varphi \in H_0(T^\star)$ and consider an arbitrary element $x \in K(T)$. According to the definition of $K(T)$, there is a sequence $(u_n) \subset X$, and a $\delta > 0$, such that $u_0 = x$, $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$. Clearly, $T^n u_n = x$ for every $n \in \mathbb{N}$. Consequently,

$$|\varphi(x)| = |\varphi(T^n u_n)| = |(T^{\star n}\varphi)(u_n)| \leq \|u_n\| \|T^{\star n}\varphi\| \leq \delta^n \|T^{\star n}\varphi\|,$$

and hence $|\varphi(x)|^{\frac{1}{n}} \leq \delta \|T^{\star n}\varphi\|^{\frac{1}{n}}$ for every $n \in \mathbb{N}$. The last term converges to 0 as $n \to \infty$, since $\varphi \in H_0(T^\star)$, and from this it follows that $\varphi(x) = 0$, i.e. $\varphi \in K(T)^\perp$. Finally, if $X = H_0(T) + K(T)$ then $\{0\} = H_0(T)^\perp \cap K(T)^\perp \supseteq H_0(T^\star) \cap K(T^\star)$, thus, by part (i), $T^\star$ has the SVEP at 0.

For the last assertion see Theorem 2.6 of [3]. ∎

An example, given in [5], of a bilateral right shift T defined in the Hilbert space $L^2(\omega)$, where $\omega := (\omega_n)_{n\in\mathbb{Z}}$ is a suitable weight sequence, shows that the SVEP at a point λ_0 does not, in general, implies that $H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\}$.

Also the finiteness of the ascent and the descent has important consequences on the SVEP. In fact we have the following implications.

Theorem 1.8. *For a bounded operator T on a Banach space X the following implications hold:*

(i) $p(\lambda_o I - T) < \infty \Rightarrow \mathcal{N}^\infty(\lambda_o I - T) \cap (\lambda_o I - T)^\infty(X) = \{0\} \Rightarrow$ *T has SVEP at λ_o.*

(ii) $q(\lambda_o I - T) < \infty \Rightarrow X = \mathcal{N}^\infty(\lambda_o I - T) + (\lambda_o I - T)^\infty(X) \Rightarrow$ *$T^\star$ has SVEP at λ_o.*

Moreover, if $\lambda_0 I - T$ is of Kato type, then all these implications are equivalences.

Proof. (i) There is no loss of generality in assuming $\lambda_o = 0$.

Let $p := p(T) < \infty$. Then $\mathcal{N}^\infty(T) = \ker T^p$ and hence, by [14, Proposition 38.1], $\mathcal{N}^\infty(T) \cap T^p(X) = \{0\}$. From $T^\infty(X) \subseteq T^p(X)$ we obtain that $\mathcal{N}^\infty(T) \cap T^\infty(X) = \{0\}$. The second implication is a consequence of Theorem 1.2, since, from the inclusions (4), we obtain that

$$\ker T \cap X_T(\emptyset) \subseteq \mathcal{N}^\infty(T) \cap T^\infty(X) = \{0\}.$$

(ii) Also here we may assume that $\lambda_o = 0$. Let $q := q(T) < \infty$. Then $T^\infty(X) = T^q(X)$ and $X = T^n(X) + \ker T^q$ for every $n \in \mathbb{N}$, by [14, Proposition 38.2]. From this it easily follows that $X = \mathcal{N}^\infty(T) + T^\infty(X)$, so the first implication of (ii) is proved.

In order to show the second implication of (ii), we first note that, if $X = \mathcal{N}^\infty(T) + T^\infty(X)$, then $\mathcal{N}^\infty(T)^\perp \cap T^\infty(X)^\perp = \{0\}$. Now, let us consider an element $x^\star \in \ker T^\star \cap X_{T^\star}(\emptyset)$. Clearly,

$$x^\star \in \ker T^\star \subseteq \ker (T^\star)^n = \ker (T^n)^\star = \overline{T^n(X)}^\perp \subseteq T^n(X)^\perp,$$

for every $n \in \mathbb{N}$ and therefore $x^\star \in T^\infty(X)^\perp$. On the other hand, from $\sigma_{T^\star}(x^\star) = \emptyset$ we obtain, by Theorem 1.5, that

$$x^\star \in K(T^\star) \subseteq (T^\star)^n(X^\star) = (T^n)^\star(X^\star) \subseteq (\ker T^n)^\perp$$

for every $n \in \mathbb{N}$. From this it follows that $x^\star \in \mathcal{N}^\infty(T)^\perp$ and therefore $x^\star \in \mathcal{N}^\infty(T)^\perp \cap T^\infty(X)^\perp$, which implies that $x^\star = 0$. Again, from Theorem 1.2 we conclude that $T^\star$ has the SVEP at 0.

For a proof of the last assertion see Corollary 2.7 of [2]. ■

Hence each one of the two conditions $p(\lambda_0 I - T) < \infty$ or $H_0(\lambda_0 I - T)$ closed implies the SVEP at λ_0. The next two examples show that in general these two conditions are independent.

Example 1.9. Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$Tx := \Big(\frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots\Big), \text{ where } x = (x_1, \cdots, x_n, \cdots).$$

It is easily seen that $\|T^k\| = \frac{1}{(k+1)!}$, from which it follows that T is quasi-nilpotent and therefore $H_0(T) = \ell^2(\mathbb{N})$. Obviously, $p(T) = \infty$.

Example1.10. In [3] it has been given an example of a direct sum of unilateral weighted shifts for which $p(T) = 0$ and $H_0(T)$ is not closed. The following simpler example is taken from [10]. Let $1 \le p < \infty$ be given and denote by $\omega = (\omega_n)_{n\in\mathbb{N}}$ a bounded sequence of positive real numbers. Let us consider the corresponding weighted unilateral right shift T on $\ell^p(\mathbb{N})$, defined by

$$Tx := \sum_{n=1}^{\infty} \omega_n x_n e_{n+1} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}),$$

where (e_n) stands for the canonical basis of $\ell^p(\mathbb{N})$. This operator has SVEP, since T has no eigenvalues and hence $p(\lambda I - T) = 0$ for all $\lambda \in \mathbb{C}$.

A routine calculation shows that the norm of T^n is given by

$$\|T^n\| = \sup_{k\in\mathbb{N}}(\omega_k \ldots \omega_{k+n-1}) \quad \text{for all } n \in \mathbb{N}.$$

Suppose now that $(\omega_n)_{n\in\mathbb{N}}$ is defined by

$$\omega_n := \begin{cases} 0 & \text{if } n \text{ is a square of an integer} \\ 1 & \text{otherwise} \end{cases}$$

It is easily seen that $\|T^n\| = 1$ for all $n \in \mathbb{N}$, so that T is not quasi-nilpotent. This excludes that $H_0(T)$ is closed, see next Theorem 2.7. It is easy to see that $K(T) = \{0\}$. Therefore, this example shows that the implication $H_0(T)$ closed $\Rightarrow$ $K(T) \cap H_0(T) = \{0\}$, noted in Theorem 1.7, cannot in general be reversed.

From Theorem 1.8 it follows that the *finite ascent property* for an operator $T \in L(X)$ defined as:

$$p(\lambda I - T) < \infty \quad \text{for every } \lambda \in \mathbb{C}$$

implies that T has SVEP. There are many examples of operators for which the condition $p(\lambda I - T) < \infty$ holds for every $\lambda \in \mathbb{C}$. For instance, every multiplier of a semi-prime Banach algebra verifies this property, see [16, p. 406], and in particular every convolution operator on a group algebra $L^1(G)$, where G is a locally compact Abelian group. Other examples of operators of this type are the generalized scalar operators, see [28], as well as several other classes of operators studied in [15]. As noted by Barnes [10], a class of operators which have this ascent property is given by the class $P(X)$ of all bounded operators on a Banach space X which satisfy a *polynomial growth condition*, where $T \in L(X)$ is said to satisfy a polynomial growth condition, if there exists a $K > 0$, a $\delta > 0$ for which

$$\|\exp(i\lambda T)\| \le K(1+|\lambda|^\delta) \text{ for all } \lambda \in \mathbb{R}.$$

The *finite descent property* for an operator $T \in L(X)$ is defined as:

$$q(\lambda I - T) < \infty \quad \text{for every } \lambda \in \mathbb{C}.$$

This property is obviously satisfied by every operator for which every spectral point is a pole of the resolvent.

THEOREM 1.11. *Let $T \in L(X)$, where X is a Banach space. Then T has the finite descent property precisely when $\sigma(T)$ is finite set of poles of the resolvent $R(\lambda, T)$.*

Proof. Clearly, if $\sigma(T)$ is finite set of poles of $R(\lambda, T)$ then $q(\lambda I - T) < \infty$ for every $\lambda \in \mathbb{C}$. Conversely, suppose that $q(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{C}$. Then $q(\lambda I - T) < \infty$ for all $\lambda \in \partial\sigma(T)$, $\partial\sigma(T)$ the boundary of $\sigma(T)$. Since T ha SVEP at every $\lambda \in \partial\sigma(T)$ then the condition $q(\lambda I - T) < \infty$ entails that every $\lambda \in \partial\sigma(T)$ is a pole of $R(\lambda, T)$, see Corollary 1 of [25]. Clearly, this implies that $\sigma(T) = \partial\sigma(T)$, so that the spectrum $\sigma(T)$ is a finite set of poles of $R(\lambda, T)$. ■

Therefore, the finite descent property implies that both T and T^* have SVEP (actually we have more, T is decomposable since it has finite spectrum). It should be noted that the proof of Theorem 1.11 shows that the finite descent property is equivalent to the apparently weaker condition $q(\lambda I - T) < \infty$ for all $\lambda \in \partial\sigma(T)$.

Theorem 1.7 suggests in a very natural way the following concept, introduced in [3]:

DEFINITION 1.12. A bounded operator $T \in L(X)$, X a Banach space, is said to have *property* (Q) if $H_0(\lambda I - T)$ is closed for every $\lambda \in \mathbb{C}$.

Clearly, every quasi-nilpotent operator has property (Q), since $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$ and $H_0(T) = X$. More generally,

$$\sigma(T) \text{ finite } \Rightarrow T \text{ has property } (Q). \tag{6}$$

Indeed, if $\lambda \in \sigma(T)$ is isolated then $H_0(\lambda I - T)$ coincides with the range of the spectral projection associated with the singleton set $\{\lambda\}$, see [14, Proposition 49.1]. In particular, the implication

$$T \text{ has finite descent property } \Rightarrow \ T \text{ has property } (Q). \tag{7}$$

holds. Since every multiplier of a semi-simple Banach algebra has property (Q), see Theorem 1.8 of [3], we see that any multiplier with a non-finite spectrum provides an example of operator which has property (Q), but not satisfies the finite descent property.

Recall that a bounded operator $T \in L(X)$, X a Banach space, is said to have *Dunford's property (C)*, shortly property (C), if the analytic subspace $X_T(\Omega)$ is closed for every closed subset $\Omega \subseteq \mathbb{C}$. It should be noted that property (β) implies property (C), see [16, Proposition 1.2.19] and it turns out, by part (ii) of Theorem 1.2, that property (C) implies that T has SVEP.

An obvious consequence of part (ii) of Theorem 1.5 is that if T has property (C) then $H_0(\lambda I - T) = X_T(\{\lambda\})$ is closed for every $\lambda \in \mathbb{C}$, so that the following implications hold:

$$T \text{ has property } (C) \;\Rightarrow\; T \text{ has property } (Q) \;\Rightarrow\; T \text{ has SVEP.} \tag{8}$$

Note that neither of the implications (8) may be reversed in general. A first counter-example, of an operator which has SVEP but not property (Q) is given by the operator T defined in Example 1.10, see also next Theorem 2.7. An example of an operator which shows that the first implication is not reversed in general, may be found among the convolution operators T_μ of group algebras $L^1(G)$, since these operators have property (Q), see [3], while may have not property (C), see Theorem 4.11.8 and Theorem 4.1.12 of [16].

2. An abstract shift condition

In this section we shall consider operators $T \in L(X)$ on a Banach space X for which $T^\infty(X) = \{0\}$. This condition may be viewed, in a certain sense, as an abstract shift condition, since it is satisfied by every weighted right shift T on $\ell^p(\mathbb{N})$. Clearly, the condition $T^\infty(X) = \{0\}$ entails that T is non-surjective and hence $0 \in \sigma(T)$. Moreover, this condition also implies that $K(T) = \{0\}$, since $K(T)$ is a subset of $T^\infty(X)$, but the quasi-nilpotent Volterra operator V on the Banach space $X := C[0,1]$, defined by

$$(Vf)(t) := \int_0^t f(s)\mathrm{d}s \text{ for all } f \in C[0,1] \text{ and } t \in [0,1],$$

shows that, in general, the converse is not true. Indeed, V is quasi-nilpotent and hence $K(V) = \{0\}$, while

$$V^\infty(X) = \{f \in C^\infty[0,1] : f^{(n)}(0) = 0 \; n \in \mathbb{Z}_+\},$$

thus $V^\infty(X)$ is not closed and, consequently, strictly larger than $K(V) = \{0\}$.

It is easily seen that the condition $T^\infty(X) = \{0\}$ has some other important consequences, for instance:

$$T^\infty(X) = \{0\} \;\Rightarrow\; p(\lambda I - T) = 0 \text{ for all } \lambda \neq 0.$$

and

$$q(\lambda I - T) = \infty \text{ for all } \lambda \in \sigma(T) \setminus \{0\}. \tag{9}$$

Indeed, ker $(\lambda I - T) = \{0\}$ for all $0 \neq \lambda \in \mathbb{C}$, since ker $(\lambda I - T) \subseteq T^\infty(X)$ for all $\lambda \neq 0$. This implies that $q(\lambda I - T) = \infty$ for all $\lambda \in \sigma(T) \setminus \{0\}$ otherwise, if were $q(\lambda I - T) < \infty$, then $q(\lambda I - T) = p(\lambda I - T) = 0$ and hence $\lambda \notin \sigma(T)$, which is impossible.

For an operator $T \in L(X)$, let

$$k(T) := \inf\{\|Tx\| : x \in X \text{ and } \|x\| = 1\}$$

be the *lower bound* of T and define

$$i(T) := \lim k(T^n)^{1/n} = \sup_{n\in\mathbb{N}} k(T^n)^{1/n}.$$

Clearly, if $r(T)$ is the spectral radius of $T \in L(X)$ then $i(T) \le r(T)$. In the sequel by $\mathbf{D}(0, i(T))$ we shall denote the closed disc centered at 0 and radius $i(T)$.

THEOREM 2.1. *If $T^\infty(X) = \{0\}$ then T has SVEP. Moreover, the following statements hold:*

(i) *$\sigma_T(x)$ is connected and*

$$\mathbf{D}(0, i(T)) \subseteq \sigma_T(x) \quad \text{for all } 0 \neq x \in X. \tag{10}$$

(ii) *$H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq \{0\}$. Consequently, T has property (Q) if and only if $H_0(T)$ is closed.*

(iii) *If $i(T) > 0$, then T has property (Q).*

(iv) *If $i(T) = r(T)$, then T has property (C).*

Proof. The SVEP may be proved in several ways, for instance from Theorem 1.2, since ker $(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$. Moreover, the local spectrum $\sigma_T(x)$ is connected, by Theorem 1 of [26]. The proof of the inclusion (10) is proved in [16, Theorem 1.6.3]. To show the statement (ii) observe first that the SVEP for T implies, by part (ii) of Theorem 1.5, that

$$H_0(\lambda I - T) = \{x \in X : \sigma_T(x) \subseteq \{\lambda\}\} \quad \text{for all } \lambda \in \mathbb{C}.$$

Now, let $\lambda \neq 0$ and suppose that there is $0 \neq x \in H_0(\lambda I - T)$. Since T has SVEP, by part (ii) of Theorem 1.2, we obtain that $\sigma_T(x) \neq \emptyset$, so that $\sigma_T(x) = \{\lambda\}$, which is impossible since $0 \in \sigma_T(x)$, by part (i). Therefore $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$.

To prove (iii) it suffices to prove, by part (ii), that $H_0(T) = \{0\}$. Since,

$$H_0(T) = \{x \in X : \sigma_T(x) \subseteq \{0\}\},$$

from the inclusion (10) we infer that the condition $i(T) > 0$ entails that each $x \neq 0$ cannot belong to $H_0(T)$. Therefore, $H_0(T) = \{0\}$.

The assertion (iv) has been proved in Proposition 1.6.5 of [16]. We give the simple proof for sake of completeness. Suppose now that $i(T) = r(T)$. Then $\mathbf{D}(0, r(T))$ is contained in $\sigma_T(x)$ for all non-zero $x \in X$, and hence

$$\sigma_T(x) = \mathbf{D}(0, r(T)) = \sigma(T) \quad \text{for all non-zero } x \in X.$$

This implies that $X_T(\Omega) = X$ for every closed set Ω which contains $\mathbf{D}(0, r(T))$, while $X_T(\Omega) = \{0\}$ otherwise. ■

Let $\sigma_w(T)$ denote the *Weyl spectrum*, i.e. the complement of the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is a Fredholm operator with index ind $T := \alpha(T) - \beta(T) = 0$. The *Browder spectrum* $\sigma_b(T)$ is defined as the complement of all $\lambda \in \mathbb{C}$ for which $\lambda I - T \in \Phi(X)$ and $p(\lambda I - T) = q(\lambda I - T) < \infty$. Note that $\sigma_w(T) \subseteq \sigma_b(T)$ and this inclusion is in general proper.

Theorem 2.2. *Let $T \in L(X)$, X an infinite-dimensional Banach space, and suppose that $T^\infty(X) = \{0\}$. Then $\sigma(T)$ is connected and*

$$\sigma(T) = \sigma_w(T) = \sigma_b(T). \tag{11}$$

Proof. By Proposition 1.3.2 of [16] we have, since T has SVEP,

$$\sigma(T) = \sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x),$$

where $\sigma_{su}(T)$ denotes the surjectivity spectrum of T. Since the local spectra $\sigma_T(x)$ are connected then $\sigma(T)$ is connected. The equality (11) has been established in [26]. We give here a simpler proof.

By Corollary 2.8 of [7], we have $\sigma_w(T) = \sigma_b(T)$, since T has SVEP. We show that $\sigma_b(T) = \sigma(T)$. The inclusion $\sigma_b(T) \subseteq \sigma(T)$ holds for all $T \in L(X)$, so it remains to establish that $\sigma(T) \subseteq \sigma_b(T)$. Observe that, if the spectral point $\lambda \in \mathbb{C}$ is not isolated in $\sigma(T)$, then $\lambda \in \sigma_b(T)$.

Suppose first that T is quasi-nilpotent. Then $\sigma_b(T) = \sigma(T) = \{0\}$, since $\sigma_b(T)$ is non-empty whenever X is infinite-dimensional. Suppose that T is not quasi-nilpotent and let $0 \neq \lambda \in \sigma(T)$. Since $\sigma(T)$ is connected and $0 \in \sigma(T)$, then neither 0 or λ are isolated points in $\sigma(T)$. Hence $\sigma(T) \subseteq \sigma_b(T)$. ∎

Let $\rho_{kt}(T)$ denote the *Kato type resolvent* of T, defined as

$$\rho_{kt}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is of Kato type}\}.$$

The set $\rho_{kt}(T)$ is an open subset of $\mathbb{C}$, see [4], so it may be decomposed in maximal connected open components.

The following two results generalize Theorem 2 of Schmoeger [26].

Theorem 2.3. *Suppose that $T \in L(X)$, where X is a Banach space, is of Kato type. Let Ω denote the connected component of $\rho_{kt}(T)$ which contains 0. Then $T^\infty(X) = \{0\}$ if and only if $p := p(T) < \infty$ and*

$$\Omega \subseteq \sigma_T(x) \quad \text{for all } x \notin \ker T^p. \tag{12}$$

Furthermore, if $T^\infty(X) = \{0\}$, then T has property (Q).

Proof. Suppose that $T^\infty(X) = \{0\}$. Since T has SVEP then $p(T) < \infty$, by Theorem 1.8, and $H_0(T) = \ker T^p$, by Corollary 2.7 of [3]. The SVEP of T also ensures, by Theorem 1.7, that $H_0(\lambda I - T)$ is closed for all $\lambda \in \Omega$. Since the mapping

$$\lambda \in \Omega \mapsto \overline{H_0(\lambda I - T)} + K(\lambda I - T) = H_0(\lambda I - T) + K(\lambda I - T)$$

is constant on Ω, see [20] or [8], we then have

$$\ker T^p = H_0(T) + K(T) = H_0(\lambda I - T) + K(\lambda I - T)$$

for all $\lambda \in \Omega$. Now, by part (iii) of Theorem 2.1, $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$ and hence

$$\ker T^p = K(\lambda I - T) = \{x \in X : \lambda \notin \sigma_T(x)\},$$

for all $\lambda \in \Omega \setminus \{0\}$. Thus, for $x \notin \ker T^p$ we obtain $\Omega \setminus \{0\} \subseteq \sigma_T(x)$ and, by part (ii) of Theorem 2.1, $0 \in \sigma_T(x)$ for all $x \neq 0$. This implies that $\Omega \subseteq \sigma_T(x)$ for all $x \notin \ker T^p$.

Conversely, assume that $p = p(T) < \infty$ and $\Omega \subseteq \sigma_T(x)$ for all $x \notin \ker T^p$. The condition $p := p(T) < \infty$ entails that T has SVEP at 0 and, by Theorem 1.8, $\mathcal{N}^\infty(T) \cap T^\infty(X) = \{0\}$. Assume that there exists $0 \neq x \in T^\infty(X)$. Then $x \notin \mathcal{N}^\infty(T) = \ker T^p$, thus $0 \in \sigma_T(x)$. On the other hand, since $T^\infty(X) = K(T)$, see Theorem 2.4 of [2], then $x \in K(T)$, so that $0 \notin \sigma_T(x)$, by Theorem 1.5; a contradiction. Therefore, $T^\infty(X) = \{0\}$.

To show the last assertion, observe that $H_0(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$, while the SVEP, by Theorem 1.7, implies that also $H_0(T)$ is closed. ∎

It should be noted that the previous result extends, in a sense, Theorem 2.1. In fact, if $\sigma_{ap}(T)$ denotes the *approximate point spectrum* of T, then

$$\sigma_{ap}(T) \subseteq \{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\},$$

see Proposition 1.6.2 of [16]. Therefore the condition $i(T) > 0$ entails that T is bounded below and hence is of Kato type.

THEOREM 2.4. *Suppose that $T \in L(X)$ is of Kato type and $T^\infty(X) = \{0\}$. If T is not quasi-nilpotent then T^* does not have the SVEP. Moreover, T is not decomposable.*

Proof. We know that $q(\lambda_0 I - T) = \infty$ for all $0 \neq \lambda \in \sigma(T)$. Indeed, if for some $\lambda_0 \in \sigma(T) \setminus \{0\}$ we have $q(\lambda_0 I - T) < \infty$. Let Ω be the component of $\sigma_{kt}(T)$ containing 0. Since $\sigma(T)$ is connected and T is not quasi-nilpotent, then $\Omega \cap (\sigma(T) \setminus \{0\}) \neq \emptyset$. Let $\mu \in \Omega \cap (\sigma(T) \setminus \{0\})$. Since $\mu I - T$ is of Kato type, the condition $q(\mu I - T) = \infty$ entails that T^* does not have the SVEP at μ, see Theorem 2.9 of [3].

The last assertion is clear, since the decomposability of T implies that $T^\star$ is decomposable and hence has SVEP, by Theorem 2.5.19 of [16]. ∎

The previous results apply to isometries, since $i(T) = r(T) = 1$ for every isometry $T \in L(X)$. Note that for every isometry $\sigma_{ap}(T)$ is contained in the unit circle, so that $\lambda I - T$ is bounded below and hence upper semi-Fredholm for every $|\lambda| < 1$, see Proposition 1.6.2 of [16]. Hence every isometry T has property (C), by Theorem 2.1 (actually we have much more, T has property (β), by Proposition 1.6.7 of [16]). In the case that an isometry T is non-invertible, for instance in the case that $T^\infty(X) = \{0\}$, the spectrum is the entire closed unit disc, while $\sigma_{ap}(T)$ is the unit circle. Furthermore, by Corollary 2.9 of [7], we have $\sigma_{ap}(T) = \sigma_{kt}(T)$.

An isometry $T \in L(X)$ for which the condition $T^\infty(X) = \{0\}$ is satisfied is said to be a *semi-shift*. Proposition 1.6.8 of [16] shows that T is a semi-shift if and only if T has *fat local spectra*, i.e. the equality $\sigma_T(x) = \sigma(T)$ holds for every $x \neq 0$, see also [23]. Examples of semi-shifts are the unilateral right shift operators of arbitrary multiplicity on $\ell^p(\mathbb{N})$, as well as every right translation operator on $L^p([0, \infty))$, see Section 1.6 of [16]. From Theorem 2.4 it follows that every semi-shift operator is not decomposable.

The following result has been established in [21]. We give an alternative proof.

Theorem 2.5. *If $K(T)=\{0\}$, the following assertions are equivalent:*
(i) *T is decomposable;* (ii) *T has property (δ);*
(iii) *T is quasi-nilpotent;* (iv) *0 is an isolated point of $\sigma(T)$;*
(v) *$q(\lambda I-T)<\infty$ for all $\lambda\neq 0$.*

Proof. Clearly (i) $\Rightarrow$ (ii). To establish the implication (ii) $\Rightarrow$ (iii), we show first that the surjectivity spectrum $\sigma_{su}(T)$ is $\{0\}$. Suppose that $\lambda\neq 0$ and choose $\varepsilon>0$ such the closed disc $\mathbf{D}(\lambda,\varepsilon)$ does not contains 0. Let $U:=\mathbb{D}(0,\varepsilon/2)$ and $V:=\mathbb{C}\setminus\mathbf{D}(0,\varepsilon)$, where $\mathbb{D}(0,\varepsilon/2)$ is the open disc centered at 0 and radius $\varepsilon/2$. Clearly, $\{U,V\}$ is an open cover of $\mathbb{C}$, so, taking into account that T has SVEP, the property (δ) implies that $X=X_T(\mathbf{D}(0,\varepsilon))+X_T(\overline{\mathbb{C}\setminus\mathbf{D}(0,\varepsilon)})$. From the inclusion $\overline{\mathbb{C}\setminus\mathbf{D}(0,\varepsilon)})\subseteq\mathbb{C}\setminus\{0\}$ we infer

$$X_T(\overline{\mathbb{C}\setminus\mathbf{D}(0,\varepsilon)})\subseteq X_T(\mathbb{C}\setminus\{0\})=K(T)=\{0\},$$

so that $X=X_T(\mathbf{D}(0,\varepsilon))$. On the other hand, by Proposition 1.2.16 of [16], we know that

$$(\mu I-T)(X_T(\mathbf{D}(0,\varepsilon))=X_T(\mathbf{D}(0,\varepsilon))\quad\text{for all } |\mu|>\varepsilon,$$

thus $\lambda\notin\sigma_{su}(T)$. Hence, $\sigma_{su}(T)=\{0\}$. On the other hand the point spectrum $\sigma_p(T)$ is contained in $\{0\}$, so that $\sigma(T)=\sigma_p(T)\cup\sigma_{su}(T)=\{0\}$.

Clearly, (iii) $\Rightarrow$ (i), so the statements (i), (ii) and (iii) are equivalent. Obviously, (iii) $\Rightarrow$ (iv). We prove that (iv) $\Rightarrow$ (iii). Suppose 0 isolated in $\sigma(T)$. Then $K(T)=\ker P$ and $H_0(T)=P(X)$, where P is the spectral projection associated with the spectral set $\{0\}$. From $K(T)=\{0\}$ we deduce that $H_0(T)=X$, so T is quasi-nilpotent. It is evident that (iii) $\Rightarrow$ (v). To show the opposite implication, assume that T is not quasi-nilpotent. Then $q(\lambda I-T)=\infty$ for all $0\neq\lambda\in\sigma(T)$, by (9), so the proof is complete. ∎

It is easy to see that, if $T^\infty(X)=\{0\}$, then

$$T \text{ has the finite descent condition } \Leftrightarrow\ T \text{ is nilpotent.}$$

In fact, if $q:=q(T)<\infty$, then $T^\infty(X)=T^p(X)=\{0\}$, while the converse is obvious.

Example 2.6. In [16, p.89] an example is given of a right shift T on $\ell^p(\mathbb{N})$ such that T has property (C), but not property (β). Therefore, T also provides an example of a shift which has property (Q), but not quasi-nilpotent.

However, the following result, established in [10], shows that the property (Q) for a right shift T is equivalent to be T quasi-nilpotent in some special case.

Theorem 2.7. *Suppose that infinitely many weights ω_n are zero. Then for the corresponding right shift T on $\ell^p(\mathbb{N})$, $1\leq p<\infty$, the following statements are equivalent:*
(i) *T is quasi-nilpotent;* (ii) *T is decomposable;*
(iii) *T has property (δ);* (iv) *T has property (β);*

(v) T *has property* (C); (vi) T *has property* (Q);
(vii) $H_0(T)$ *is closed.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) have been proved in Theorem 2.5. The implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are satisfied by every bounded operator.

(vi) $\Rightarrow$ (i) Suppose that $H_0(T)$ is closed. Since $Te_n = \omega_n e_{n+1}$ for all $n \in \mathbb{N}$, if $\omega_n = 0$ then $e_n \in \ker T \subseteq H_0(T)$. Suppose that $\omega_n \neq 0$ and let k be the smallest integer such that $\omega_{n+k} = 0$. It is easy to check that

$$T^{k+2} e_n = \omega_n \omega_{n+1} \cdots \omega_{n+k} \; e_{n+k+1} = 0,$$

so $e_n \in \ker T^{k+2} \subseteq H_0(T)$. This shows that $H_0(T) = \ell^p(\mathbb{N})$ and hence T is quasi-nilpotent. Therefore (vi) $\Leftrightarrow$ (i). Since the equivalence (vi) $\Leftrightarrow$ (vii) has been proved in part (ii) of Theorem 2.1, the proof is complete. ■

We conclude this paper by mentioning a characterization of property (Q), established by Bourhim [10, Proposition 4.5], in the case that the weighted right shift T is injective. It is easily seen that, if $(\omega_n)_{n\in\mathbb{N}}$ is the weight sequence, then T is injective if and only if none of the weights ω_n is zero; so that Theorem 2.7 does not apply to this case.

THEOREM 2.8. *Suppose that the right shift operator* T *on* $\ell^p(\mathbb{N})$, *for some* $1 \le p < \infty$, *is injective. Then the following statements are equivalent:*

(i) T *has property* (Q);

(ii) *Either* T *is quasi-nilpotent or* $d(T) := \lim_{n\to\infty} \sup(\omega_1 \cdots \omega_n)^{1/n} > 0$. ■

Observe that the quantity $i(T)$ for a right shift operator T with weight $(\omega_n)_{n\in\mathbb{N}}$ may be computed as

$$i(T) = \lim_{n\to\infty} \inf_{k\in\mathbb{N}} (\omega_k \cdots \omega_{k+n-1})^{1/n}$$

and $i(T) \le d(T)$, so, to prove the preceding theorem, we cannot apply the result from Theorem 2.1.

ACKNOWLDGEMENT. The first author thanks the organizers of the Symposium V. Rakočević and D. Đorđevic for this fruitful session.

REFERENCES

[**1**] P. Aiena, O. Monsalve, *Operators which do not have the single valued extension property*, J. Math. Anal. Appl. **250** (2000), 435–448.

[**2**] P. Aiena, O. Monsalve, *The single valued extension property and the generalized Kato decomposition property*, Acta Sci. Math. (Szeged) **67** (2001), 461–477.

[**3**] P. Aiena, M. L. Colasante, M. Gonzalez, *Operators which have a closed quasi-nilpotent part*, Proc. Amer. Math. Soc. **130** (2002), 2701–2710.

[**4**] P. Aiena, M. Mbekhta, *Characterization of some classes of operators by means of the Kato decomposition*, Boll. Un. Mat. It. **10-A** (1996), 609–621.

[**5**] P. Aiena, T. L. Miller, M. M. Neumann, *On a localized single valued extension property*, to appear in Proc. Royal Irish Acad. (2002).

[**6**] P. Aiena, E. Rosas, *The single valued extension property at the points of the approximate point spectrum*, to appear in J. Math. Anal. Appl.

[**7**] P. Aiena, C. Carpintero, *Single-valued extension property and semi-Browder spectra*, preprint (2002).

[**8**] P. Aiena, F. Villafãne, *Components of resolvent sets and local spectral theory*, to appear on Contemporary Math. (2002).

[**9**] B. A. Barnes, *Operators which satisfy polynomial growth conditions*, Pacific. J. Math. **138** (1989), 209–219.

[**10**] A. Bourhim, *On the local spectral properties of weighted shift operators*, A. Salam International Center for Theoretical Physics (2002), preprint.

[**11**] S. R. Caradus, W. E. Pfaffenberger, B. Yood, *Calkin Algebras of Operators and Algebras of Operators on Banach Spaces*, M. Dekker, New York, 1974.

[**12**] I. Colojoară, C. Foiaş, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York, 1968.

[**13**] J. K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.

[**14**] H. Heuser, *Functional Analysis*, Wiley, New York, 1982.

[**15**] K. B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–36.

[**16**] K. B. Laursen, M. M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.

[**17**] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.

[**18**] M. Mbekhta, *Sur la théorie spectrale locale et limite des nilpotents*, Proc. Amer. Math. Soc. **110** (1990), 621–631.

[**19**] M. Mbekhta, *Local spectrum and generalized spectrum*, Proc. Amer. Math. Soc. **112** (1991), 457–463.

[**20**] M. Mbekhta, A. Ouahab, *Perturbation des opérateurs s-réguliers*, Topics in operator theory, operator algebras and applications, Timisoara (1994), Rom. Acad. Bucharest, 239–249.

[**21**] T. L. Miller, V. G. Miller, M. M. Neumann, *On operators with closed analytic core*, to appear in Circolo Mat. Palermo (2000).

[**22**] V. Müller, *On the regular spectrum*, J. Operator Theory **31** (1994), 363–380.

[**23**] M. M. Neumann, *Banach algebras and local spectral theory*. Proc. Int. Workshop on Operator Theory, (Cefalú), Italy, July 1997, Suppl. Rend. Circ. Matem. Palermo **56** (1998), 15–25.

[**24**] V. Rakočević, *Generalized spectrum and commuting compact perturbations*, Proc. Edinburgh Math. Soc. **36** (2) (1993), 197–209.

[**25**] C. Schmoeger, *On isolated points of the spectrum of a bounded operator*, Proc. Amer. Math. Soc. **117** (1993), 715–719.

[**26**] C. Schmoeger, *Semi-Fredholm operators and local spectral theory*, Demonstratio Math. **4** (1995), 997–1004.

[**27**] P. Vrbová, *On local spectral properties of operators in Banach spaces*, Czechoslovak Math. J. **23** (98) (1973a), 483–492.

[**28**] P. Vrbová, *The structure of maximal spectral spaces of generalized scalar operators*, Czechoslovak Math. J. **23** (98) (1973a), 493–496.

[**29**] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Roy. Irish Acad. **87 A**, N.2 (1987), 137–146.

(received 29.10.2002)

P. Aiena, Dipartimento di Matematica ed Applicazioni, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo (Italy)

E-mail: paiena@mbox.unipa.it

M. T. Biondi, Departamento de Matemáticas, Facultad de Ciencias, Universidad UCLA de Barquisimeto (Venezuela),

E-mail: mtbiondi@hotmail.com